The Geometry of Geometrical Optics

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2022

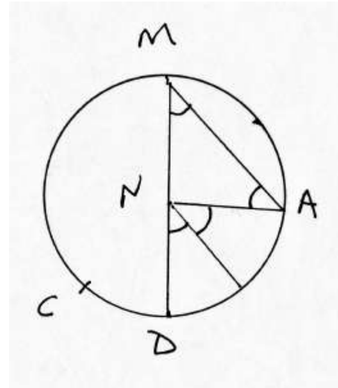
This work is dedicated to William Brown, OD, PhD., who always taught the geometry first.

Contents:

Introductory Geometry

$\angle DNA = 2\angle DMA$

$\angle DNC = 2\angle DMC$

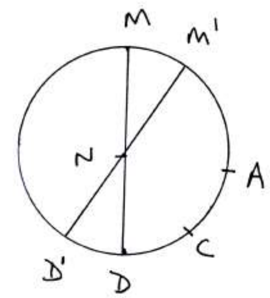


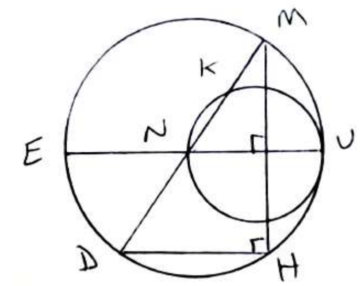
$\angle ANC$

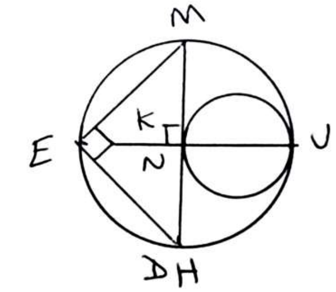
$= \angle DNA \ +/- \ \angle DNC$

$= 2(\angle DMA +/- \angle DMC)$

$= 2\angle AMC = 2\angle AM'C$

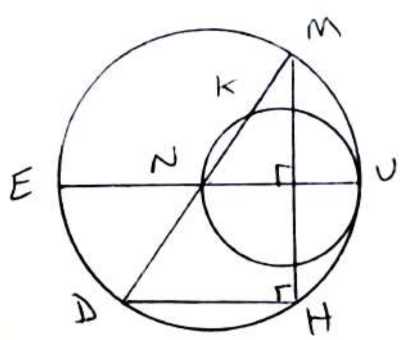


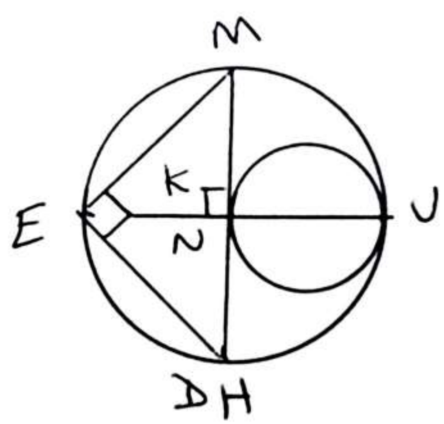




~UK/UN = ~MH/MD = 2~UM/UE = 2~UM/2UN

~UK = ~UM





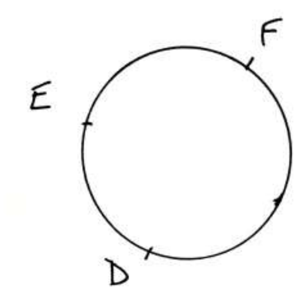
As K ⇒ N, and D ⇒ H:

$2\sim KU/UN = 2\angle MNU = \angle MNH \Rightarrow \pi$

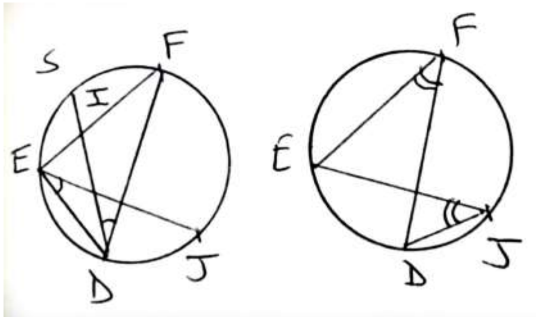
$\angle FDE = \sim EF/DM$

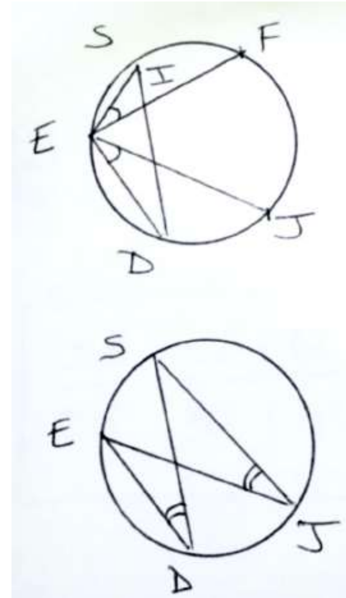
$\angle DEF = \sim DF/DM$

$\angle EFD = \sim DE/DM$



$\angle FDE + \angle DEF + \angle EFD = \pi$





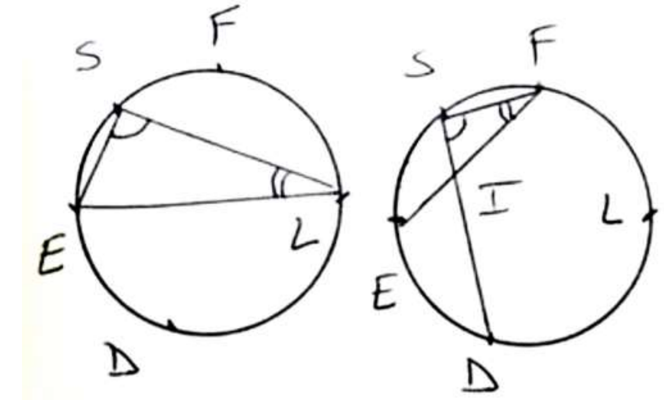
SD || FJ
ΔEJD ≅ ΔDFI, FD/FI = JE/JD
ΔEJS ≅ ΔEDI, EI/ED = ES/EJ
(FD)(EI) / (FI)(ED)
= (JE)(ES) / (JD)(EJ) = SE/SF

IE/IF = (SE)(DE) / (SF)(DF)

LD || FE

DE/DF = LF/LE

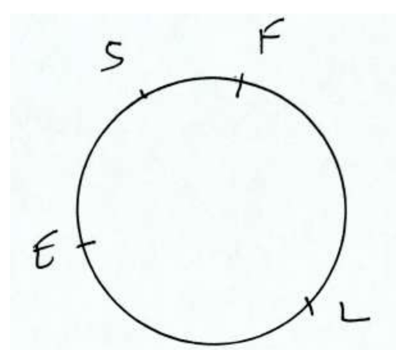
IE/IF
= (SE)(LF) / (SF)(LE)



FE/FI = {(SE)(LF) + (SF)(LE)} / (SF)(LE)

LD || FE, ~EL = ~FD, ΔLSE ≅ ΔFSI
LS/FS = LE/FI, LS = FS(LE) / FI

Ptolemy's Theorem:
(FE)(LS) = (SE)(LF) + (SF)(LE)



Pythagorean's Theorem can be shown when the cyclic quadrilateral SELF is a rectangle, and the law of cosines can be shown when it is a trapezoid.

When the cyclic quadrilateral SELF is a trapezoid, and:

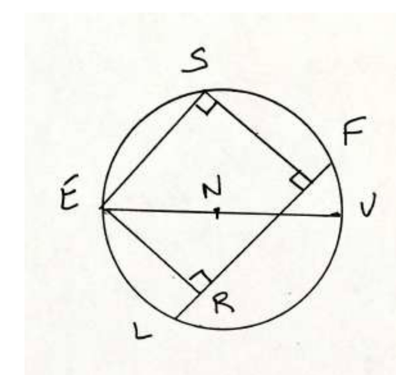
LF > ES || LF

∠ELF = ~ESF/EU < ~EU/EU = π/2

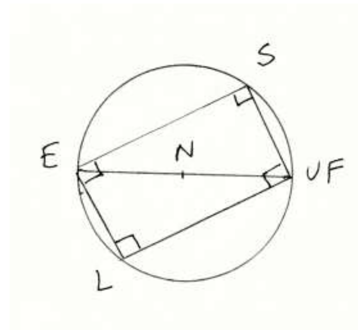
$EF^2 = EL^2 + LF(ES)$

LF(ES) = LF[LF - 2(EL)(LR/LE)]

LR/LE = UF/UE = cosine ∠ELF



When the cyclic quadrilateral SELF is a rectangle, so:



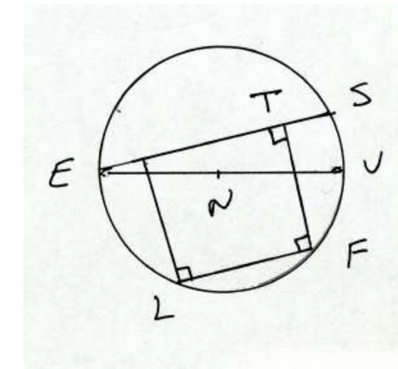
$LF = ES \parallel LF$

$\angle ELF = \sim ESF/EU = \sim EU/EU = \pi/2$

$EF^2 = EL^2 + LF(ES)$

$LF(ES) = LF^2$

When the cyclic quadrilateral SELF is a trapezoid, and:



$LF < ES \parallel LF$

$\angle ELF = \sim ESF/EU > \sim EU/EU = \pi/2$

$EF^2 = EL^2 + LF(ES)$

$LF(ES) = LF[LF + 2(EL)(TS/SF)]$

$TS/SF = UF/UE = \text{cosine } \angle ELF$

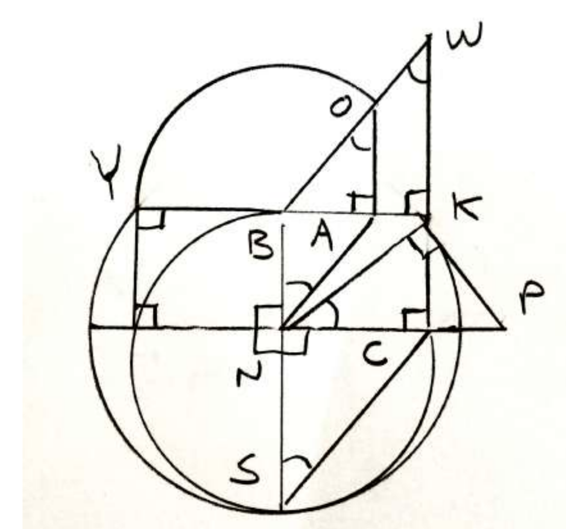
Refraction Along a Line

Let:
$(NK/NC) = (CN/CK)$

When:
$\Delta CKP \cong \Delta KNP$
$= \Delta NSC = \Delta KWB$,

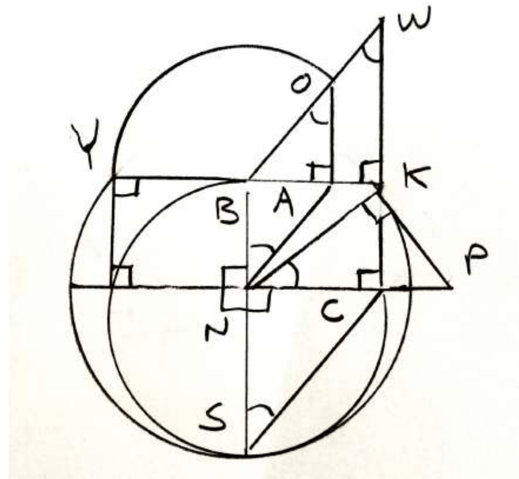
$\Delta CKP = \Delta BNA = \Delta AOB$

and $KW = YN$



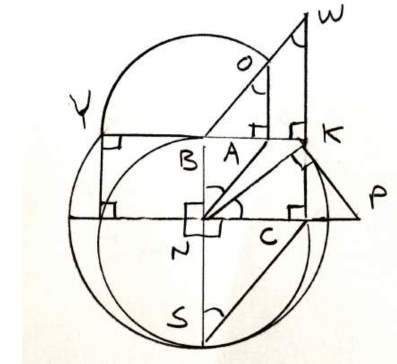
But also, ***whenever***:

