Crossed Ophthalmic Cylinders

Gregg Baldwin, OD
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It is useful to know the meridian of maximum axial refraction when combining the effects of two ophthalmic cylinders crossed obliquely. To do this, we need to first describe how the axial radius of curvature of an ophthalmic cylinder changes from infinity along its axis to its minimum value perpendicular to that axis. Ophthalmic cylinder meridional sections are ellipses of variable shape that transform from initial front and back parallel lines along the cylinder axis to a circular section perpendicular to that axis.

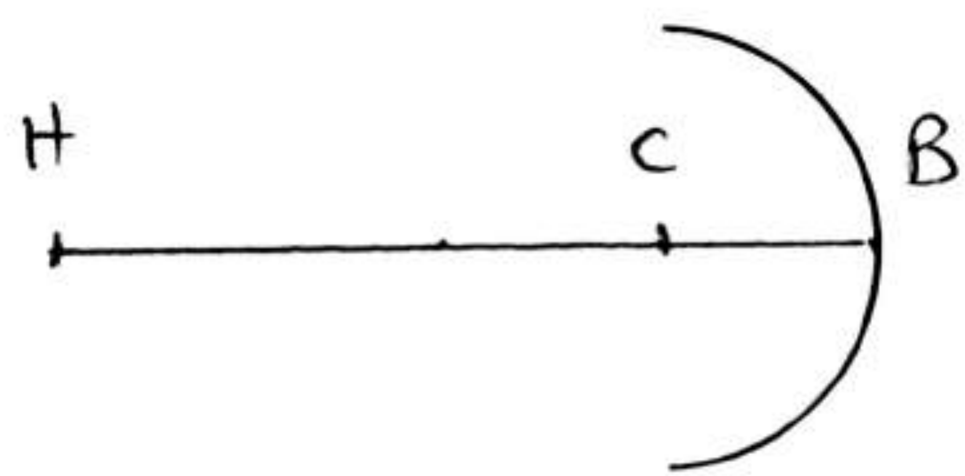
Assume that the meridian of minimum ophthalmic cylinder radius occurs in a parabolic section, rather than a circular one.

Now assume that meridional sections maintain a parabolic shape as they vary towards a single tangential point represented as the cylinder axis with an infinite radius of curvature relative to that point.

This will allow for the following *relatively* easy approximation of the axial radii of curvature of meridional sections. If these approximate axial radii of curvature are expressed in forms that are additive in terms of refraction, we can then approximate the sum of those expressions for any meridional section of obliquely crossed ophthalmic cylinders, and we can approximate the maximum sum of those expressions with the required meridional axis.

We know that with any axial radius of curvature CB, and index of refraction $\mathbb{R}$, the axial image of a distant object lies at H when:

$\mathbb{R}$ = HB/HC



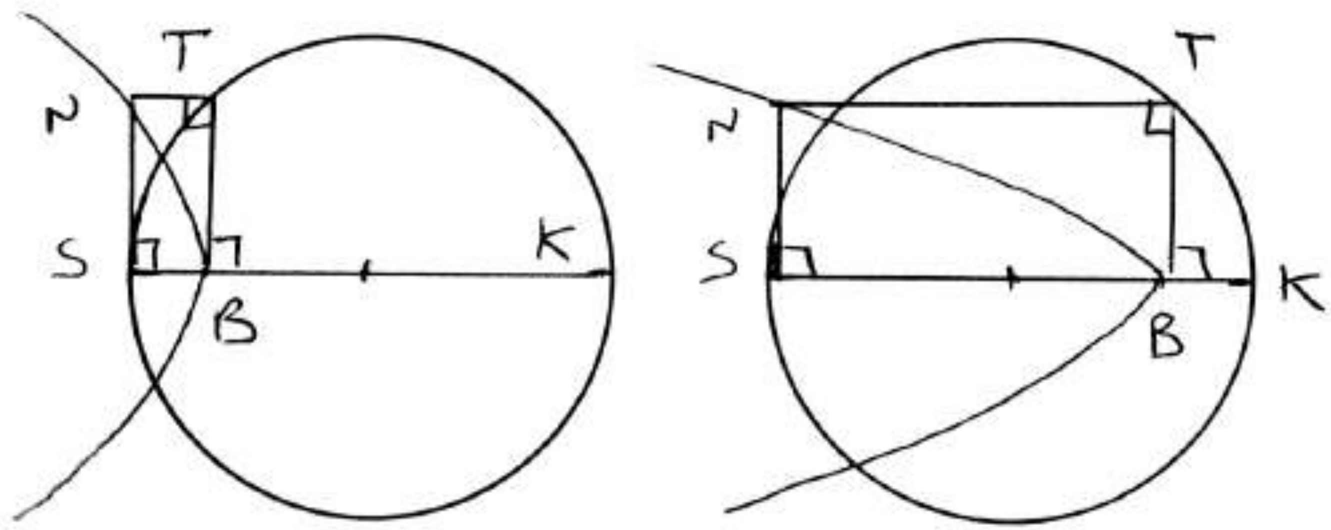
We also know that the axial refractive effects of compound refractive surfaces at B are additive only as their refractive "powers," which equal:

$$\mathbb{R}/HB \;=\; 1/HC \;=\; [(HB - HC)/HC]/CB \;=$$

$$(\mathbb{R} - 1)/CB$$

All parabolas have the same shape, in the same way that all circles have the same shape. However, while circles have a single (internal) determining constant, the radius of curvature, parabolas have both a determining constant internal and external to the curve, and can be defined by either.

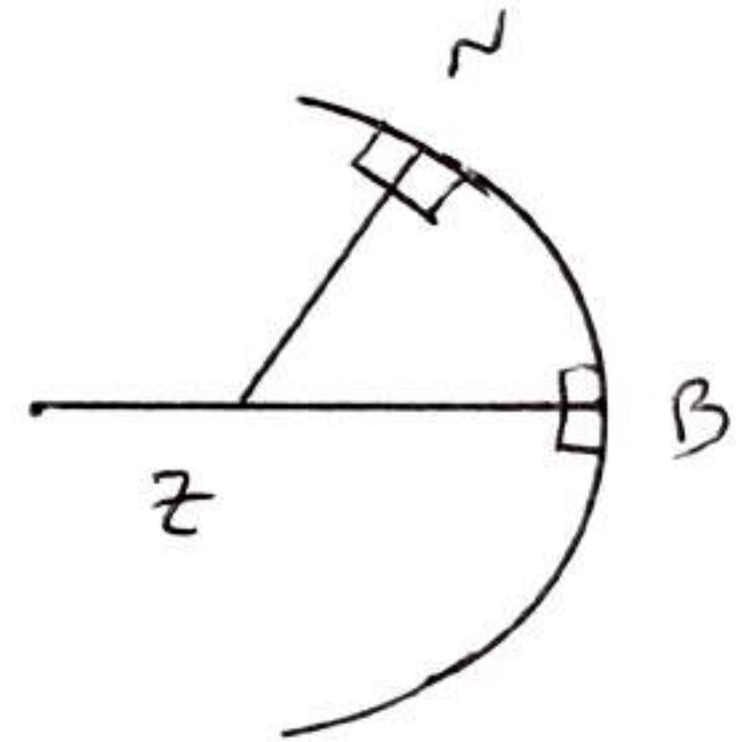
For example, a parabola's external determining constant equals BK when:



$$\frac{SB}{BT} = \frac{BT}{BK}$$

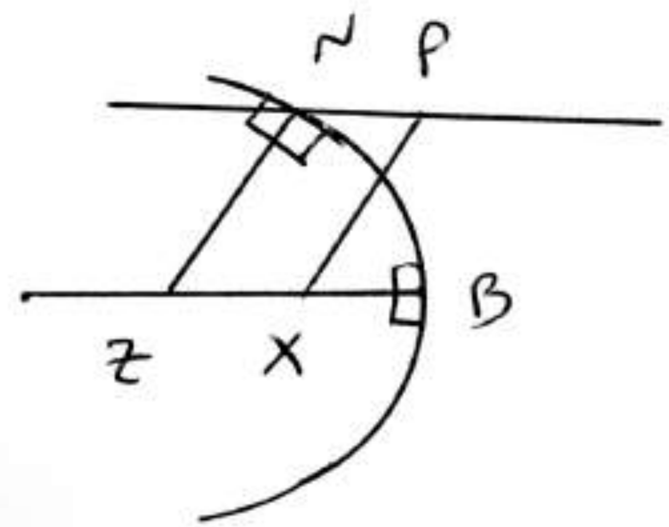
Both these curves have the same shape. The one on the left simply represents a "zoomed in" look at the vertex of the one on the right.

We can set up the necessary off-axis conditions to determine a parabola's axial center of curvature in terms of its internal determining constant XB, by involving ZN in the geometric solution for XB.



In order to keep the determining geometrical relationships axial as N ⇒ B, they should also

depend on line NP being parallel to the axis, and XP being parallel to ZN.



We know X lies between Z and B, since parabolas flatten in their periphery.

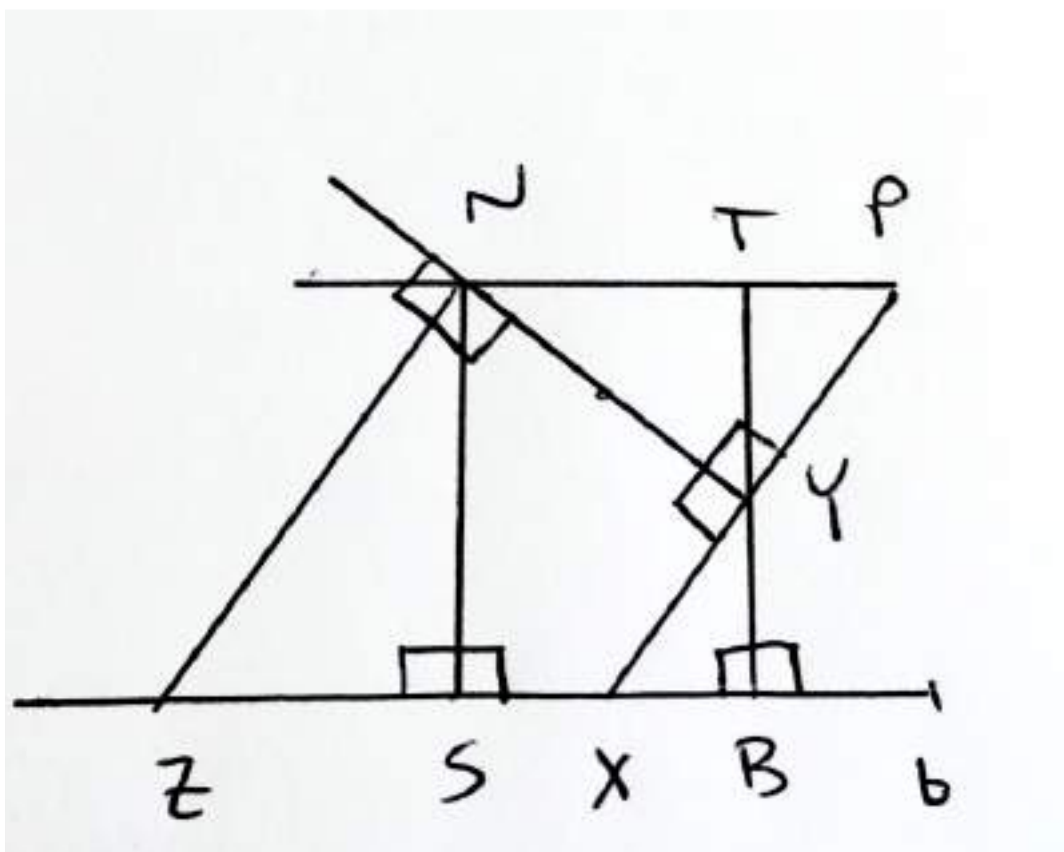
Since as N $\Rightarrow$ B,

Z $\Rightarrow$ C by definition,

and since XP = ZN,

P will remain external to the curve, and X can therefore not be its axial center of curvature, but must instead lie somewhere along CB.