$KB^2 = KN^2 - BN^2$
$= KN^2 - (AN^2 - AB^2)$
$= (KN^2 - AN^2) + AB^2$



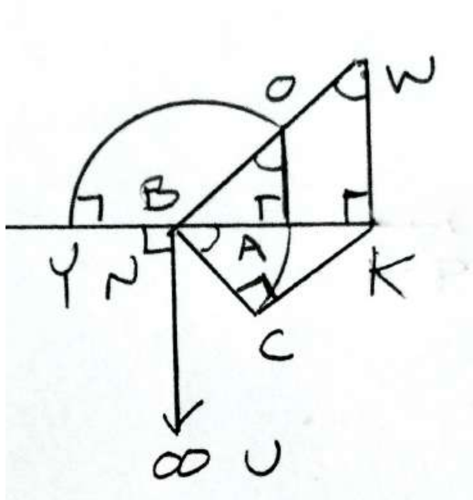
and:
$AN^2 - BN^2 = BO^2 - AO^2$
so:
$(AO^2 + AN^2)$
$= (BO^2 + BN^2) = YN^2$

if:
$(KB/KW) = (AB/AO) = (CK/CN$
so:
KB^2/KW^2
$= (AB^2 + CK^2)/(AO^2 + CN^2)$



and if:
$AN = CN$,
then:
$KW^2 = (AO^2 + CN^2) = YN^2$
$KW = YN$

Under these conditions, it can also be shown that:

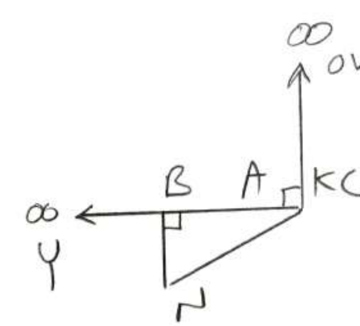


As N ⇒ B, KW ⇒ YN

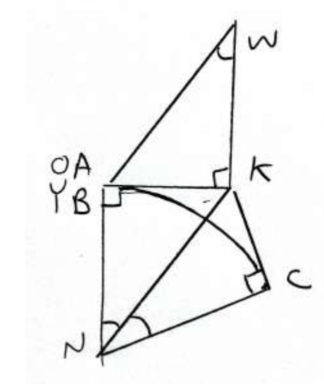
because:
$KW/OA \Rightarrow NK/NA$
$= NK/NC$
$= OB/OA$
$= WB/WK$

so that:
$KW \Rightarrow OB \Rightarrow YN$

and both that:



As A ⇒ K,
KW ⇒ YN

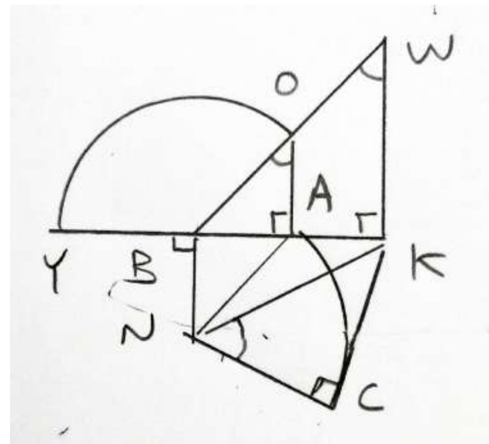


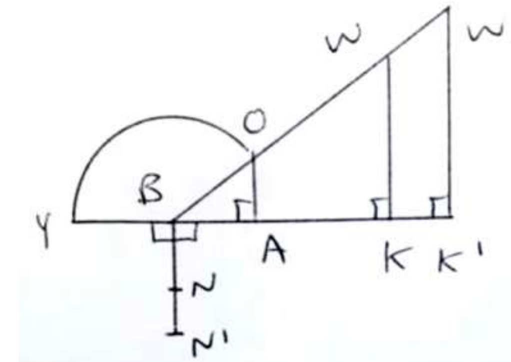
As A ⇒ B,
KW ⇒ YN

Therefore, whenever
A lies on KB
of right triangle ΔKBN,

if:
ΔCNK ≅ ΔAOB
≅ ΔKWB,
and NA = NC,

then KW = YN

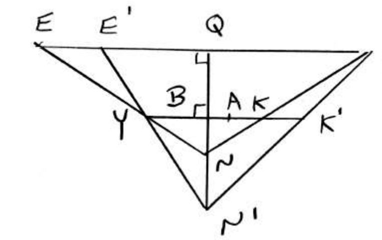




OB/OA = NK/NA = N’K’/N’A

KW = YN
K’W’ = YN’

KB/YN = K’B/YN’



QX/EN = KB/YN
= K’B/YN’ = QX/E’N’

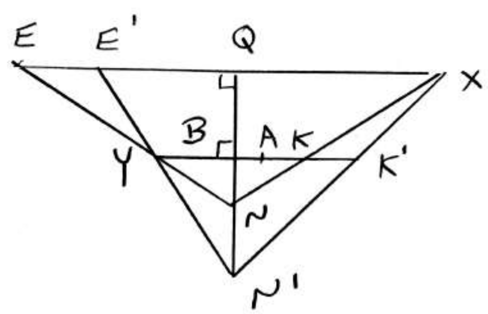
EN = E’N’

Only one N’K’X exists for NKX since only one E’N’ exists equal to EN.

When EN is changed to become the smallest segment through Y,
bound by the right angle EQN:

E’ lies at E, and
N’ lies at N.

Also, QX varies with EN because:
QX/EN = KB/YN
= KB/KW, which is a constant.

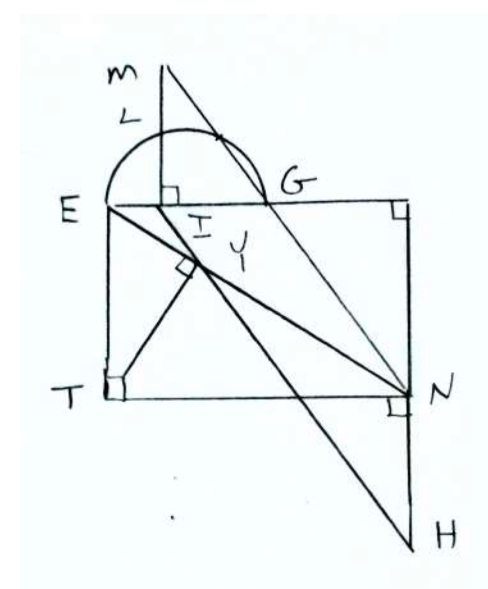


To specify EN as the shortest hypotenuse through Y:

NE || GL
TY || EL
HI || NM
HI = NM > NL

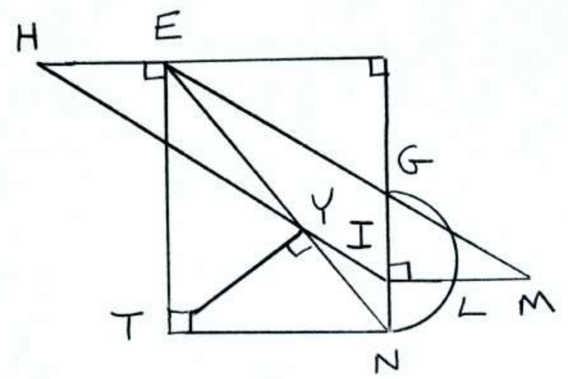
NL is the hypotenuse of right triangle NEL, so:

NL > NE
HI > NE



But also:

NE || GL
TY || NL
HI || EM
HI = EM > EL

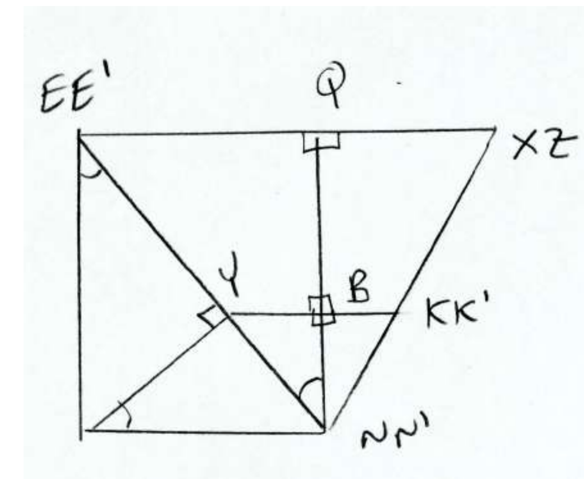


EL is the hypotenuse of right triangle ENL, so:

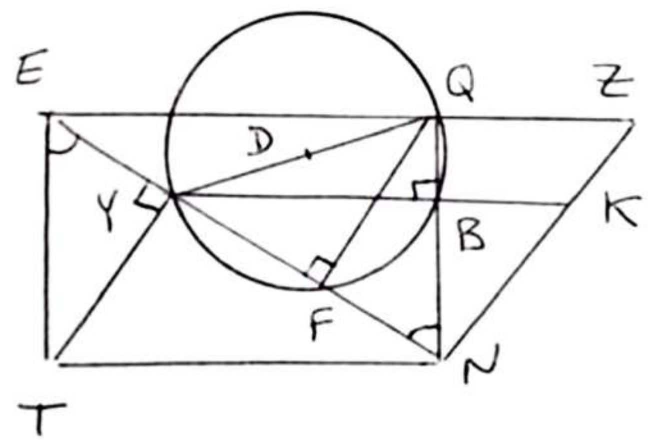
EL > EN
HI > EN

Let X = Z when EN is the shortest segment through Y included in right angle EQN.

In order to find Z given ΔYBN, we must find E = E’ using:

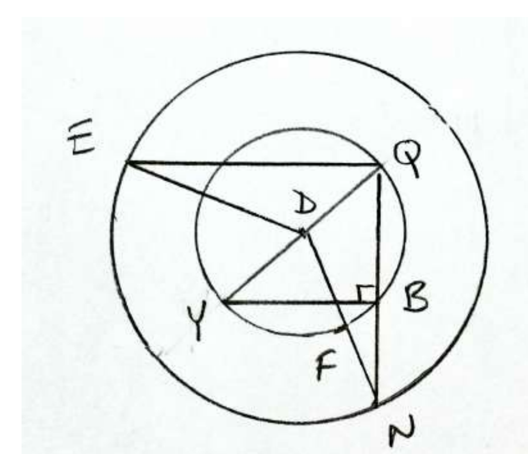


ΔYBN ≅ ΔNYT ≅ ΔNTE



In order to find Z given ΔYBQ, we must find EN = E’N’ by making ΔTYE a right triangle.

Draw a concentric circle around ⊙YBQ using its center at D, (the midpoint of hypotenuse YQ), containing an arc ~EN, so that YF lies on its chord EN. The arc intercepted by ∠DEN then equals that intercepted by ∠DNE.



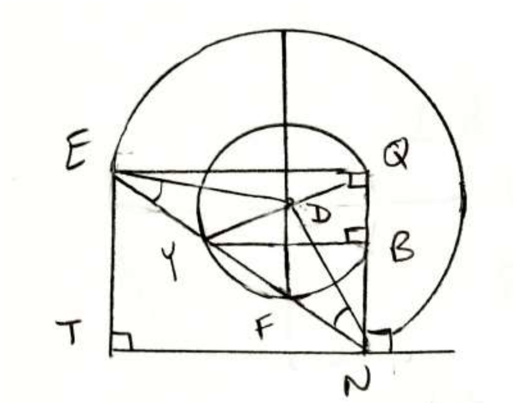
∠DEY = ∠DNF

DY = DF ; DE = DN

ΔEDY = ΔNDF

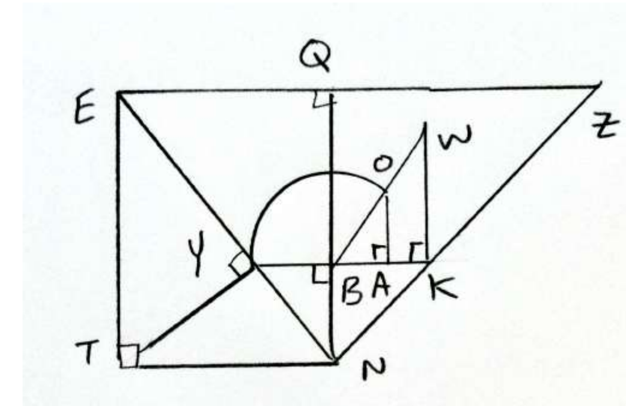
EY = NF

Since ΔQFN is a right triangle, so is ΔTYE.



WK = YN

Given   ΔBAO:



use ΔBNY to find ΔBKW and ΔQBY,

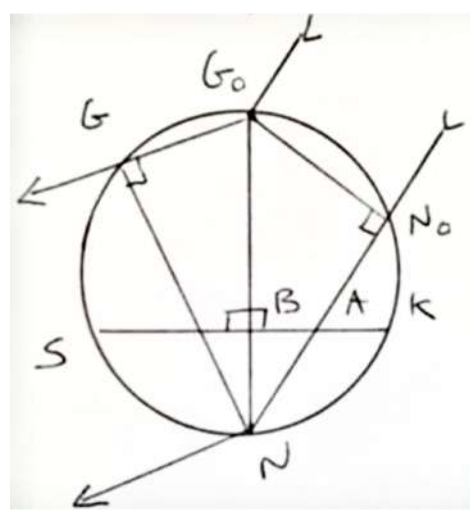
use ΔQBY or ΔBKW to find ΔBNY.

$\Delta N_oNK \cong \Delta KNA$

because:

~NS = ~NK

across diameter G_oN.



Wavefront G_oN_o refracts into wavefront GN along G_oN, since it travels G_oG in the same time it travels N_oN.

$$\mathbb{R} = NN_o/GG_o = NN_o/NK = NK/NA$$

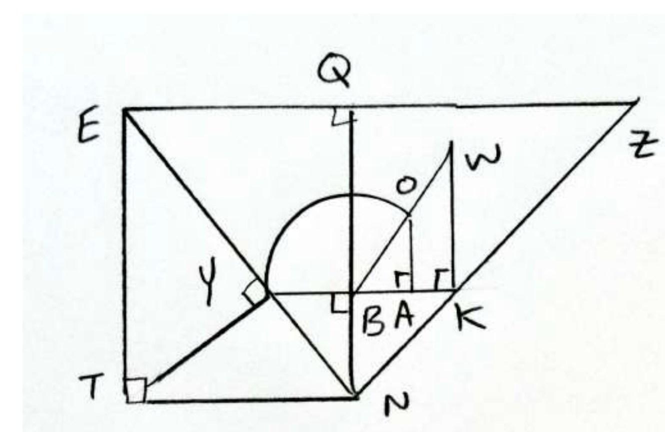
Therefore, if

$\mathbb{R}$ = OB/OA,

and WK = YN;

then,

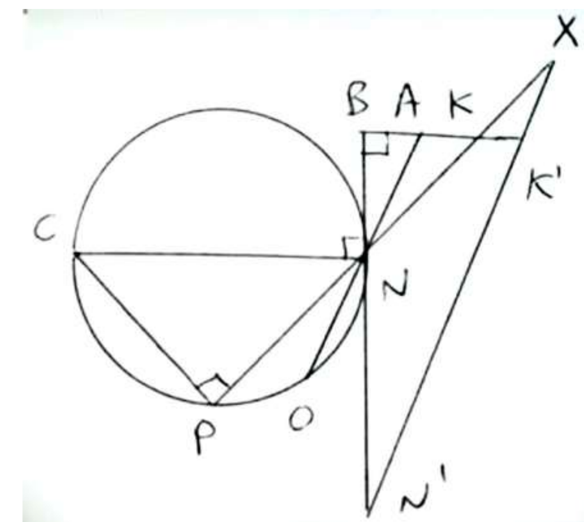
$\mathbb{R}$ = NK/NA



and Z is the clear image of object A refracted at N (= N’), along BN, because the two possible refracted rays through Z coincide at N.

Refraction Along a Circle

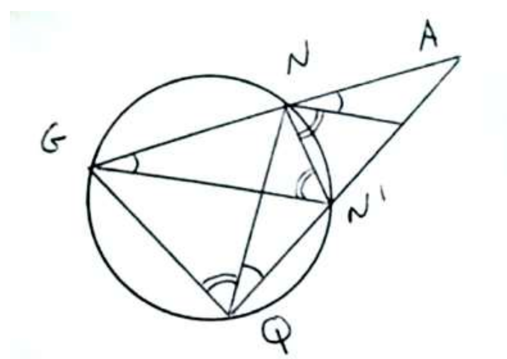
ΔKNA ≅ ΔOCP
ℝ = NK/NA
= N'K'/N'A
= CO/CP

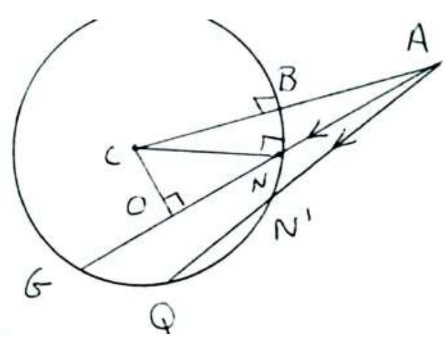


Real object A:

ΔANN' ≅ ΔAQG
AG/AN' = QG/NN'

(AG + AN')/2AN'
= (QG + NN')/2NN'

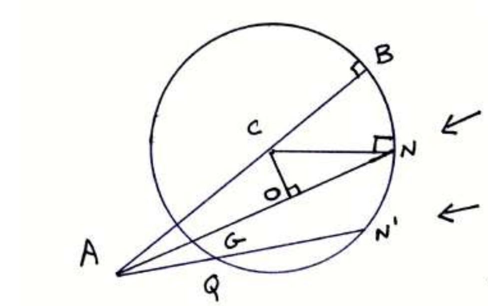




Virtual object A, which can not be projected on a screen due to refraction at BN:

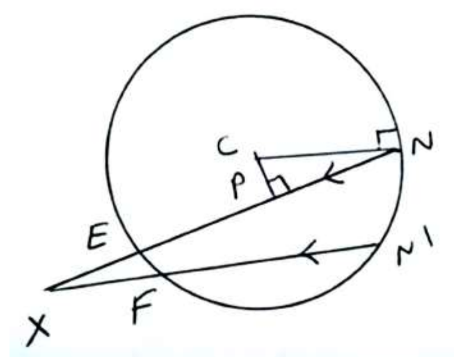
ΔANN' ≅ ΔAQG
AG/AN' = QG/NN'

(AG + AN')/2AN'
= (QG + NN')/2NN'



Real image at X,
(will be defined as clear
as N’ ⇒ N, and X ⇒ Z),

can be projected on a
screen:



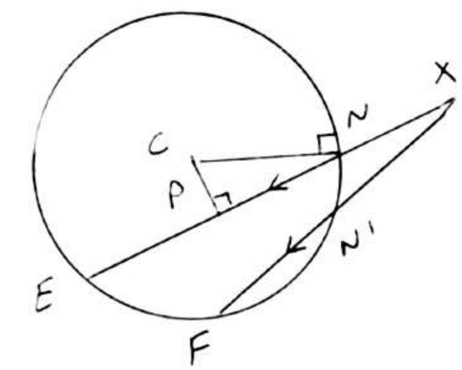
ΔXNN’ ≅ ΔXFE

XE/XN’ = EF/NN’

(XE + XN’)/2XN’
= (EF + NN’)/2NN’

Virtual image at X,
(will be defined as clear
as N’ ⇒ N, and X ⇒ Z),

can not be projected on
a screen:



ΔXNN’ ≅ ΔXFE

XE/XN’ = EF/NN’

(XE + XN’)/2XN’
= (EF + NN’)/2NN’

(AG + AN’)/2AN’ = (QG + NN’)/2NN’
(XE + XN’)/2XN’ = (EF + NN’)/2NN’

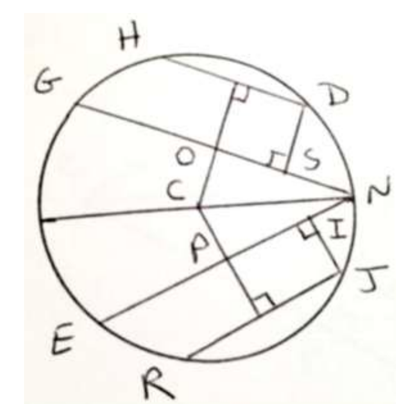
(QG + NN’)/(EF + NN’)
= [(AG + AN’)/2AN’][2XN’/(XE + XN’)]

As N’ ⇒ N, X ⇒ Z, and:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ (QG + NN’)/(EF + NN’)

⇒ (AO/AN)(ZN/ZP)

Also, when HD = QN’
and RJ = FN’

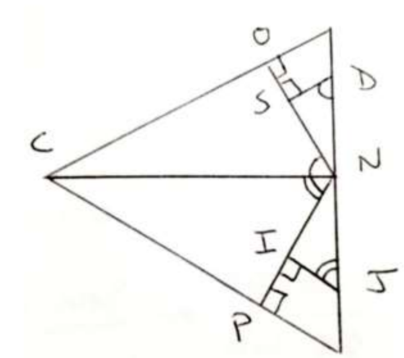


(~QG + ~NN’)/(~EF + ~NN’)
= 2(~ND)/2(~NJ) = ~ND/~NJ

As N’ ⇒ N, X ⇒ Z, and:

~DJ ⇒ line segment DJ, so:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ ND/NJ



DS/JI = CO/CP
JI/JN = NP/NC
DN/DS = NC/NO
ND/NJ = (NP/NO)(CO/CP)

As N’ ⇒ N, X ⇒ Z, and:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ (NP/NO)(CO/CP)

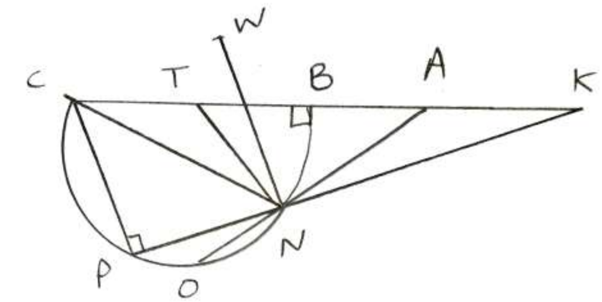
and therefore:
(AO/AN)(ZN/ZP) ⇒ (NP/NO)(CO/CP)

Thus ℝ = CO/CP, and Z, (along both NP and CW), is the clear image of A refracted along ~BN, when:

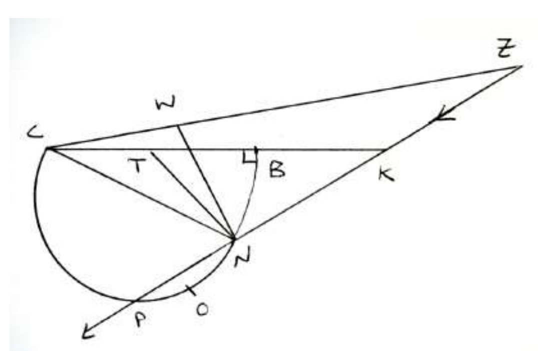
NT||CO, so:
AO/AN = CO/NT and:

NW||CP, so:
ZN/ZP = NW/CP
and:

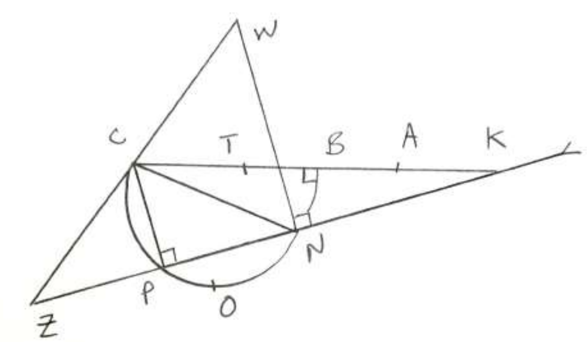
NW/NT = NP/NO
(ΔWNT ≅ ΔPNO)

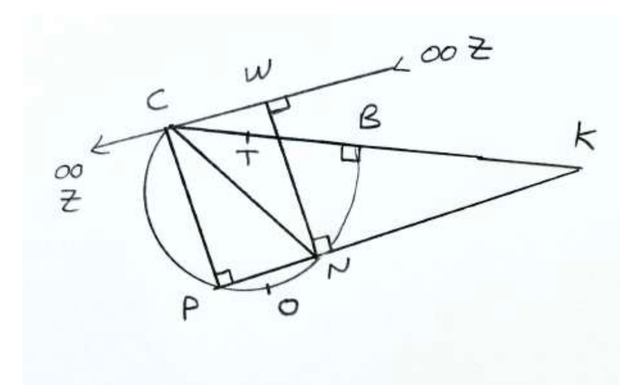


The off-axis rays from any on-axis object A, (real or virtual), can not form a virtual on-axis image at Z because NW must be less than CP for Z to be virtual; but NW must also be greater than NT.