In order to maintain ZN perpendicular to the parabola at N as N ⇒ B, the same geometrical relationships must exist that allow for that when N lies at B.



In other words:

YP = YX and
Bb = BX so
CB = 2(XB)

$$\frac{TN}{TB} = \frac{TN}{2(TY)} = \frac{YB}{2(XB)} = \frac{YB}{CB} = \frac{TB}{2(CB)}$$

Since TN = SB, the external determining constant BK equals 2(CB).
Since TB = 2(YB), the internal determining constant XB equals (CB)/2.

Refracting power equals: $(\mathbb{R} - 1)/CB$

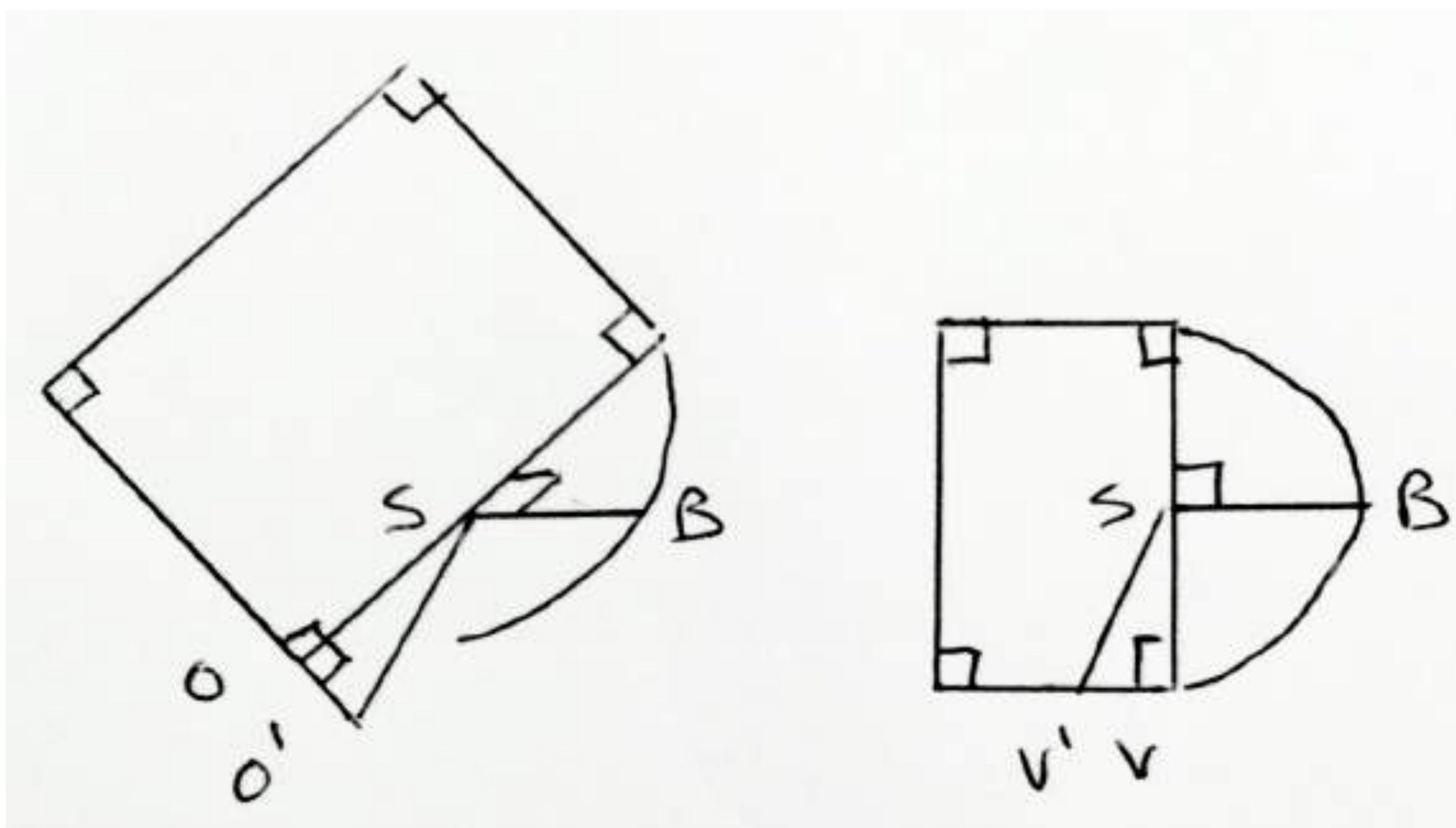
If $\mathbb{R} = 1.5$, this equals: $1/[2(CB)]$

For a parabola: $SB/BT = BT/BK = BT/[2(CB)]$

so its axial refracting power then equals:

$SB/TB^2 = SB/SN^2 = 1/BK$

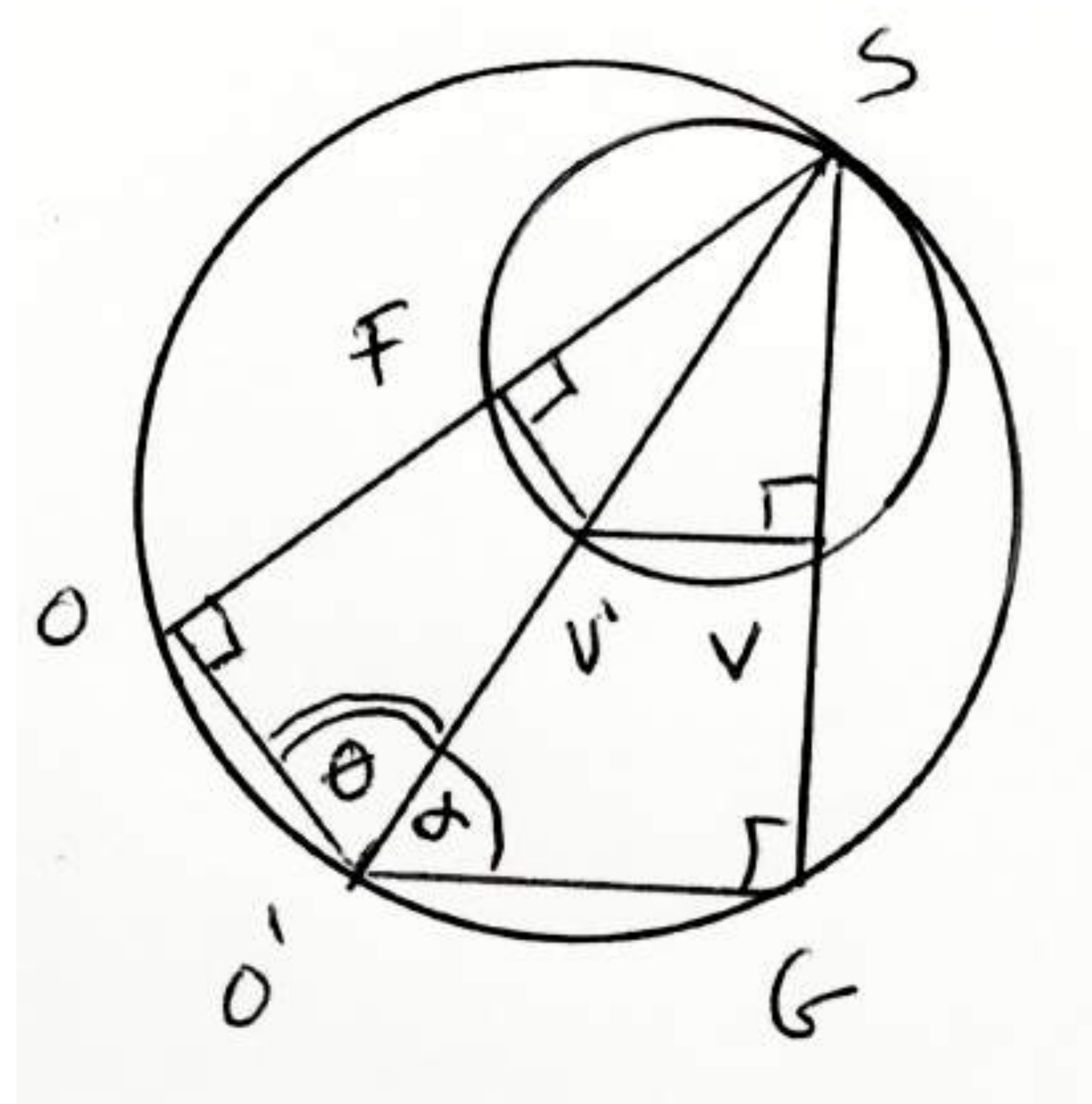
When 2(SO) equals the minimum sagitta of an oblique parabolic cylinder, and when with equal sagittal depth SB, 2(SV) equals the minimum sagitta of a more highly curved parabolic cylinder with a horizontal axis:



Keeping ΔOSV constant, as we rotate circle SOG with variable diameter SV'O' around point S:

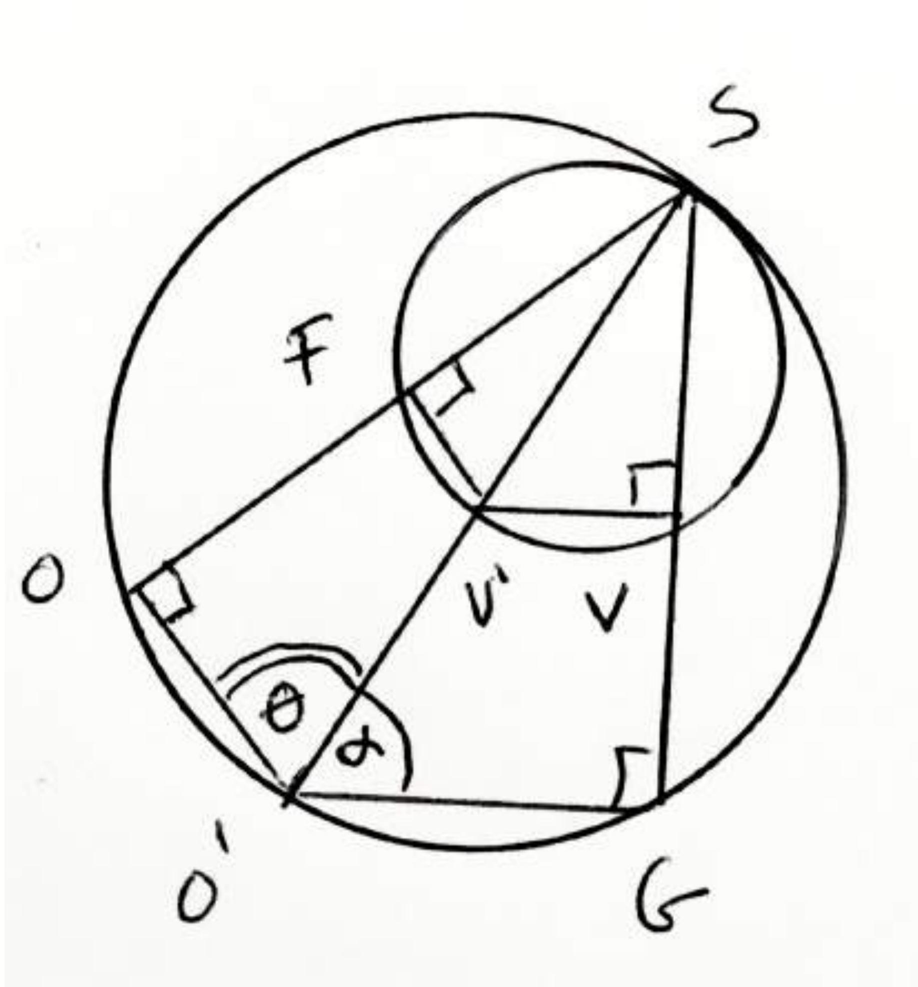
∠OO'G is constant because ∠OSG is constant,