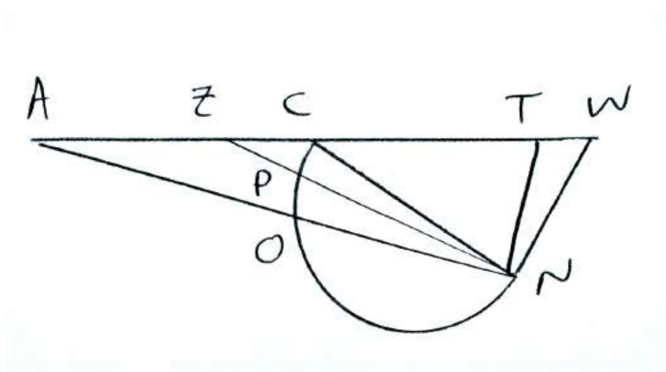


The off-axis rays from any real on-axis object A can not form a real on-axis image at Z because NW must be greater than (or equal to) CP for Z to be real; but NW must also be greater than NT.

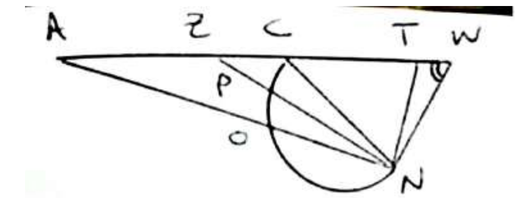




The off-axis rays from a virtual on-axis object A *can* form a real on-axis image at Z, if NW is greater than CP, and WT lies along the axis.

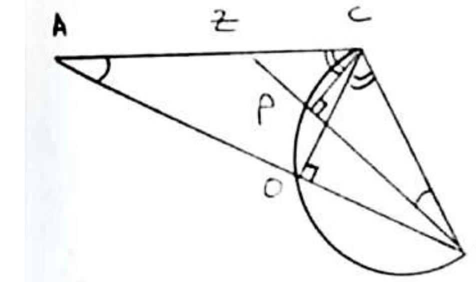


Since:
∠NWT = ∠NPO = ∠NCO
and NW || CP

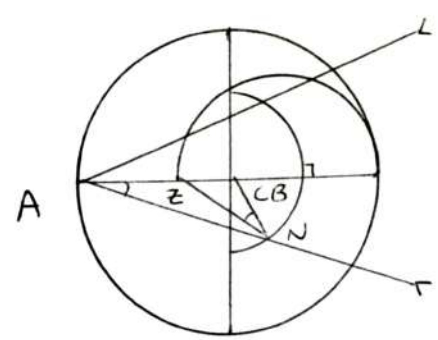


WT lies along the axis when:

ΔNCO ≅ ΔZCP



When off-axis rays from a virtual on-axis object A form a real on-axis image Z, this occurs at all points N because:

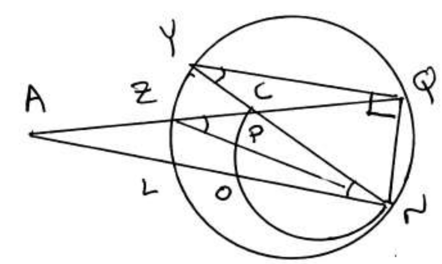


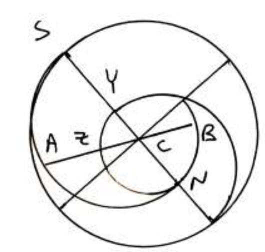
ΔACN ≅ ΔNCZ for all N,
(since they share proportional sides around a common angle).

This can also be demonstrated using similar right triangles:
ΔSAN ≅ CON, and ΔYZN ≅ ΔCPN,
so that: (AO/AN)(ZN/ZP) = (SC/SN)(YN/YC).

Since: CY/CN = CN/CS = (CY + CN)/(CN + CS) = NY/NS
(SC/SN) = (NC/NY), and:

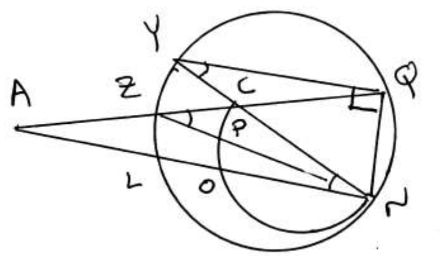
(AO/AN)(ZN/ZP) = CN/CY

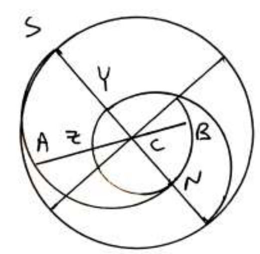




But it is also true that:
(CO/CP)(NP/NO) = CN/CY, because:

(CO/CP)(NP/NO) = (LY/LN)(PN/PC) =
= (QN/QY)(PN/PC) = (QN/QY)(ZN/ZY) =
QN (ZN)/QY(ZY) which, by the property of cyclic quadrilaterals shown in slide #7, equals CN/CY





Keeping:

ℝ = (CO/CP) = (NO/NP)(AO/AN)(ZN/ZP)

constant, as N ⇒ B:

(BC/BC)(AC/AB)(ZB/ZC) ⇒ ℝ

Refraction Through a Circle's Center

(Axial Refraction)

Refraction through a circle's center occurs when N lies at B, so that an object's ray from A to N lies along ABC, and an image ray lies along BCZ. The locations of the object A and image Z along the optic axis BC are described by the equation:

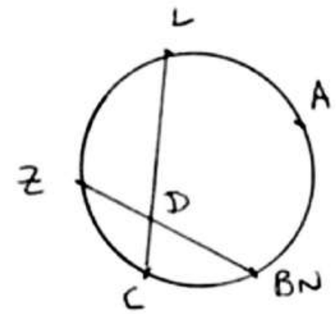
ℝ = CO/CP = (AC/AB)(ZB/ZC)

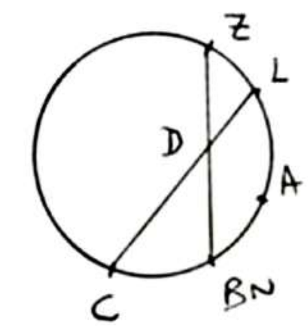
If we draw A and Z along the optic axis BC *as if* it were a circle, and draw CDL so that AL || ZB:

$\Delta ACB \cong \Delta ZCD$, and:

$(AC/AB)(ZB/ZC) =$
$(ZC/ZD)(ZB/ZC) =$
(ZB/ZD)
so as the reference circle's radius $\Rightarrow \infty$,

$(ZB/ZD) \Rightarrow \mathbb{R}$



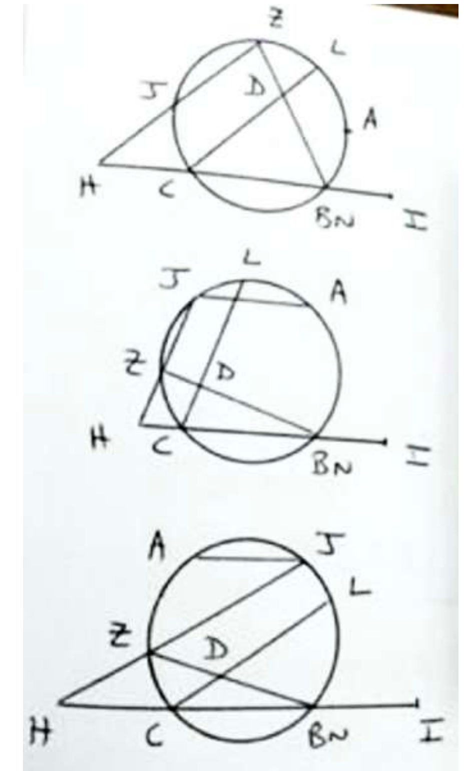


$AL \parallel ZB$
$AZ = BL$
$\sim AZ = \sim BL$

$HZ \parallel CL$
$ZC = LJ$
$\sim ZC = \sim LJ$

$\sim AZ + \sim ZC = \sim AZC$
$\sim BL + \sim LJ = \sim BLJ$

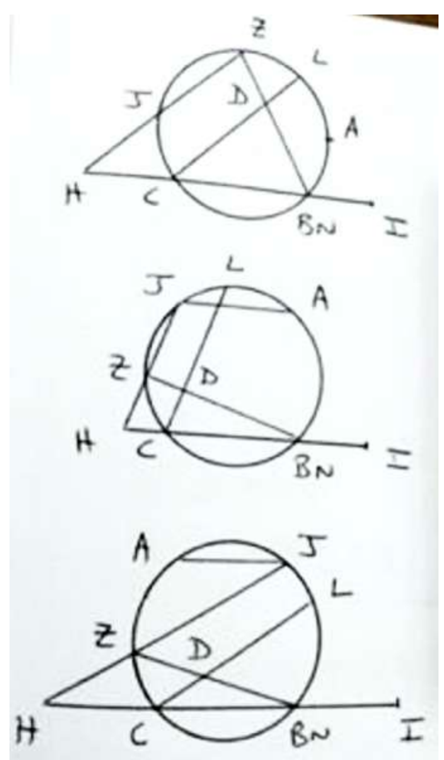
$\sim AZC = \sim BLJ$
$AJ \parallel CB$



$HZ \parallel CL$
$ZB/ZD = HB/HC$
$\Delta HBZ \cong \Delta HJC$
when $\Delta HJC = \Delta IAB$:
$HC = IB$, and:
$IB/IA = HZ/HB$

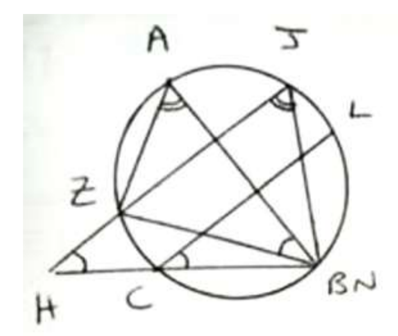
This results in Newton's Equation:
as the reference circle radius $\Rightarrow \infty$,

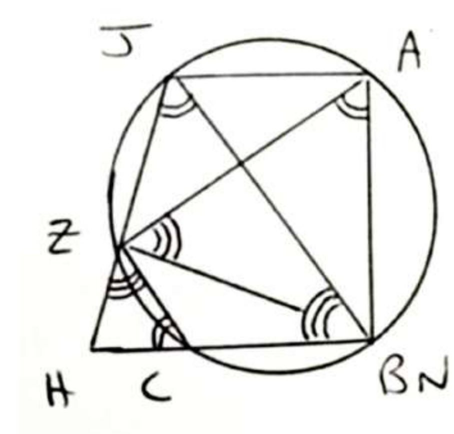
$(AI)(ZH) = (BI)(BH)$

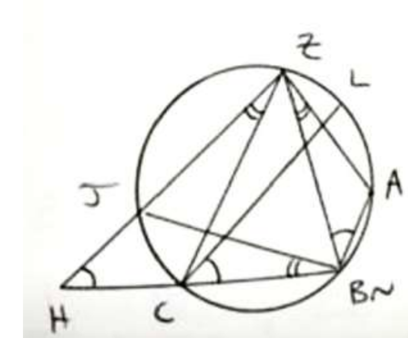


$\Delta HCZ \cong \Delta HJB \cong \Delta BAZ$

$(HC/HZ) = (BA/BZ)$
$[1/(HZ)(BA)] = [1/(HC)(BZ)]$







as the reference circle's radius ⇒ ∞,

$[1/(HZ)(BA)] = [1/(HC)(BZ)] \Rightarrow \mathbb{R}/(HB)(BZ)$

and the resulting possible sums occur:

$HZ = HB + BZ$
$HB = HZ + BZ$
$BZ = HZ + HB$

which, when multiplied by the above three factors, form the conjugate foci equations.

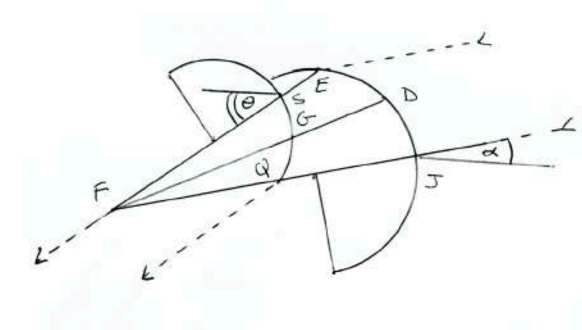
The conjugate foci equations allow for the effect of axial refraction at a circle to be expressed as the term:

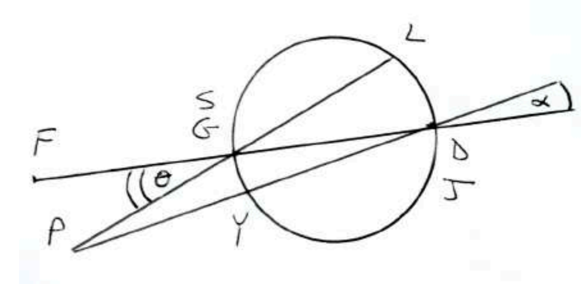
$(1/HC) = (\mathbb{R}/HB)$

which is then additive with object vergence, defined as $(1/BA)$; or image vergence, defined as $(\mathbb{R}/BZ)$.

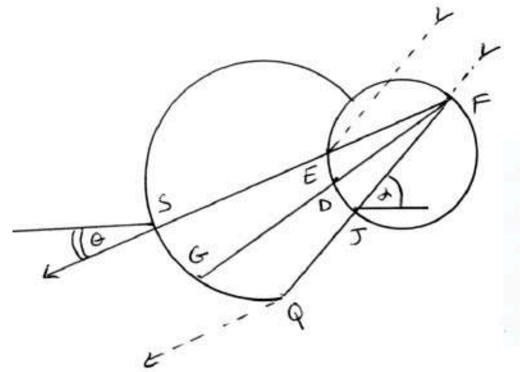
Afocal Angular Magnification/Minification

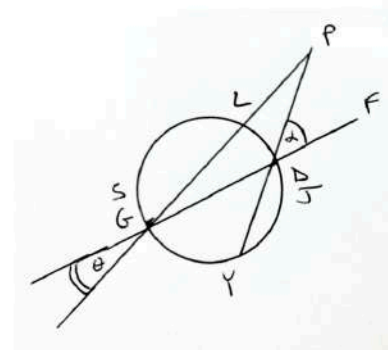
When off-axis distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis DGF as shown;
as ∠JFD = ∠SFG,
and both approach zero:





Or when off-axis distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis FDG, as shown;
as ∠JFD = ∠SFG,
and both approach zero:





$\theta/\alpha \Rightarrow (\sim LD/GD)/(\sim YG/GD)$ as $P \Rightarrow F$

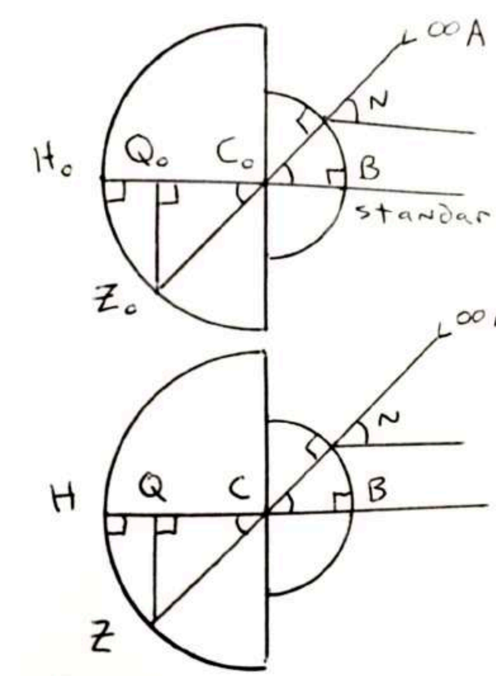
$\theta/\alpha \Rightarrow (FD/FG)$ as $P \Rightarrow F$

so that afocal axial angular magnification/minification equals:

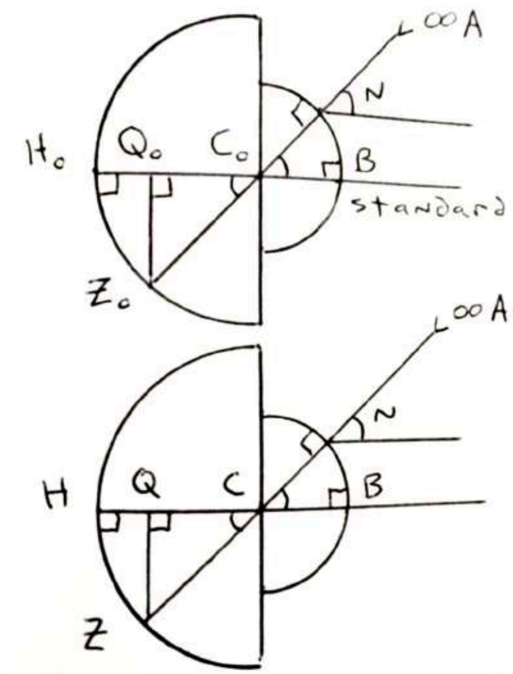
FD/FG

Retinal Image Size Magnification/Minification

The top diagram illustrates a standard single-surfaced eye with a distant object A, and resulting retinal image size H_oZ_o.



The bottom diagram illustrates any single-surfaced eye with a distant object A, and resulting retinal image size HZ.



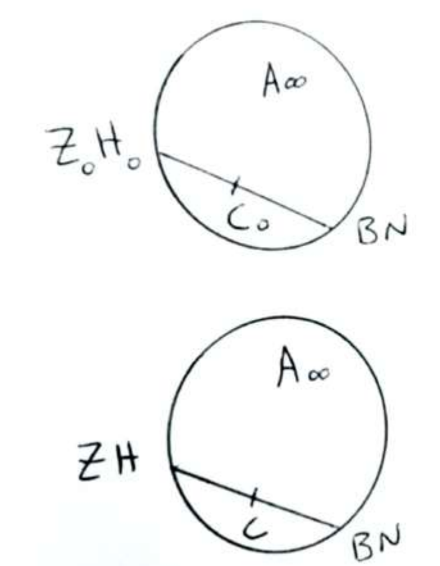
As N ⇒ B, the retinal image size magnification, ZH/Z_oH_o, (relative to an arbitrary standard which factors out with subsequent comparisons), then approaches its axial value:

$ZQ/Z_oQ_o = ZC/Z_oC_o = HC/H_oC_o$

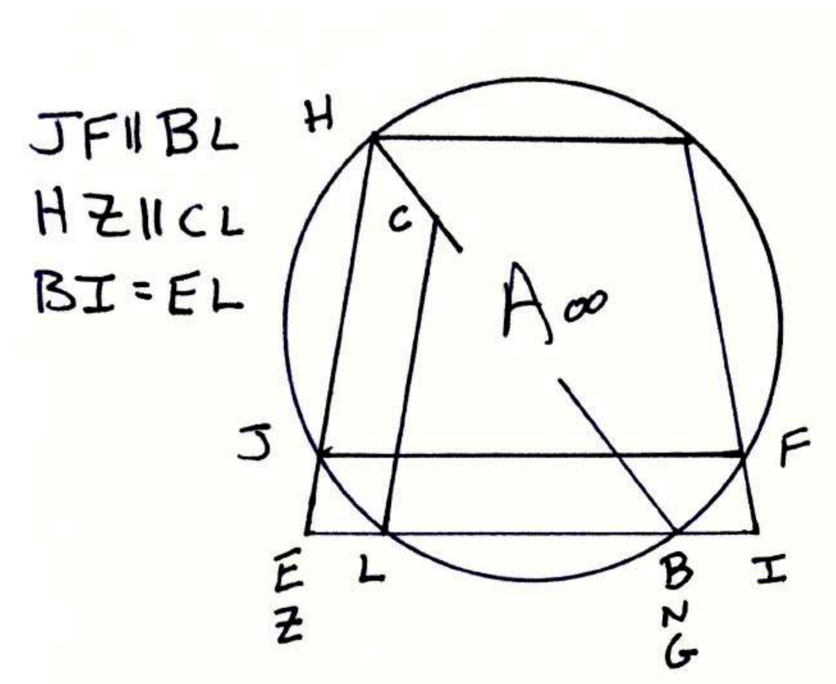
$= (BH/\mathbb{R})/(BH_o/\mathbb{R}) = BH/BH_o$

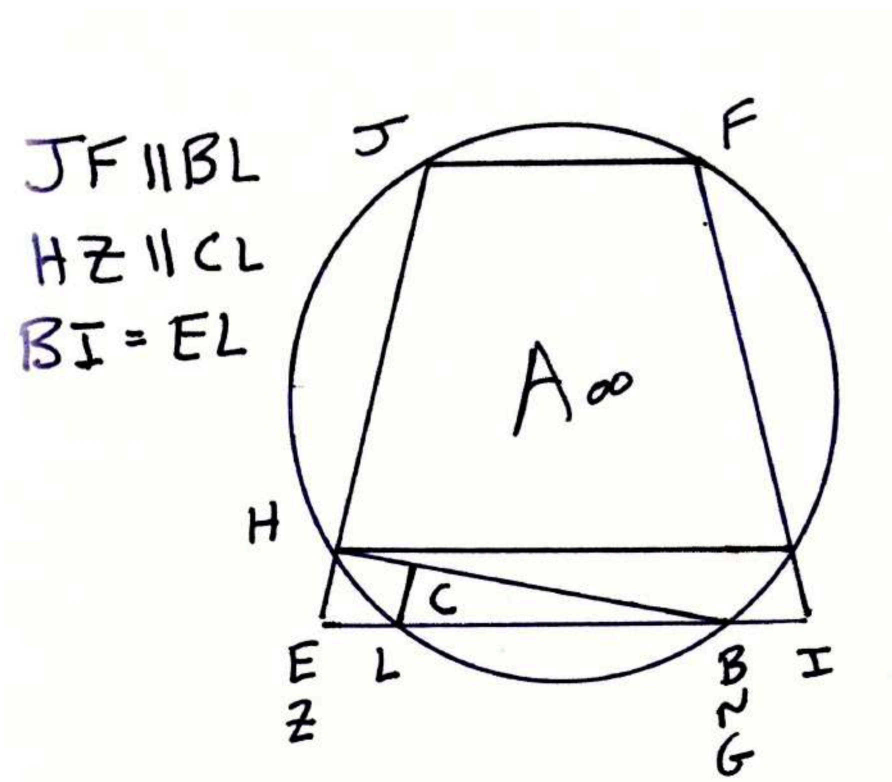
Distance Correction Magnification/Minification

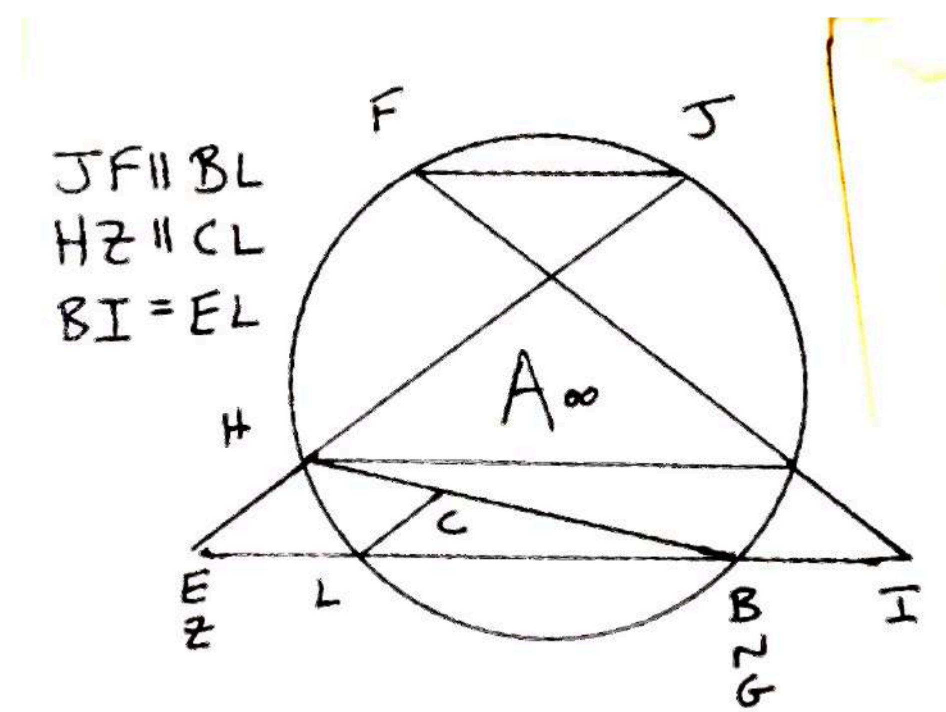
Once again representing the optic axis BCZ as a circle of infinite radius, the distant object A at ∞ is focused by the radius BC of the presumed single refracting surface towards the axial image Z, which lies at the retina H when there is no distance refractive error. (BH_o represents the standard axial length, and BC_o represents the standard single refracting curvature radius).