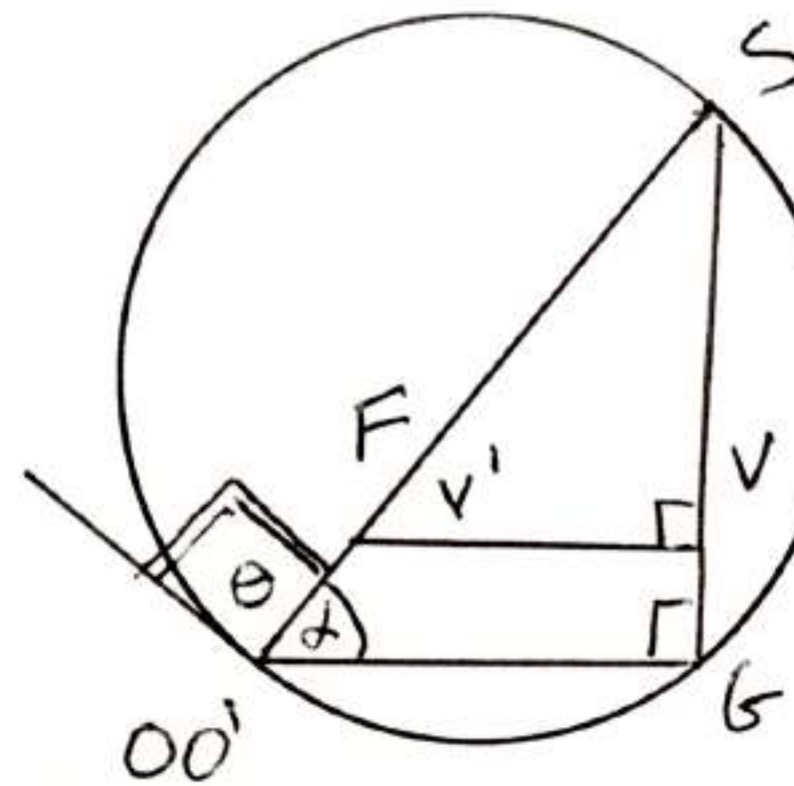
so $\Delta\theta = -\Delta\alpha$



Keeping ΔOSV constant,   as O' ⇒ O:

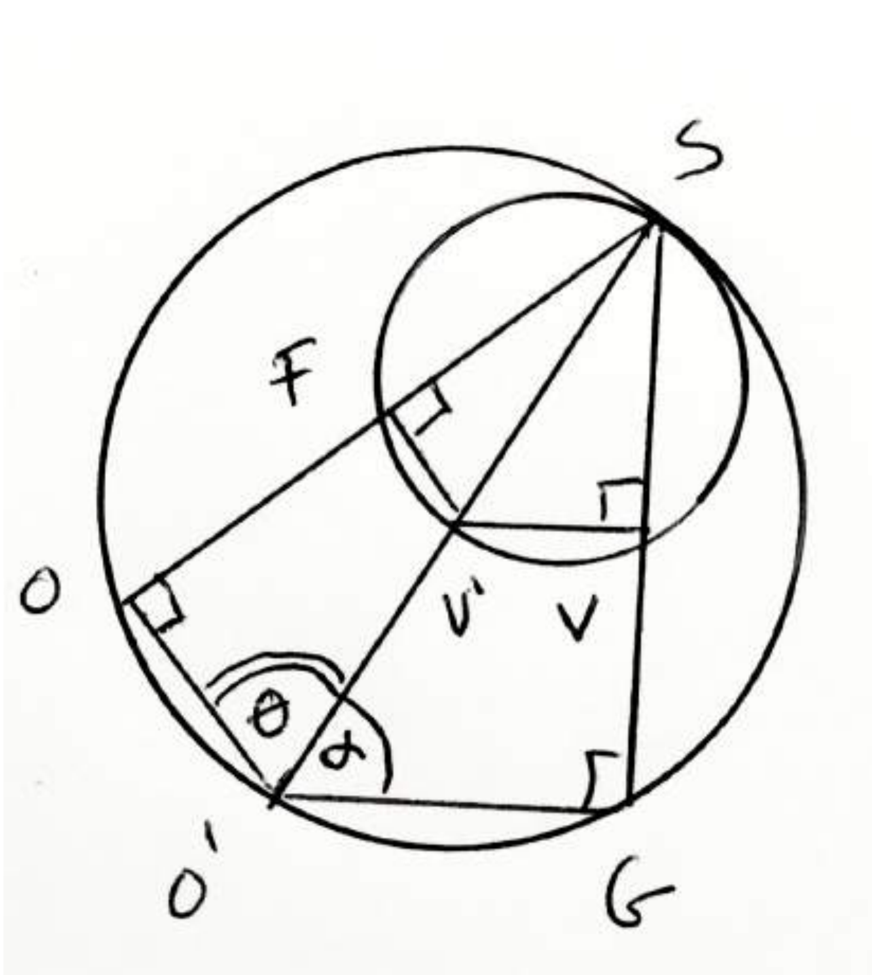
SG and diameter SO' decrease.
SV' increases more than SO' decreases.

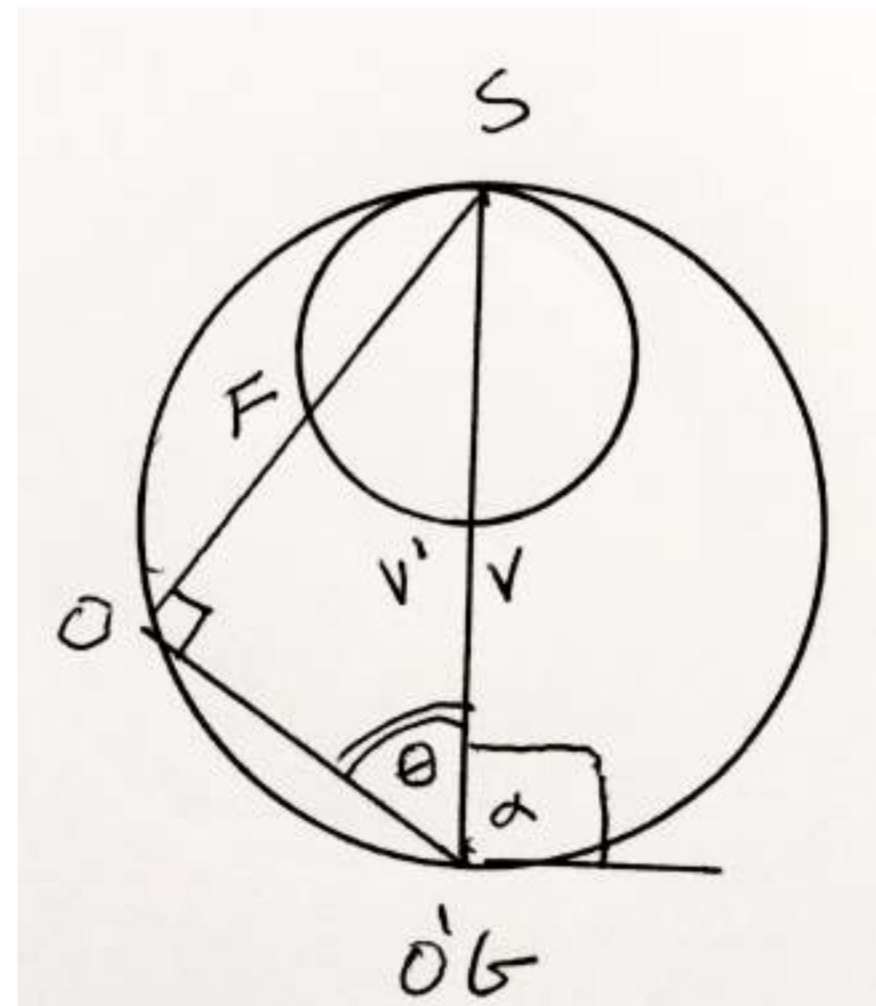




Keeping ΔOSV constant, as V' ⇒ V:

SG and diameter SO' increase.
SO' increases more than SV' decreases.





Since the sum (SO' + SV') increases when either:

O' ⇒ O,   or   V' ⇒ V

there must be a specific SV'O' within ΔOSV producing a minimum sum (SO' + SV'), which must be near where small rotations of SV'O' about S produce only minimal changes in the sum (SO' + SV').

Since as when one term of the sum (SO' + SV') increases, the other always decreases, the minimum (SO' + SV') must occur near where small rotations of SV'O' within ΔOSV produce equal but opposite changes in SO' and SV'. Therefore, the minimum (SO' + SV') can be found by finding the position of SV'O' where:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta(SO') = \lim_{\Delta\alpha \Rightarrow 0} \Delta(SV')$$

However, the combined effects of refraction are additive only as refractive powers, which, when $\mathbb{R} = 1.5$, equal:

$SB/(SO')^2$ and $SB/(SV')^2$

Therefore, the meridian with the maximum combined effects of this refraction can be found by finding the position of SV'O' where:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta \left[SB/(SO')^2\right] = \lim_{\Delta\alpha \Rightarrow 0} \Delta \left[SB/(SV')^2\right]$$

To solve this equation, each expressed limit must be transformed into the variable that approaches zero, so the equation must be transformed into:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta\{[SB(SO/SO')^2]/SO^2\} = \lim_{\Delta\alpha \Rightarrow 0} \Delta\{[SB(SV/SV')^2]/SV^2\}$$

$$\lim_{\Delta\theta \Rightarrow 0} \Delta\{[(SB)\sin^2\theta]/SO^2\} = \lim_{\Delta\alpha \Rightarrow 0} \Delta\{[(SB)\sin^2\alpha]/SV^2\}$$

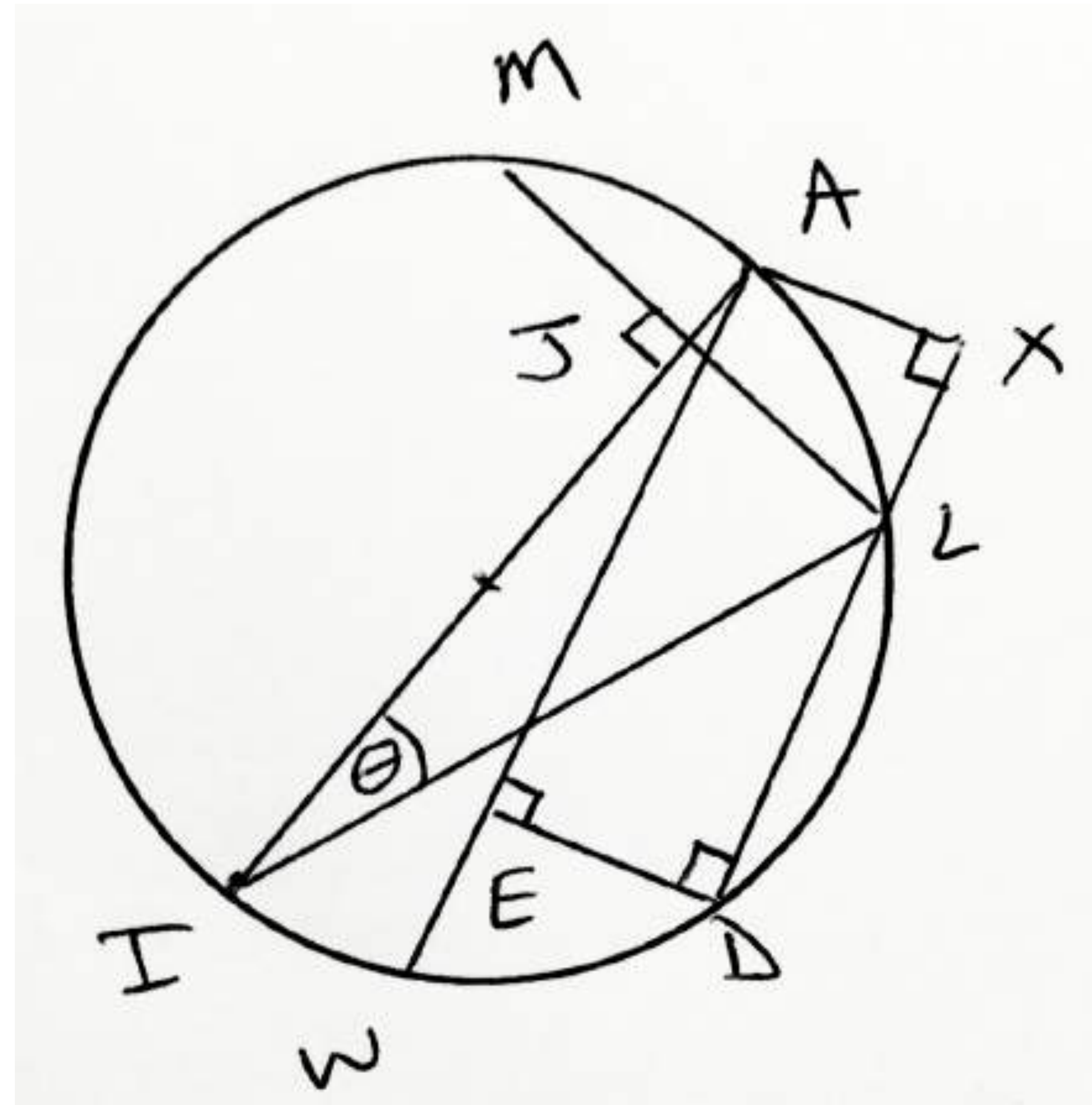
$$(SB/SO^2) \lim_{\Delta\theta \Rightarrow 0} \{\Delta\sin^2\theta\} = (SB/SV^2) \lim_{\Delta\alpha \Rightarrow 0} \{\Delta\sin^2\alpha\}$$

$$\frac{\text{Limit as } \Delta\theta \Rightarrow 0 \text{ of } (\Delta \sin^2 \theta)}{\text{Limit as } \Delta\alpha \Rightarrow 0 \text{ of } (\Delta \sin^2 \alpha)} = SO^2/SV^2$$