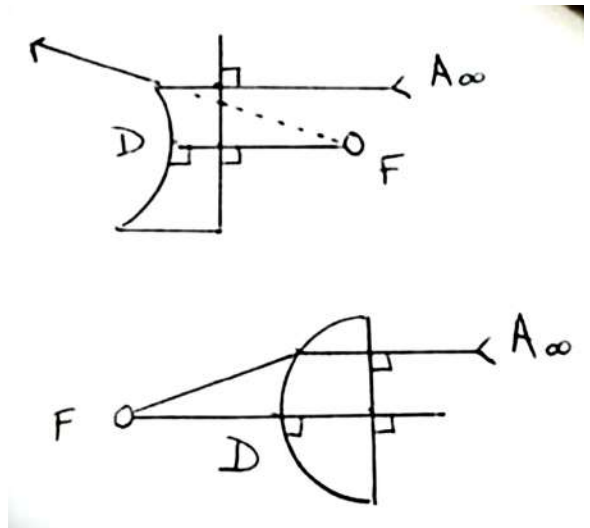
As pictured in the next three slides, additional refraction G (at B) will create an “ametropic” eye, with “distance refractive error,” and a combination curvature effect with total radius BL instead of BC, moving image Z from the retina at H to its erroneous location at E. The “front focal point” of the “ametropic” eye is defined as point I.
A “distance correction” must focus the distant object towards F, so that JF || BL, in order to move image Z back to the retina H.







The distance correction at D:



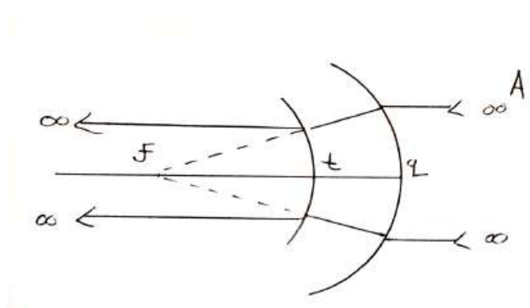
Since the distance correction D moves image Z from E to retina H, rays leaving the refractive error G (at B) after this correction is in place must be afocal. This results in afocal axial angular magnification equaling:

FD/FG (= FD/FB)

Therefore, the total axial magnification of distance correction equals:

$M = (BH/BH_o)(FD/FB)$

When the front surface of a spectacle lens that corrects distance refractive error is not flat, it is convex; and adds an additional “shape” factor, (fq/ft), to the afocal axial magnification of distance correction. (Point “t” lies at D, and FD/FB remains the “power” factor of the afocal axial magnification of distance correction).



“Axial Ametropia” occurs when E is at H_o, (and point I is therefore at I_o, the front focal point of the standard eye). The distance refractive error is then completely due to an axial length BZ, (or BH), that is not standard.

$\Delta H_oBH = \Delta EBH \cong \Delta EJL = \Delta I_oFB$

$(BH/BH_o) = (FB/FI_o)$

$M = (FB/FI_o)(FD/FB) = FD/FI_o$

Therefore, in the case of axial ametropia, there is no total axial magnification of distance correction if the correction D lies at I_o.

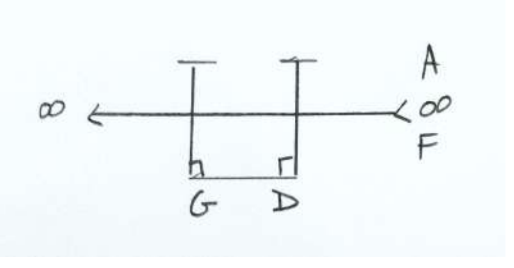
“Refractive Ametropia” occurs when the retina H is at at H_o. The distance refractive error at G moving image Z to E is then completely due to a refracting radius BL that is not the standard BC_o.

When the distance correction D lies at B:

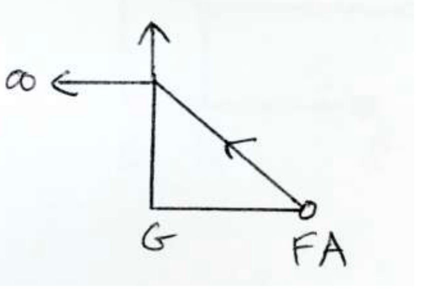
$M = (BH/BH_o)(FD/FB) = 1$

Near Correction Magnification

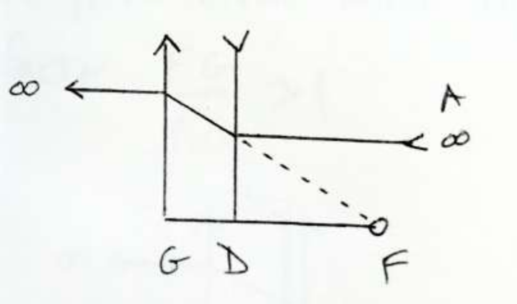
There is no afocal axial angular magnification of distance correction with a distant object “A,” and an emetropic eye whose refractive error at G (at B) is by definition zero, (with its focal point F at infinity).



There is also no afocal axial angular magnification when object A is at the front focal point F of an uncorrected ametropic eye as shown, since this “myopic” system is not afocal, and involves only one refracting element G.



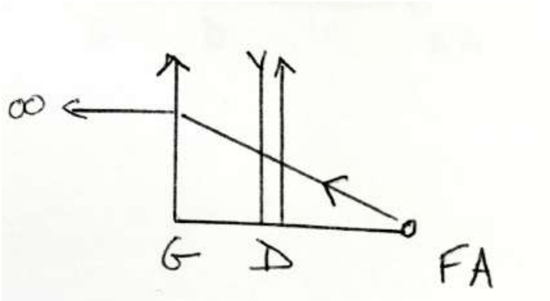
A distance myopic correction at D creates afocal axial angular minification:



FD/FG < 1

and this is relative to either the myopic eye with object A at its front focal point F, or the emetropic eye with object A at distance.

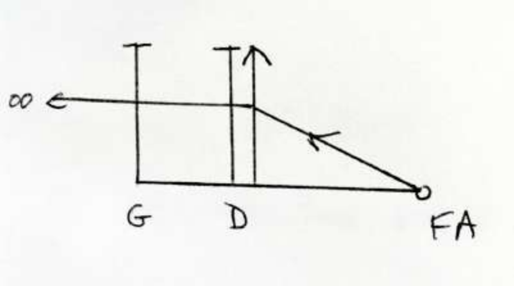
Removing the myopic distance correction at D with a converging lens at D removes this afocal axial angular magnification with the factor:



FG/FD > 1

and this magnification of near correction is relative to the distance corrected myope.

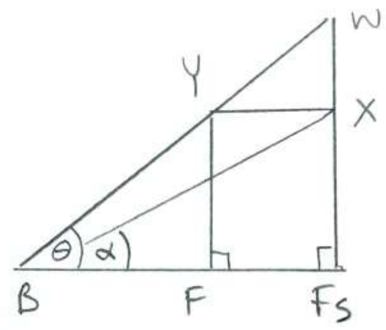
If additional converging power is added to the converging lens so that the near focal point is in focus for an emetropic eye, which we then consider to be the reference eye, the magnification of near correction is still that which is removed with the factor:



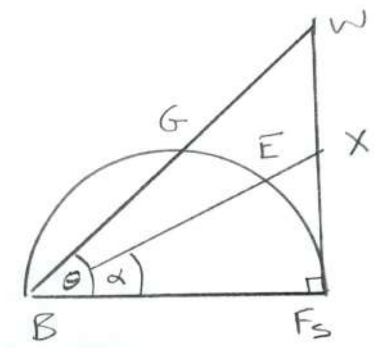
FG/FD > 1

Near Object Positional Magnification

When an object at a standard distance Fs is moved to F:



The object angular subtense magnification equals:



θ/α = (~GFs/BFs)/(~EFs/BFs)

as XFs ⇒ 0

the object angular subtense magnification approaches its axial value:

θ/α ⇒ WFs/XFs = WFs/YF = BFs/BF

which equals the axial
object angular subtense magnification.

Total Near Magnification

The ratio describing axial object angular subtense magnification:

BFs/BF

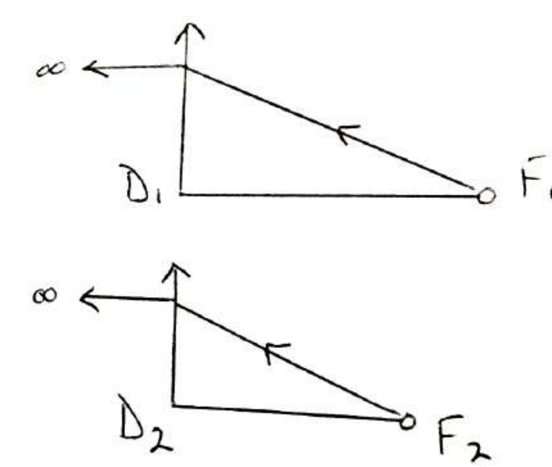
when multiplied by the ratio describing near magnification due to a single converging lens producing parallel light for an emmetropic eye:

FB/FD

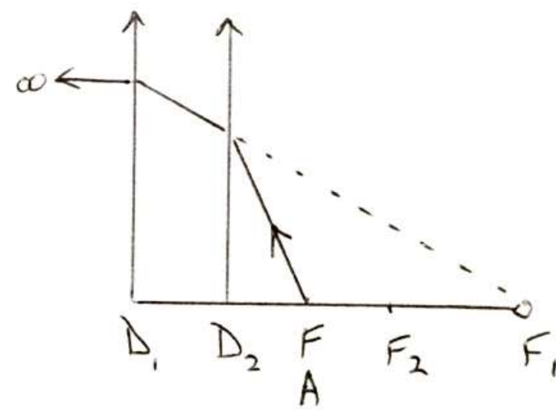
produces a ratio which factors out the object's actual distance to the eye, confirming that when a converging lens is used with its front focal point at the object, so that parallel light leaves the converging lens from the object, the image size is the same regardless of the object-to-eye distance.

Double Refraction Systems

When the converging lens at D is split into two converging lenses:



with the same combined focus F:



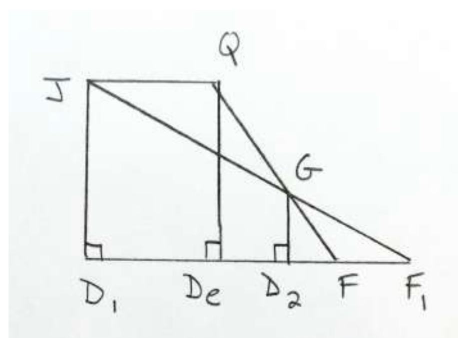
the ratio describing axial near magnification due to a single converging lens producing parallel light for an emmetropic eye:

FB/FD

must be expressed ***as if*** all convergence occurred at a single unknown axial point De:

FB/FDe

De can be located using triangles.



$D_2G/D_2F = DeQ/DeF$

$D_2G/D_2F_1 = D_1J/D_1F_1$

$D_2F(DeQ/DeF) = D_2F_1(D_1J/D_1F_1)$

$DeQ/DeF = (D_2F_1/D_2F)(D_1J/D_1F_1)$

$1/DeF = (D_2F_1/D_2F)(1/D_1F_1)$

$FB/FDe = (D_2F_1/D_2F)(FB/D_1F_1)$

Multiplying the axial object subtense magnification by the axial magnification of near correction (relative to the same eye without refractive error) produces:

$BFs/FDe = (D_2F_1/D_2F)(BFs/D_1F_1)$

The converging lens D_2 creates a virtual image F_1 of an object at F. When considering a stand magnifier with lens D_2, constant stand height D_2F, and reading spectacle add or ocular accommodation D_1, the stand magnifier's (constant) enlargement of the object at F equals:

$$E = D_2F_1/D_2F$$

The stand magnifier's axial magnification is its (constant) enlargement factor E, multiplied by what would be produced by D_1 alone, if the object A were at F_1.

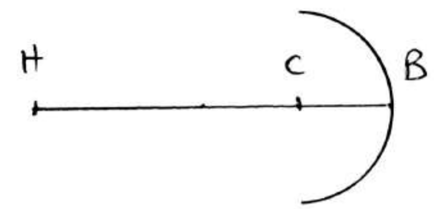
Crossed Cylinders

It is useful to know the meridian of maximum axial refraction when combining the effects of two cylindrical refracting surfaces at an oblique axis. To do this, we need to first describe how their axial radii of curvature change with various meridional cross sections. Meridional cross sections of cylindrical surfaces are ellipses until they become parallel lines along the cylinder axis.

However, assuming a cylinder is parabolic rather than spherical, and that meridional cross sections are parabolic until they rotate into a single line parallel to the cylinder axis, allows for an approximation of the axial radii of curvature of these meridional cross sections. When these axial radii of curvature are expressed in forms that are additive in terms of refraction, we can then find the maximum sum of those expressions in terms of the meridional axis.