Solve for

Limit $\Delta \sin^2 \theta$
$\Delta\theta \Rightarrow 0$

on the reference circle:



AW ≥ LD || AW

∠ALD = ~AID/AI

≥ ~AI/AI = π

First establish the necessary functions of θ in terms of arcs and chords.

$\theta = \sim AL/AI$

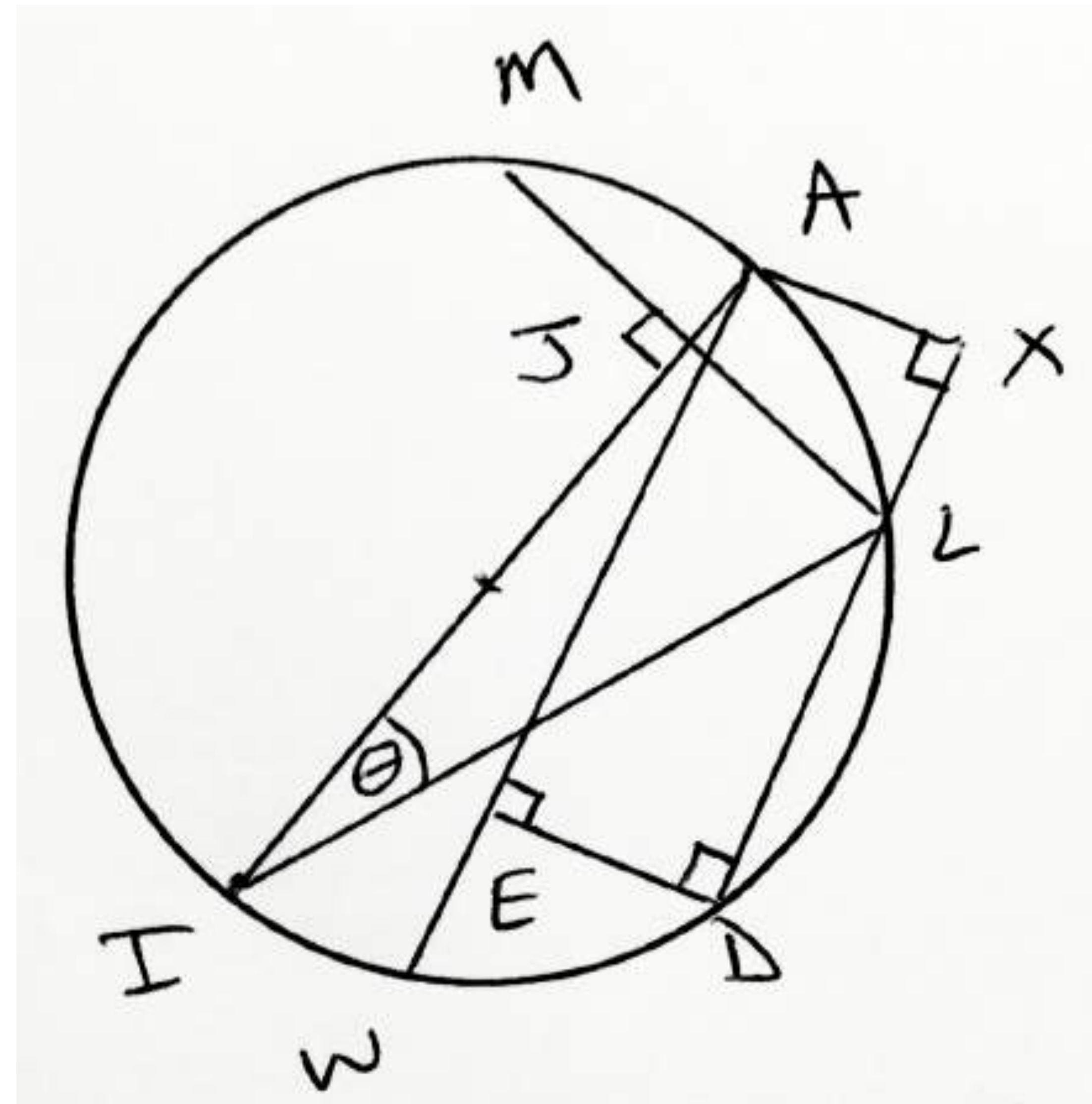
$\sin^2 \theta = AL^2/AI^2$

$\Delta \theta = \sim LD/AI$

$\sin^2 \Delta \theta = LD^2/AI^2$

$(\theta + \Delta \theta) = \sim ALD/AI$

$\sin^2 (\theta + \Delta \theta) = AD^2/AI^2$



$\cos\theta = IL/AI$

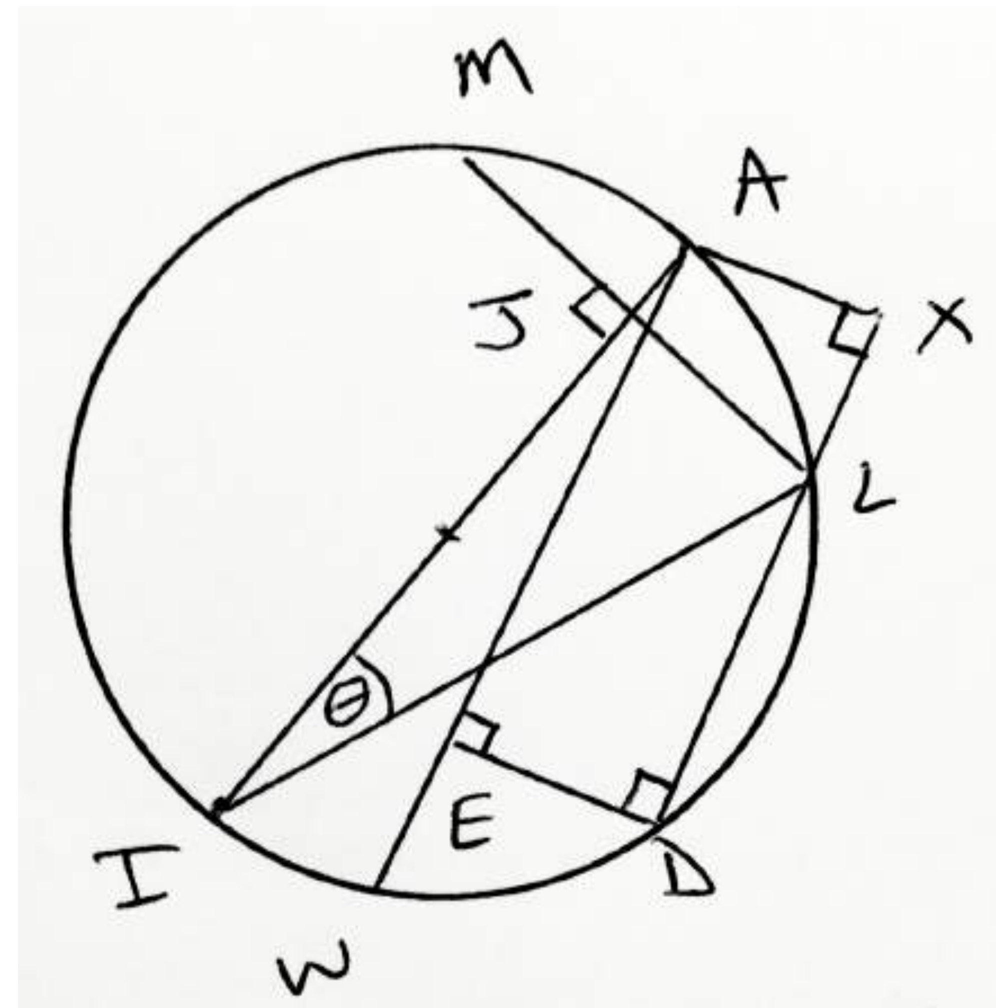
$\cos(\theta + \Delta\theta) = DI/AI$

$\sin\theta = AL/AI = JL/IL$

$\sin\theta\cos\theta = (JL/IL)\ (IL/AI)$

$2(\sin\theta\cos\theta) = ML/AI$

$2(\sin\theta\cos\theta) = \sin 2\theta$



Then consider the following property of the cyclic quadrilateral circle ALDW:

$$AD\,(LW) = AL\,(DW) + LD\,(AW)$$
$$AD^2 = AL^2 + LD\,(AW)$$

$$AW = LD + 2\,(XL) = LD + 2\,(AL)\,(XL/AL)$$

$$\Delta DIA \cong \Delta EWD = \Delta XLA$$

$$AW = LD + 2\,(AL)\,(ID/IA)$$

$$\mathbf{AD^2 - AL^2 = LD^2 + 2\,(LD)\,(AL)\,(ID/IA)}$$

$$\mathbf{AD^2 - AL^2 = LD^2 + 2\,(LD)\,(AL)\,(ID/IA)}$$

$$AD^2/AI^2 - AL^2/AI^2 =$$
$$LD^2/AI^2 + 2\,(LD/AI)\,(AL/AI)\,(ID/IA)$$

$$\sin^2(\theta + \Delta\theta) - \sin^2\theta =$$
$$\sin^2\Delta\theta + 2(\sin\Delta\theta)\,(\sin\theta)\cos(\theta + \Delta\theta)$$

$$\Delta(\sin^2\theta) = \sin^2(\theta + \Delta\theta) - \sin^2\theta =$$
$$\sin^2\Delta\theta + 2(\sin\Delta\theta)\,(\sin\theta)\cos(\theta + \Delta\theta)$$

$$\lim_{\Delta\theta \Rightarrow 0} \frac{\Delta(\sin^2 \theta)}{\Delta\theta} =$$

$$\lim_{\Delta\theta \Rightarrow 0} \frac{\sin^2 \Delta\theta + 2(\sin \Delta \theta)\,(\sin \theta)\,\cos\,(\theta + \Delta\theta)}{\Delta\theta} =$$

$$= 2 \sin \theta\,(\cos \theta) = \sin 2\theta$$

because:

$$\lim_{\Delta\theta \Rightarrow 0} \frac{\sin^2 \Delta\theta}{\Delta\theta} = 1 \; ; \quad \lim_{\Delta\theta \Rightarrow 0} \frac{\sin \Delta\theta}{\Delta\theta} = 1$$

Therefore, the meridian with the maximum combined effects of refraction can be found using:

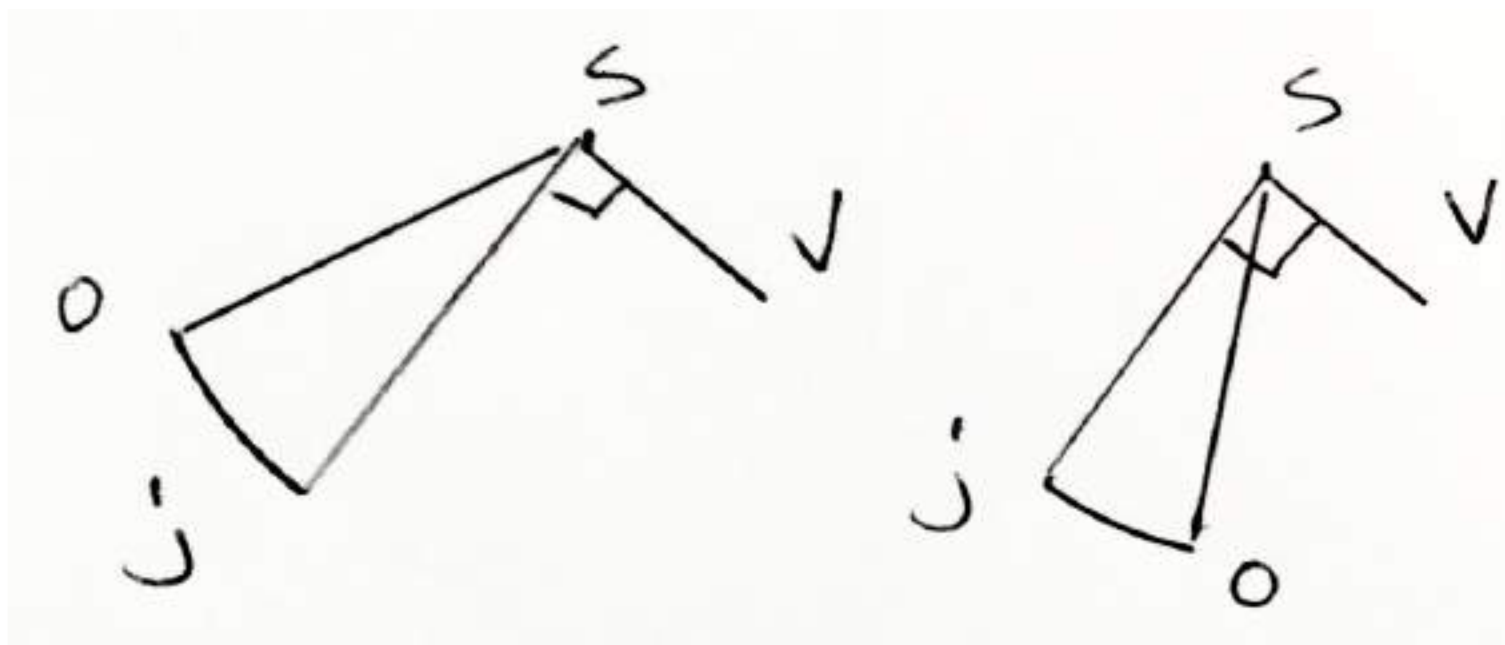
$$\frac{\sin 2\theta}{\sin 2\alpha} = \frac{SO^2}{SV^2}$$

The first step to solve this problem is to divide SV into SaV so that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

Make SO = Sj ⊥ SV

to construct:



Draw Sb so that:

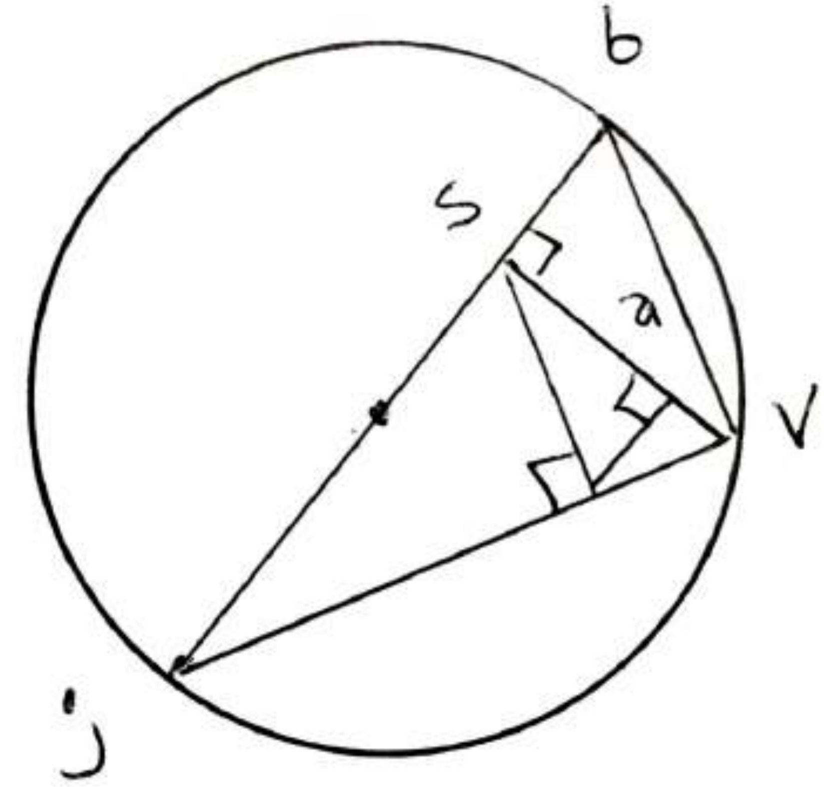
$SO^2/SV^2 = Sj^2/SV^2 = Sj/Sb$

by making:

$Sj/SV = SV/Sb$

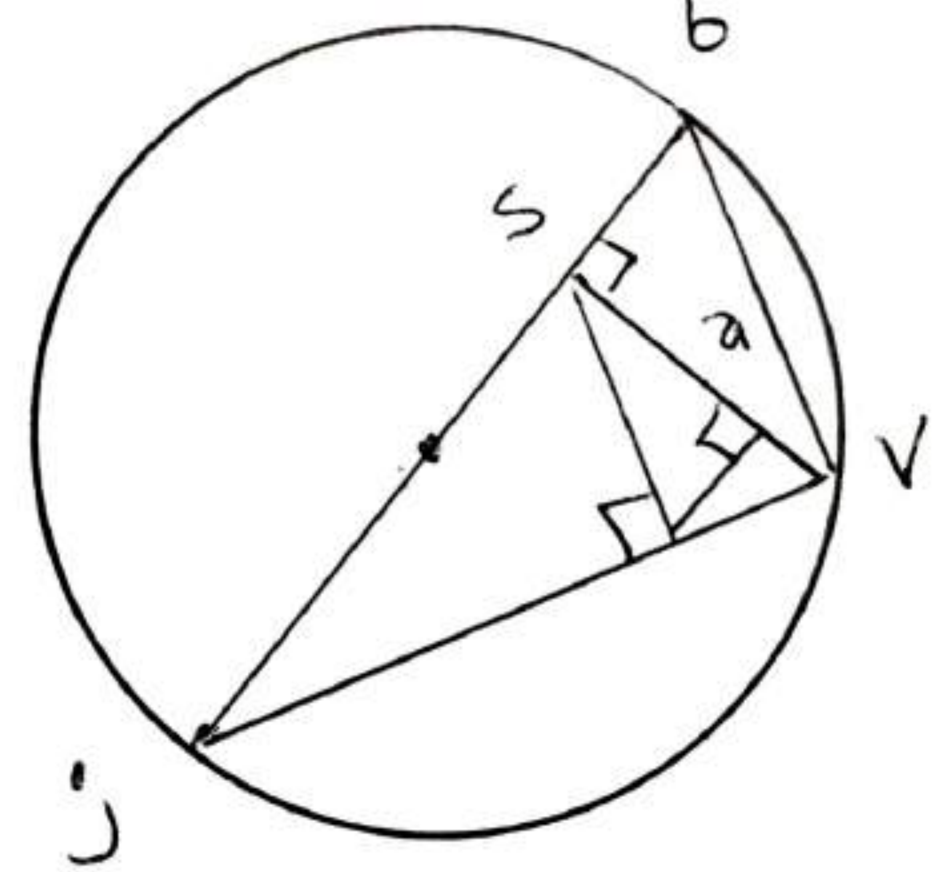
so that:

$Sj^2/SV^2 = Sj/Sb = SO^2/SV^2$



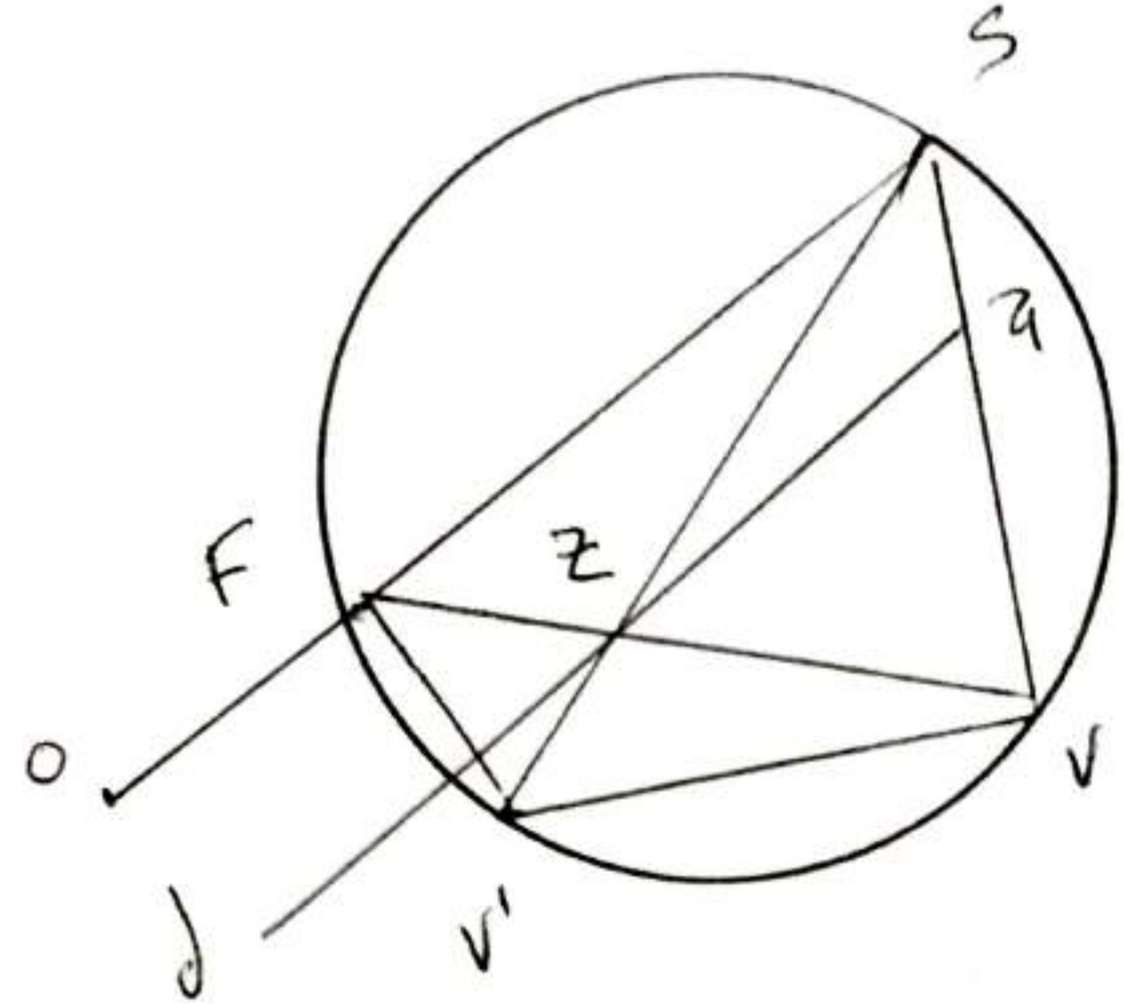
Similar triangles
then show that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

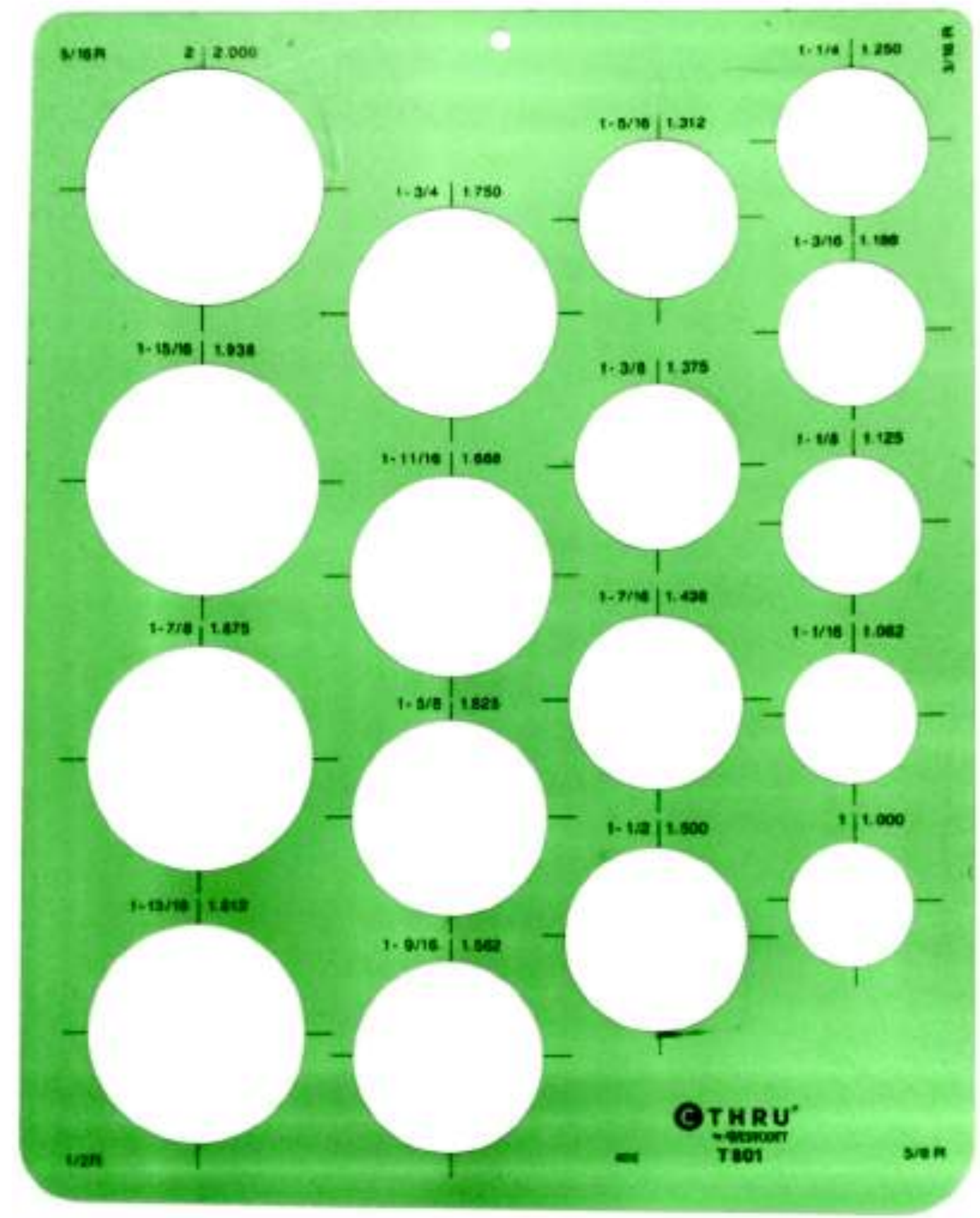


Draw ad || SO

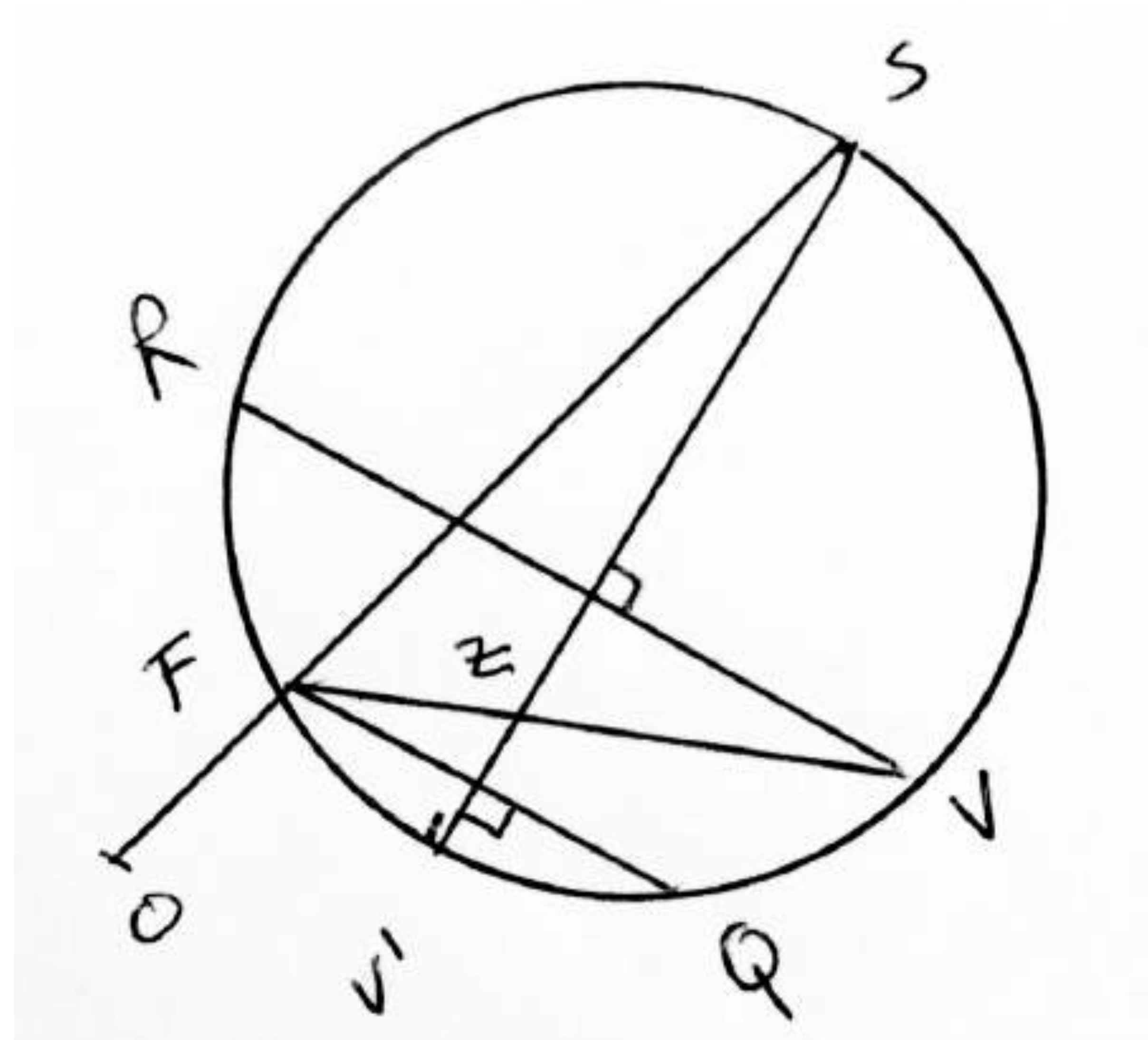
Choose a circle through S and V with a variable diameter SV' so that FZV lies on a common chord.



The easiest way to do this involves a template of various circles, each with the location of their diameters already marked.



SV' is the meridian with the maximum combined effects of refraction because:



$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{FZ}{ZV} = \frac{FQ/2}{RV/2} = \frac{FQ}{RV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

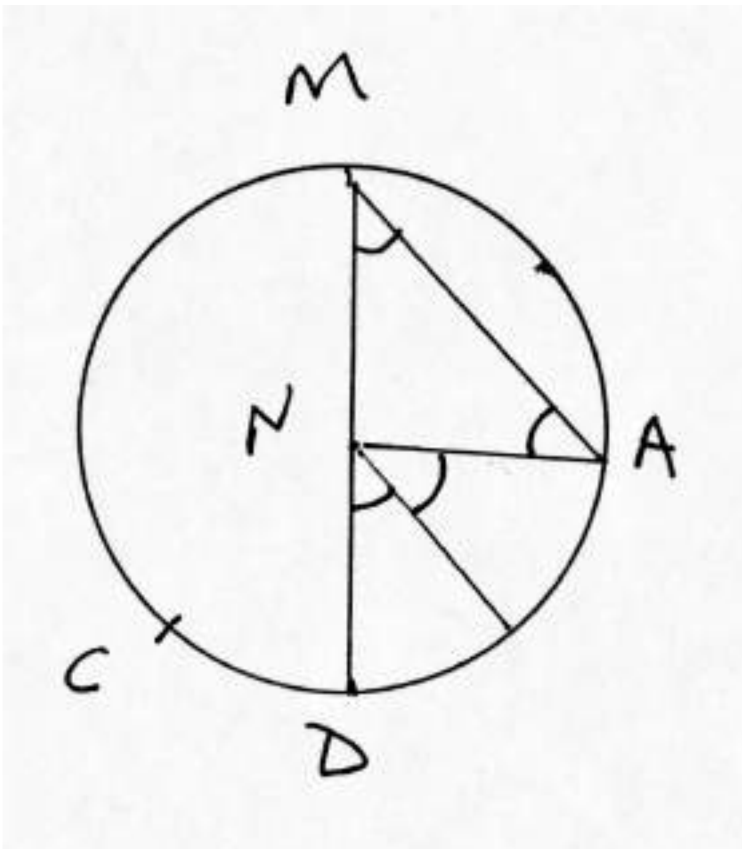
Double-angle Method

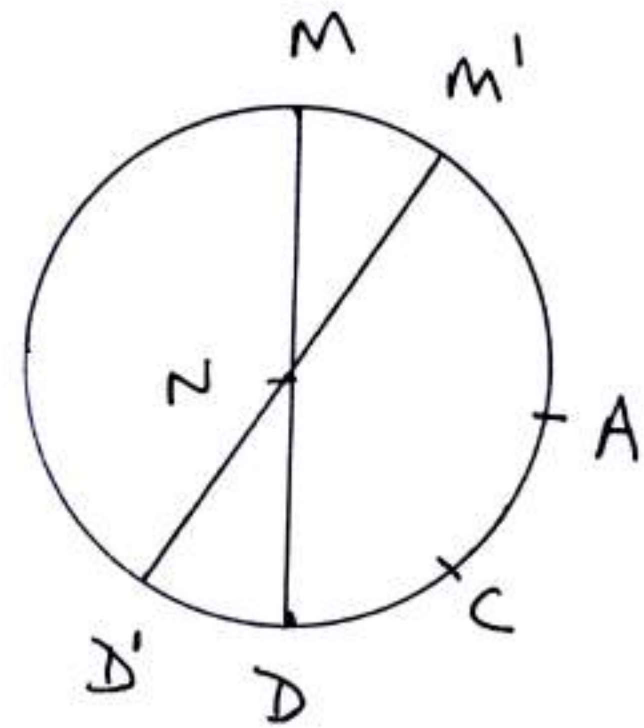
We have already shown how to find angle θ, and angle α, so that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

An additional method, the double angle method, employs the fact that an arc subtends twice the angle at a circle's center as it does at its circumference, and that the entirety of a circle subtends π radians any a point on its circumference. To illustrate:

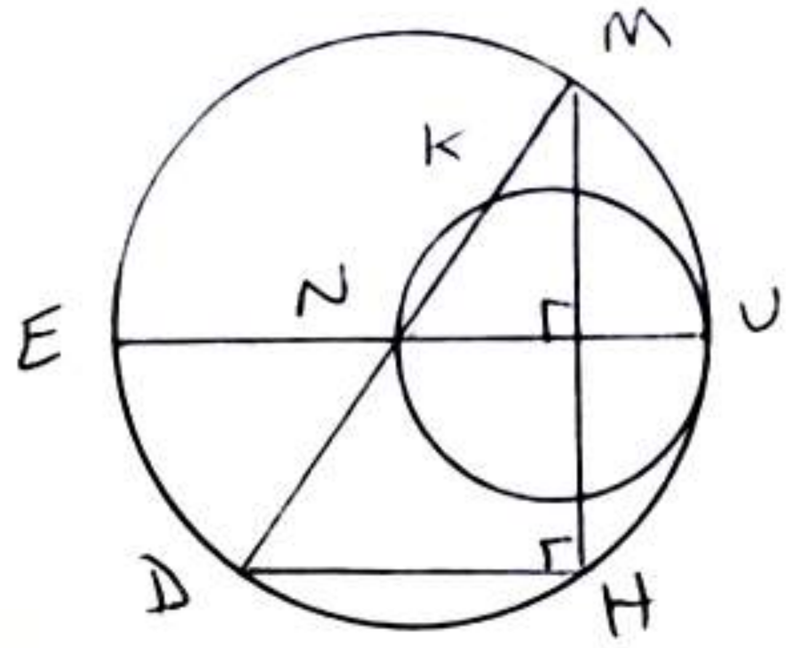
$\angle DNA = 2\angle DMA$; $\angle DNC = 2\angle DMC$

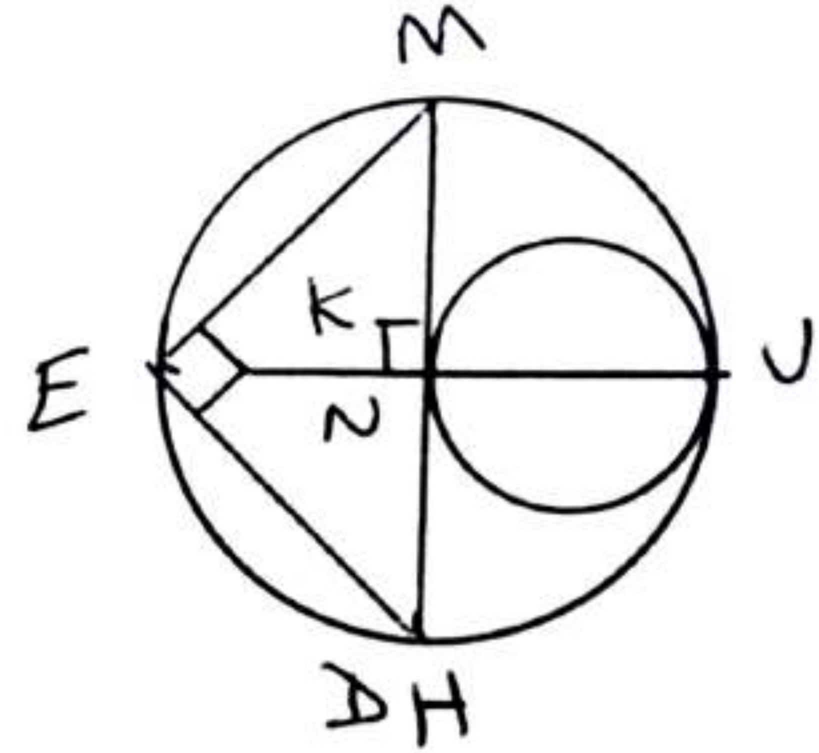




$\angle ANC = \angle DNA +/- \angle DNC = 2(\angle DMA +/- \angle DMC) =$

$2\angle AMC = 2\angle AM'C$





~UK/UN = ~MH/MD = 2~UM/UE = 2~UM/2UN

~UK = ~UM

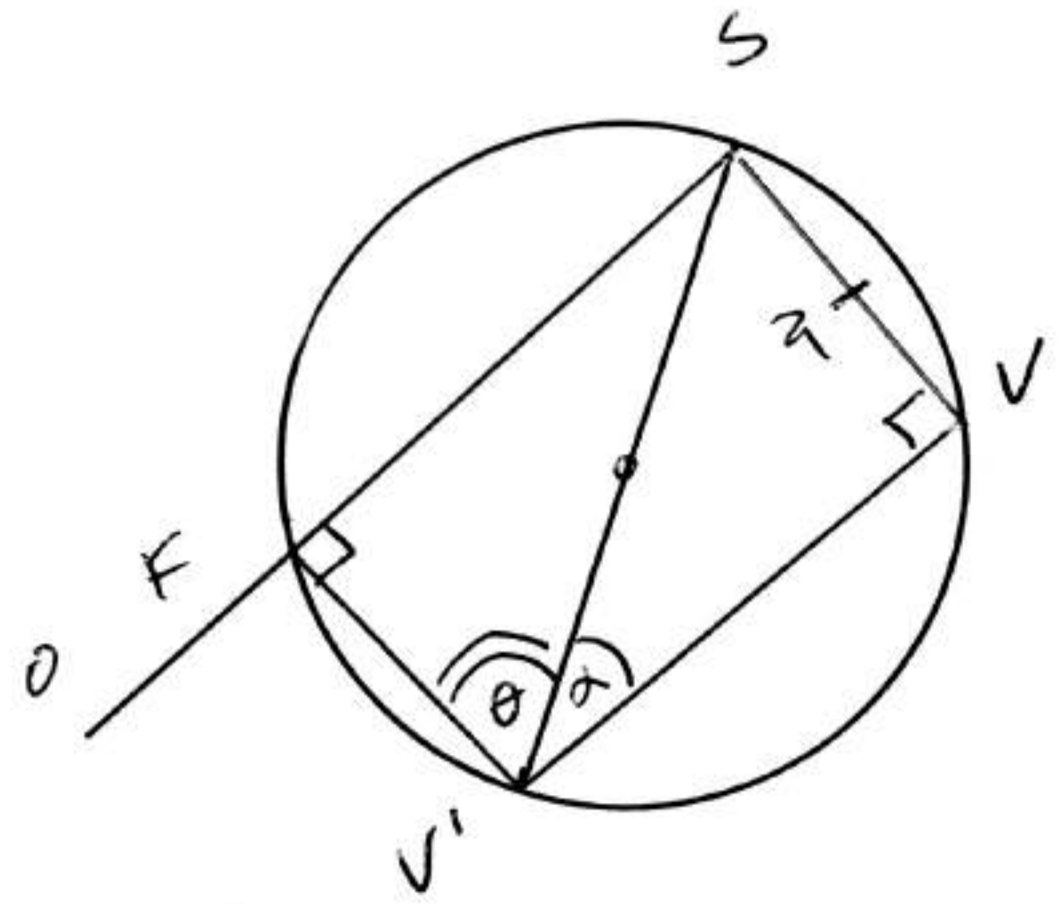
As K ⇒ N, and D ⇒ H:

2~KU/UN = 2∠MNU = ∠MNH ⇒ π radians

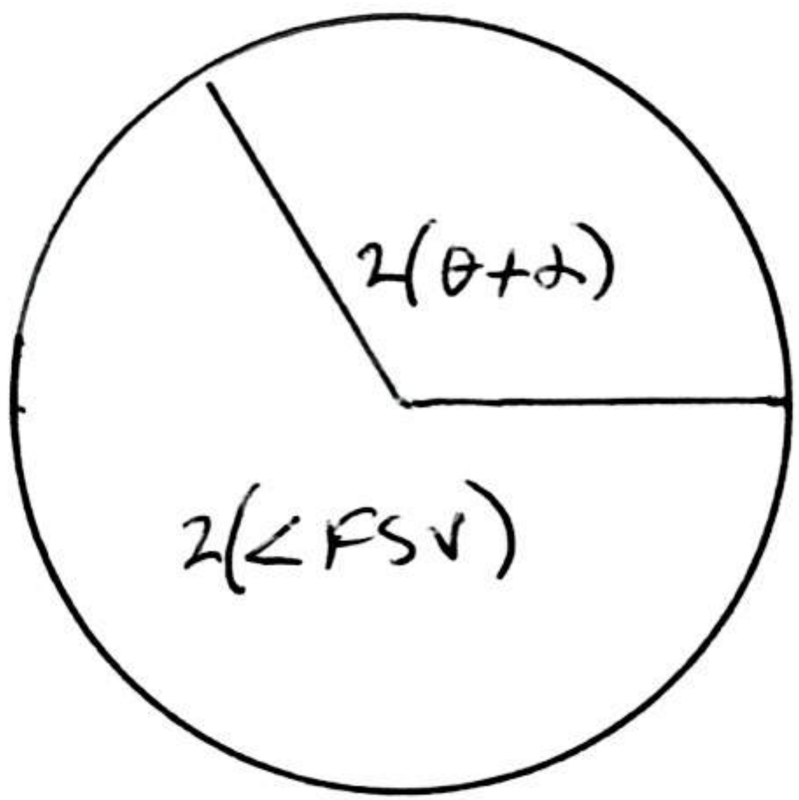
Given constant ΔOSV:
∠FSV is constant.

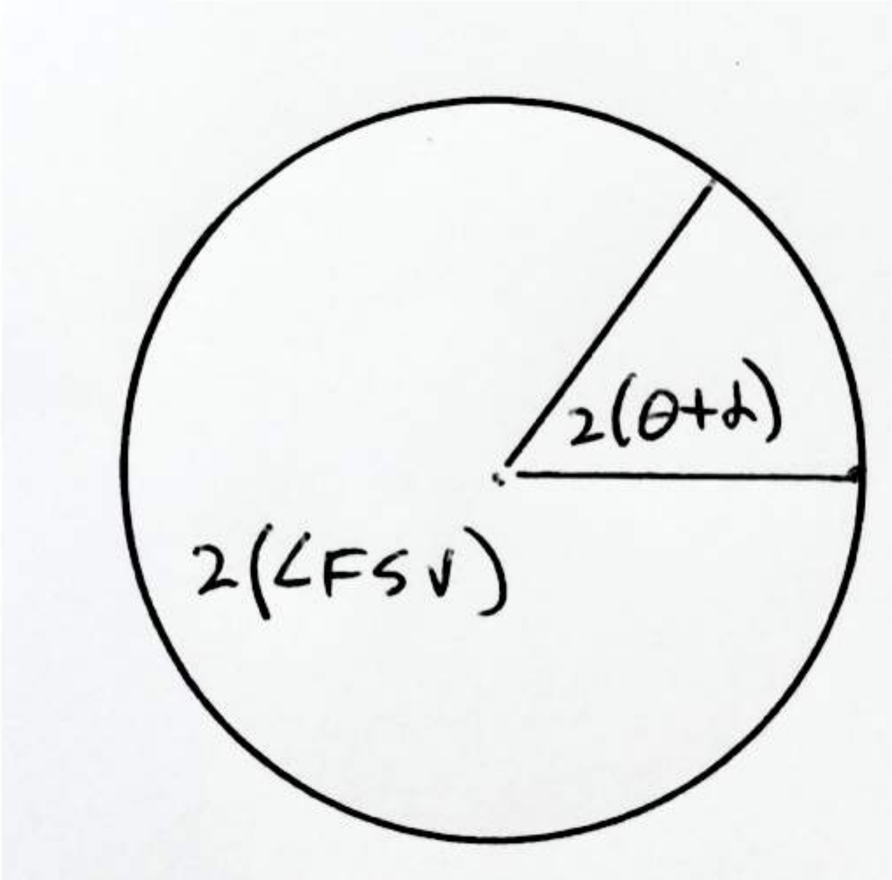
and since:
∠FSV + (θ + α) =
π radians,

(θ + α) is also
constant.