With any axial radius of curvature CB, and index of refraction ℝ, the axial image of a distant object lies at H when:

ℝ = HB/HC

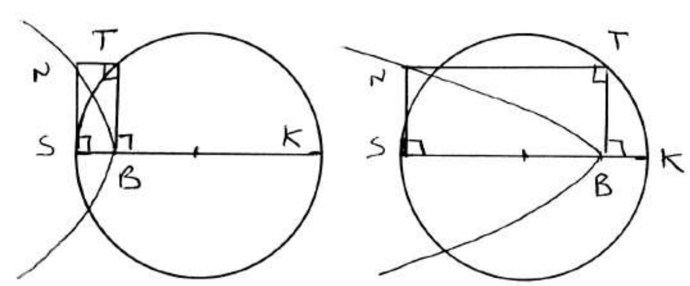


The axial refractive effects of compound refractive surfaces at B are additive only as their refractive "powers," which equal:

ℝ/HB = 1/HC = [(HB - HC)/HC]/CB = (ℝ- 1)/CB

All parabolas have the same shape, in the same way that all circles have the same shape. However, while circles have a single (internal) determining constant, the radius of curvature, parabolas have both a determining constant internal and external to the curve, and can be defined by either.

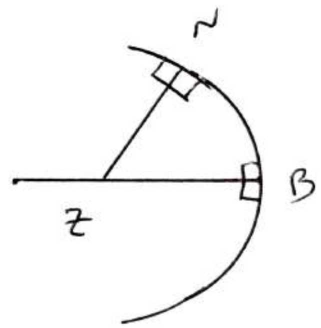
For example, a parabola's external determining constant equals BK when:



$$\frac{SB}{BT} = \frac{BT}{BK}$$

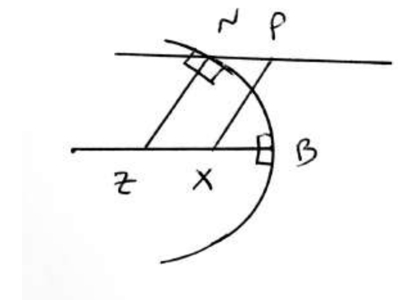
[2(SN) equals the sagitta corresponding to the sagittal depth SB].

We can set up the necessary off-axis conditions to determine a parabola's axial center of curvature in terms of its internal determining constant XB, by involving ZN in the geometric solution for XB.



In order to keep the determining geometrical relationships axial as N ⇒ B, they should also

depend on line NP being parallel to the axis, and XP being parallel to ZN.



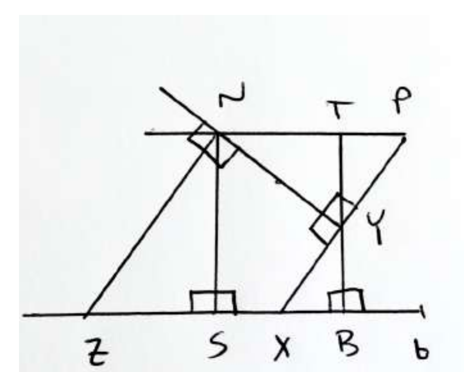
We know X lies between Z and B, since parabolas flatten in their periphery.

Since as N ⇒ B, Z ⇒ C by definition, and

since XP = ZN, P will remain external to the curve, and X can therefore not be its axial center of curvature, but must instead lie somewhere along CB.

In order to maintain ZN perpendicular to the parabola at N as N ⇒ B, the same geometrical

relationships must exist that allow for that when N lies at B.



In other words:

YP = YX and
Bb = BX so
CB = 2(XB)

Since:

$$\frac{TN}{TB} = \frac{TN}{2(TY)} = \frac{YB}{2(XB)} = \frac{YB}{CB} = \frac{TB}{2(CB)}$$

We know the external determining constant BK equals 2(CB), and the internal determining constant XB equals (CB)/2.

Axial refracting power equals $(\mathbb{R}-1)/CB$

Since for a parabola:

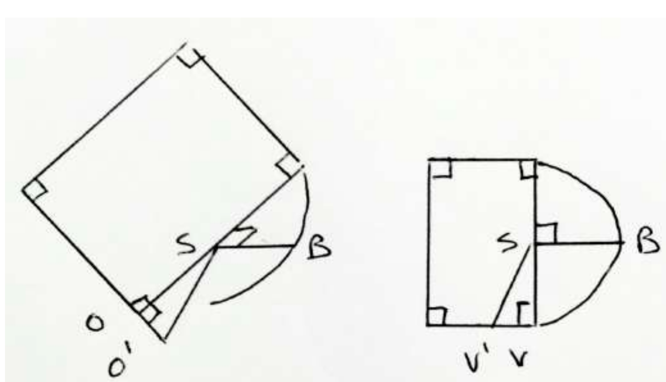
SB/SN = SB/TB = TB/[2(CB)]

If $\mathbb{R} = 1.5$

The axial refracting power of a parabola equals:

$1/[2(CB)] = SB/SN^2 = 1/BK$

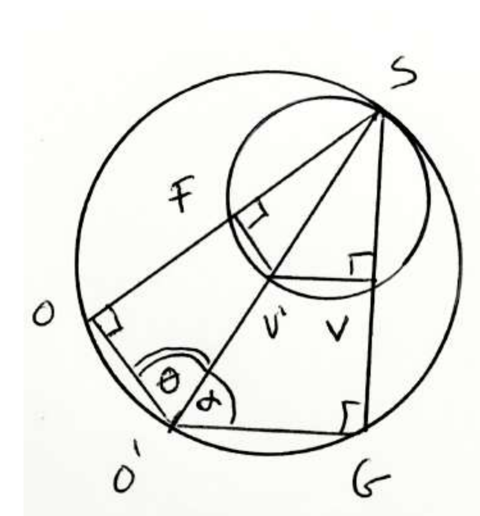
When 2(SO) equals the minimum sagitta of an oblique parabolic cylinder, and when with equal sagittal depth SB, 2(SV) equals the minimum sagitta of a more highly curved parabolic cylinder with a horizontal axis:



Keeping ΔOSV constant, as we rotate circle SOG with variable diameter SV'O' around point S:

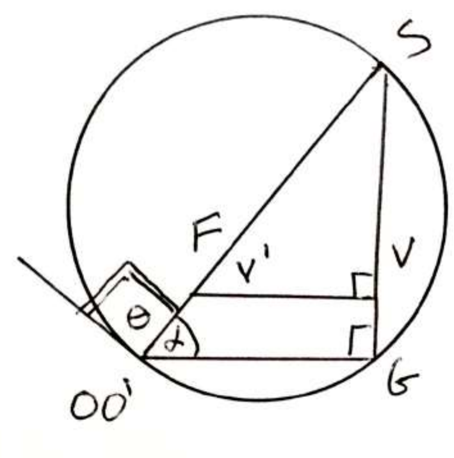
∠OO'G is constant because ∠OSG is constant,

so $\Delta\theta = -\Delta\alpha$



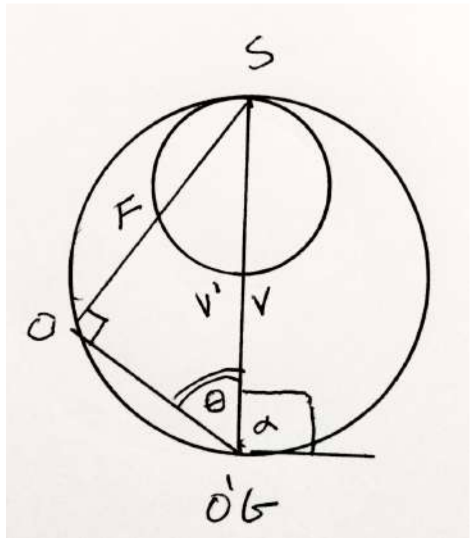
As O' ⇒ O

SV' increases more than SO' decreases



As V' ⇒ V

SO' increases more than SV' decreases



Since the sum (SO' + SV') increases when either:

O' ⇒ O, or V' ⇒ V

there must be a specific SV'O' within ΔOSV producing a minimum sum (SO' + SV'), which must be near where small rotations produce only minimal changes in (SO' + SV').

Since as when one term of the sum (SO' + SV') increases, the other always decreases, this process can be taken to its limits to determine the meridian with minimum (SO' + SV') using:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta(SO') = \lim_{\Delta\alpha \Rightarrow 0} \Delta(SV')$$

However, the combined effects of refraction are additive only as refractive powers, which, when ℝ = 1.5, equal:

$SB/(SO')^2$ and $SB/(SV')^2$

Therefore, the meridian with the maximum combined effects of this refraction can be found using:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta\ [SB/(SO')^2] = \lim_{\Delta\alpha \Rightarrow 0} \Delta\ [SB/(SV')^2]$$

To solve this equation, all variables must be expressed in terms of the variables approaching zero, so:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta\{[SB(SO/SO')^2]/SO^2\} = \lim_{\Delta\alpha \Rightarrow 0} \Delta\{[SB(SV/SV')^2]/SV^2\}$$

$$\lim_{\Delta\theta \Rightarrow 0} \Delta\{[(SB)\sin^2\theta]/SO^2\} = \lim_{\Delta\alpha \Rightarrow 0} \Delta\{[(SB)\sin^2\alpha]/SV^2\}$$

$$(SB/SO^2) \lim_{\Delta\theta \Rightarrow 0} \{\Delta\sin^2\theta\} = (SB/SV^2) \lim_{\Delta\alpha \Rightarrow 0} \{\Delta\sin^2\alpha\}$$

$$\{\text{Limit as } \Delta\theta \Rightarrow 0 \text{ of } [\Delta\sin^2\theta]\}/\{\text{Limit as } \Delta\alpha \Rightarrow 0 \text{ of } [\Delta\sin^2\alpha]\}$$

$$= [SO^2/SV^2]$$

Solve for

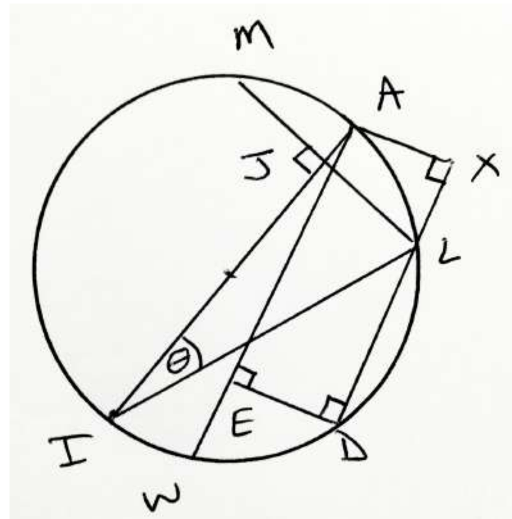
$$\lim_{\Delta\theta \Rightarrow 0} \Delta \sin^2\theta$$

on the reference circle:

AW ≥ LD || AW

$\angle ALD = \sim AID/AI$

$\geq \sim AI/AI = \pi$



Establish the necessary functions of θ in terms of line segments and chords.