$$2(\angle FSV) + 2(\theta + \alpha) = 2\pi$$

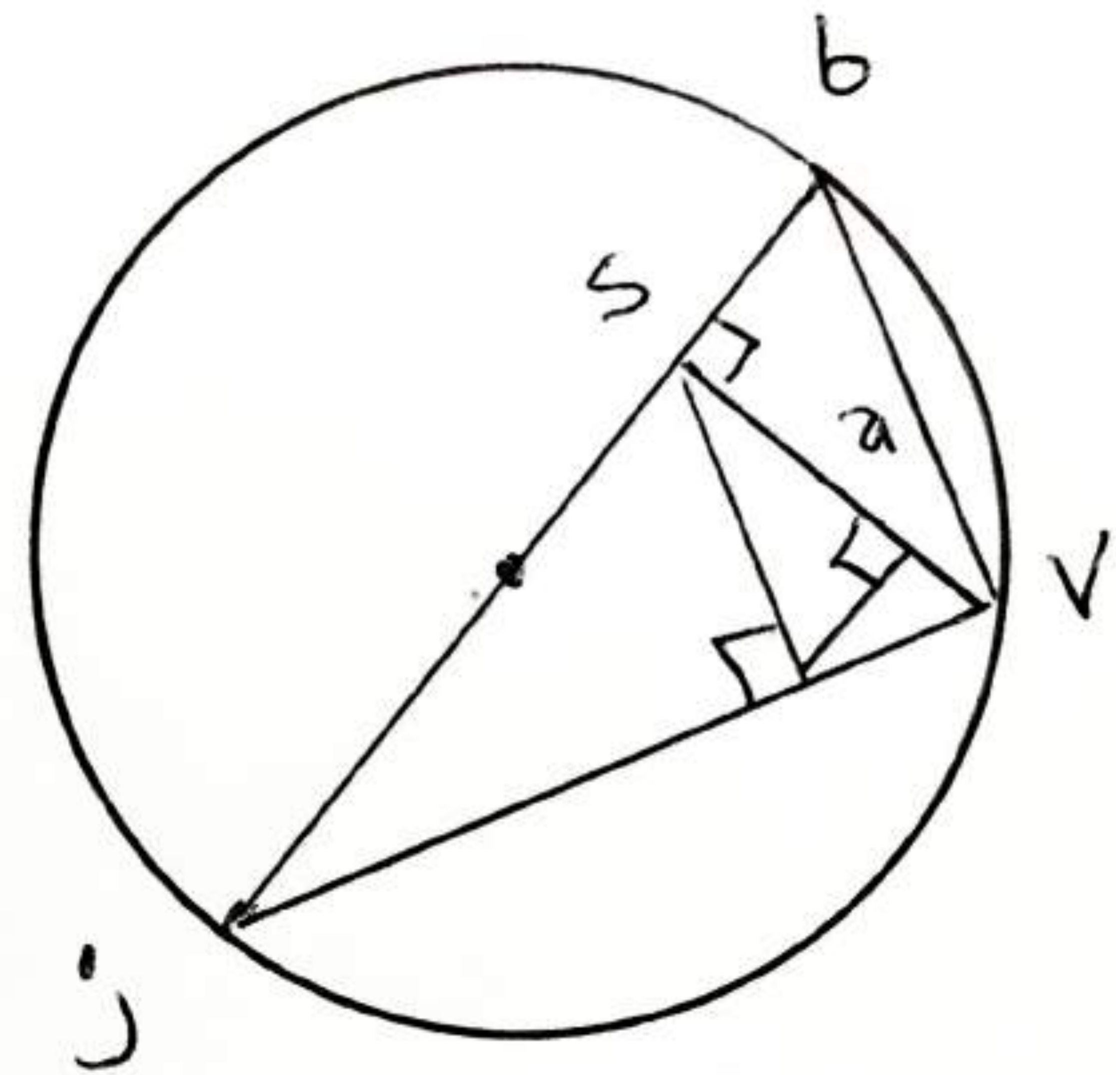




When:

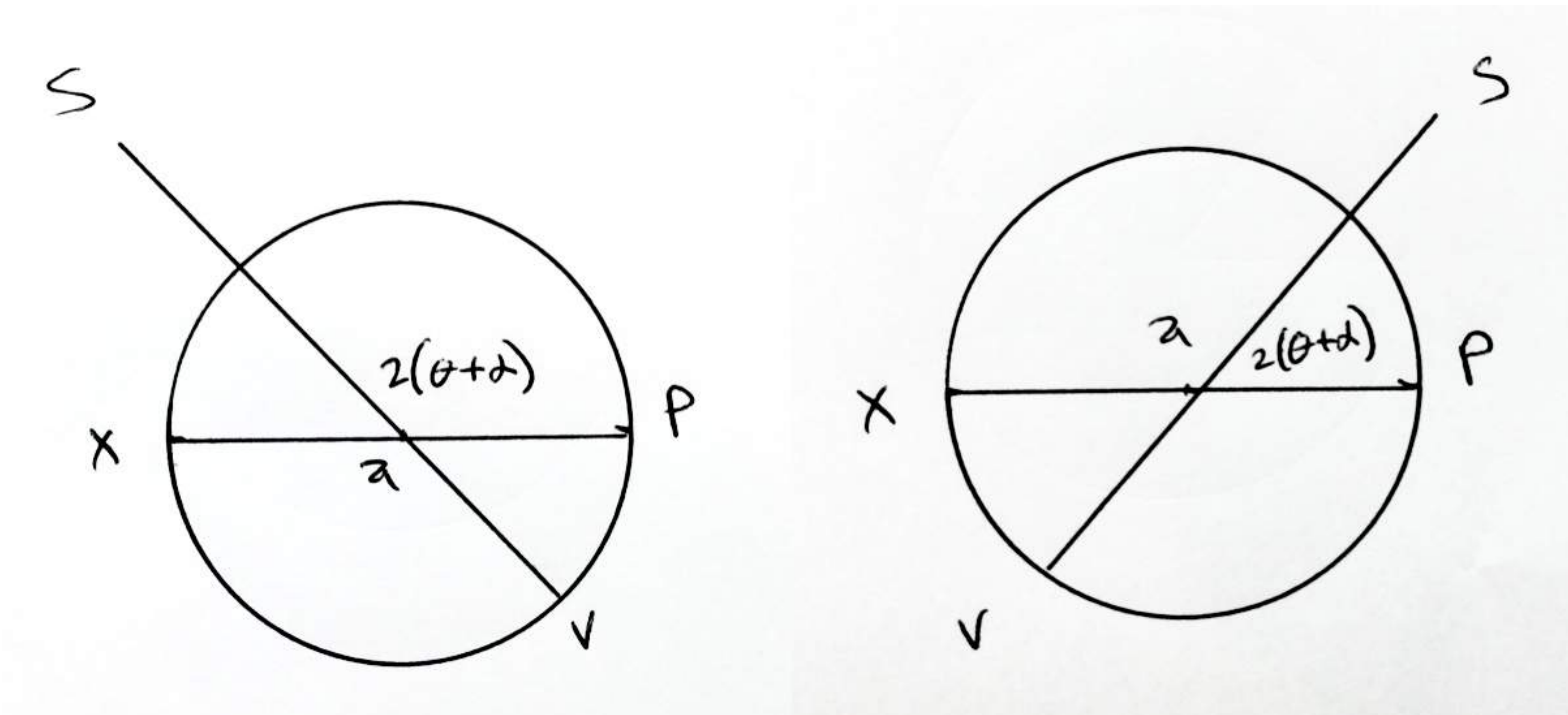
$$\frac{SO^2}{SV^2} = \frac{Sj^2}{SV^2} = \frac{aS}{aV}$$

as drawn:

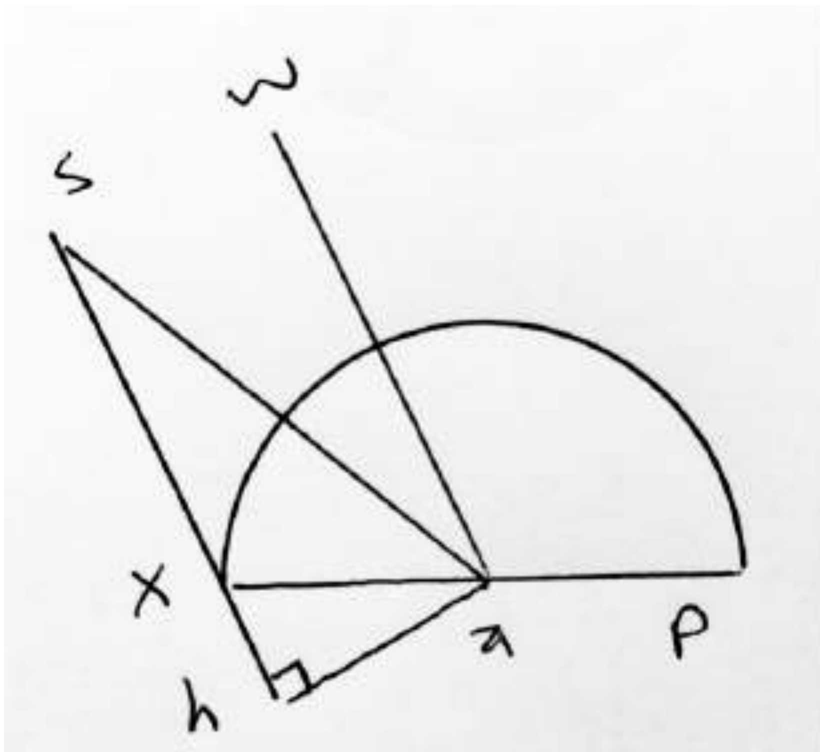


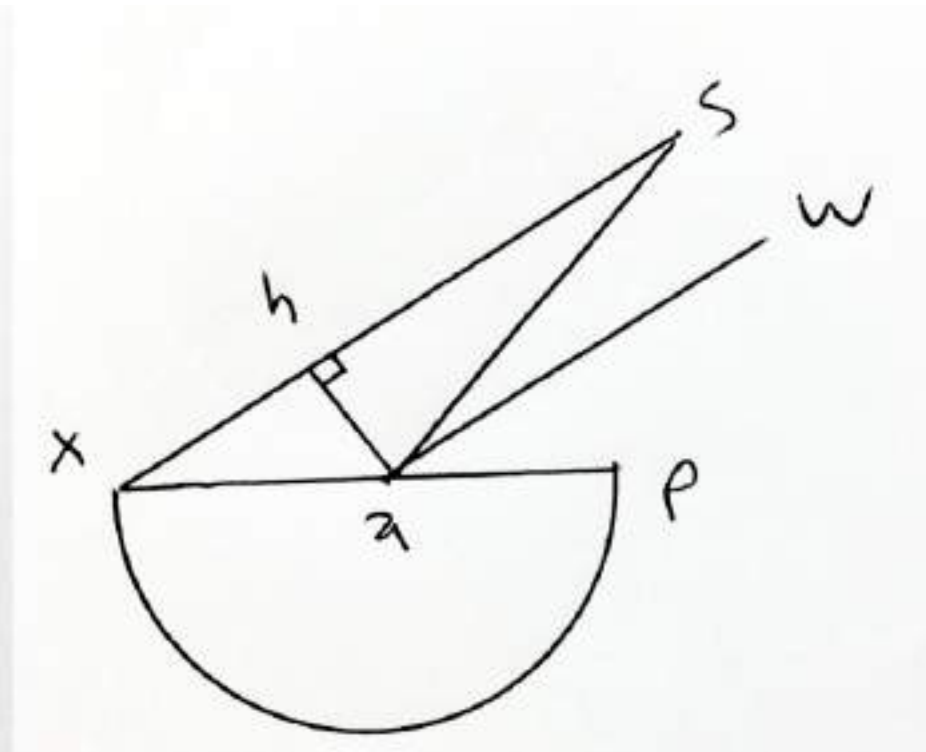
If we draw diameter XaP so:

$aX = aV$, and $\angle SaP = 2(\theta + \alpha)$



$$\frac{SO^2}{SV^2} = \frac{aS}{aX} = \frac{ah/aX}{ah/aS} = \frac{\sin 2\theta}{\sin 2\alpha}$$





When aw || sX, we have divided the doubled angle
2 (θ + α) = ∠SaP
into 2θ = ∠WaP, and 2α = ∠WaS.