$\theta = \sim\frac{AL}{AI}$; $\sin^2\theta = \frac{AL^2}{AI}$

$\Delta\theta = \sim\frac{LD}{AI}$; $\sin^2\Delta\theta = \frac{LD^2}{AI}$

$(\theta + \Delta\theta) = \sim\frac{ALD}{AI}$; $\sin^2(\theta + \Delta\theta) = \frac{AD^2}{AI}$

$\cos\theta = \frac{IL}{AI}$; $\cos(\theta + \Delta\theta) = \frac{DI}{AI}$

$\sin\theta = \frac{AL}{AI} = \frac{JL}{IL}$; $\sin\theta\cos\theta = \frac{JL}{IL}\,\frac{IL}{AI}$

$2(\sin\theta\cos\theta) = \frac{ML}{AI} = \sin 2\theta$

Then consider the following property of the cyclic quadrilateral circle ALDW: AD(LW) = AL(DW) + LD(AW)

$\Delta DIA \cong \Delta EWD = \Delta XLA$; $AD^2 = AL^2 + LD(AW)$

$AW = LD + 2(AL)\frac{LX}{LA}$; $AW = LD + 2(AL)\frac{ID}{IA}$

$AD^2 - AL^2 = LD^2 + 2(LD)(AL)\frac{ID}{IA}$

$AI[\sin^2(\theta + \Delta\theta) - \sin^2\theta] =$

$AI[\sin^2\Delta\theta] + 2(LD)(AL)\cos(\theta + \Delta\theta) =$

$AI[\sin^2\Delta\theta] + 2(LD)[(AI)\sin\theta]\cos(\theta + \Delta\theta)$

Divide both sides by AI:

$\sin^2(\theta + \Delta\theta) - \sin^2\theta = \sin^2\Delta\theta + 2(LD)\sin\theta\cos(\theta + \Delta\theta)$

$$\lim_{\Delta\theta \Rightarrow 0} \frac{\Delta(\sin^2\theta)}{\sim LD} = 2\sin\theta(\cos\theta) = \sin 2\theta$$

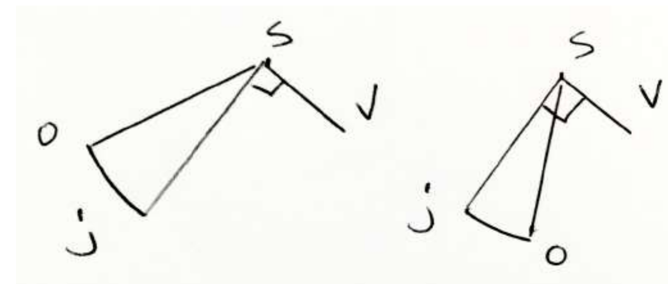
Therefore, the meridian with the maximum combined effects of refraction can be found using:

$$\frac{\sin 2\theta}{\sin 2\alpha} = \frac{SO^2}{SV^2}$$

The first step to solve this problem is to divide SV into SaV so that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

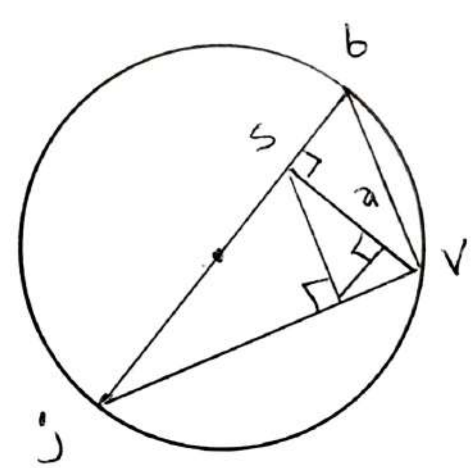
Make SO = Sj ⊥ SV to construct:



$$\frac{Sj}{SV} = \frac{SV}{Sb} \quad ; \quad \frac{Sj^2}{SV^2} = \frac{Sj}{Sb} = \frac{SO^2}{SV^2}$$

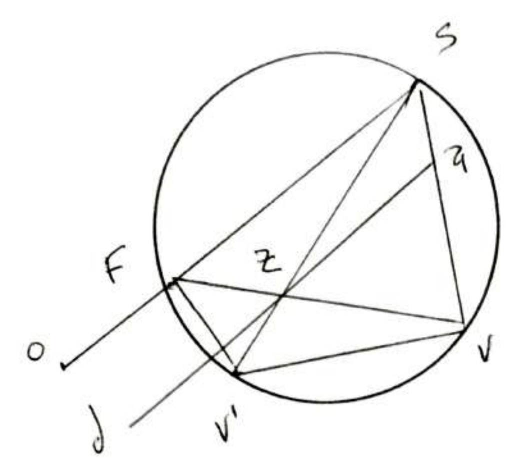
Similar triangles show that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

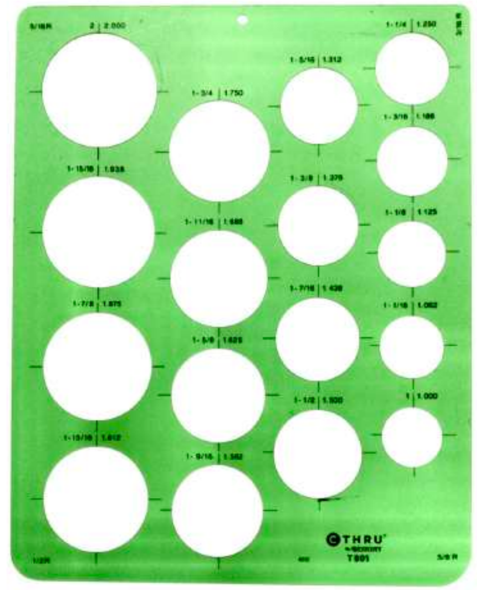


Draw ad || SO

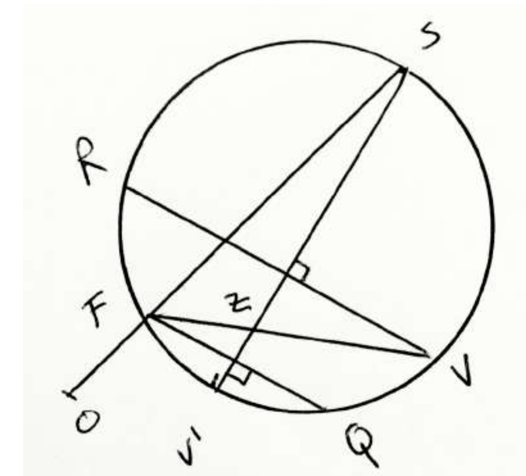
Choose a circle through S and V with a variable diameter SV' so that FZV lies on a common chord.



The easiest way to do this involves a template of various circles, each with the location of their diameters already marked.



SV' is the meridian with the maximum combined effects of refraction because:



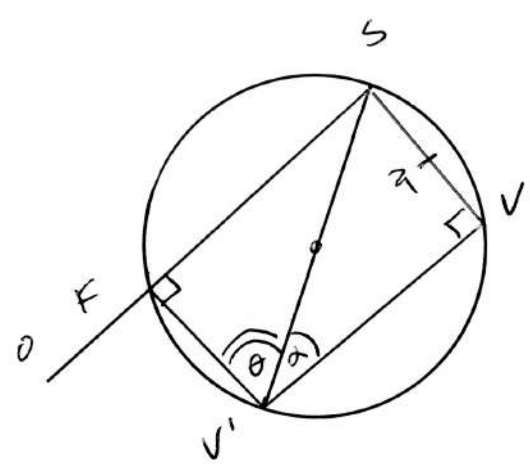
$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{FZ}{ZV} = \frac{FQ/2}{RV/2} = \frac{FQ}{RV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

Double-angle Method:

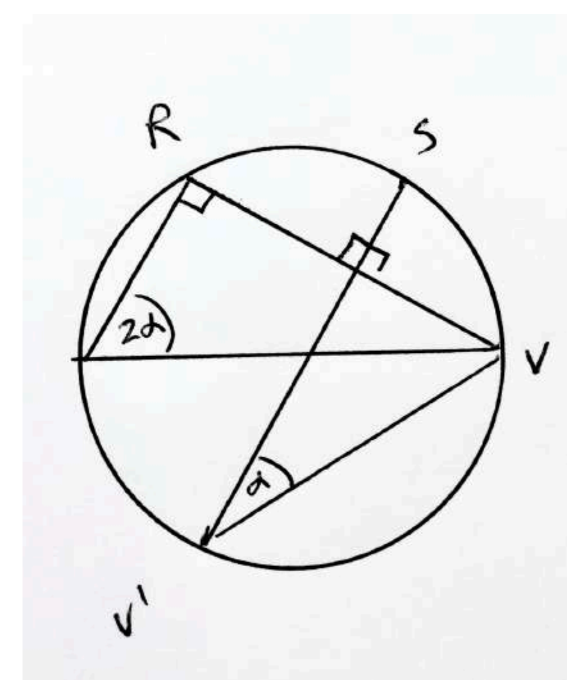
Given constant ΔOSV:
$\angle FSV$ is constant
$\angle FSV + (\theta + \alpha) = \pi$
$(\theta + \alpha)$ Is constant

We have already shown how to find single angles $\theta + \alpha$ so that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

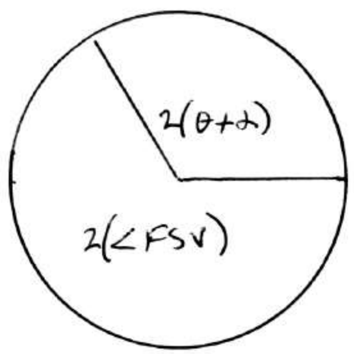


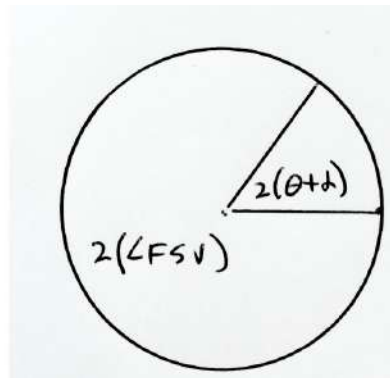
An angle on a circle equals its inscribed arc, divided by the arc's diameter. Since the sum of all angles measured on a circle's circumference add to π, when measured from a circle's center they add to 2π.



Therefore:

$$2(\angle FSV) + 2(\theta + \alpha) = 2\pi$$

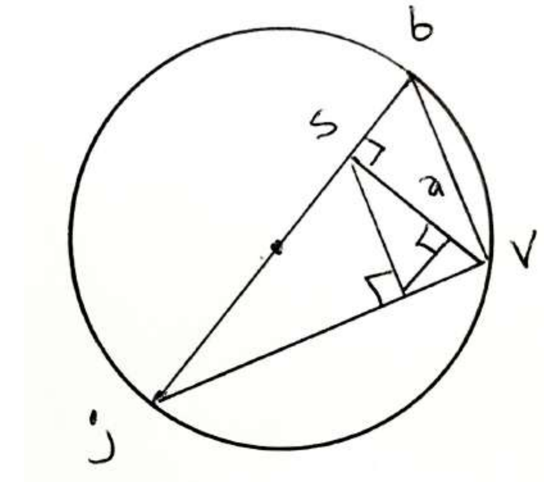




When:

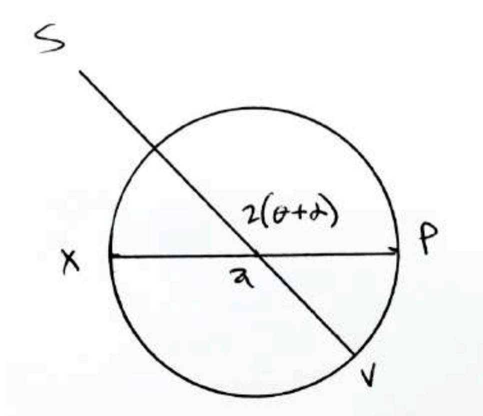
$$\frac{SO^2}{SV^2} = \frac{Sj^2}{SV^2} = \frac{aS}{aV}$$

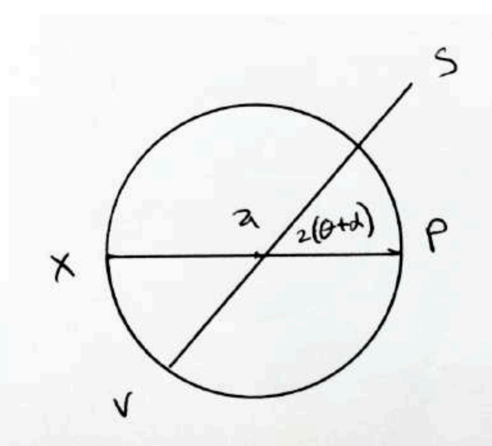
as drawn:



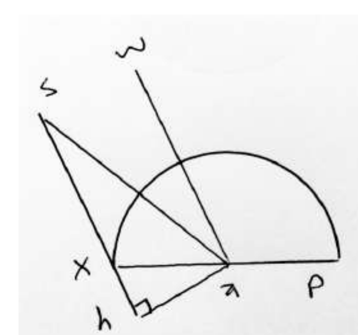
If we draw diameter XaP so:

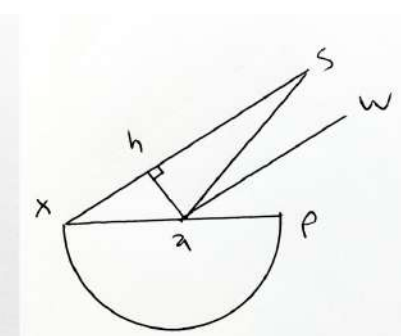
$aX = aV$, and $\angle SaP = 2(\theta + \alpha)$





$$\frac{SO^2}{SV^2} = \frac{aS}{aX} = \frac{ah/aX}{ah/aS} = \frac{\sin 2\theta}{\sin 2\alpha}$$





When aw || sX, we have divided the doubled angle
$2(\theta + \alpha) = \angle SaP$
into $2\theta = \angle WaP$, and $2\alpha = \angle WaS$.