Geometrical Optics

2021

Gregg Baldwin OD

Dedicated to my Geometrical Optics professor, William Brown, OD, PhD, who always taught the geometry first.

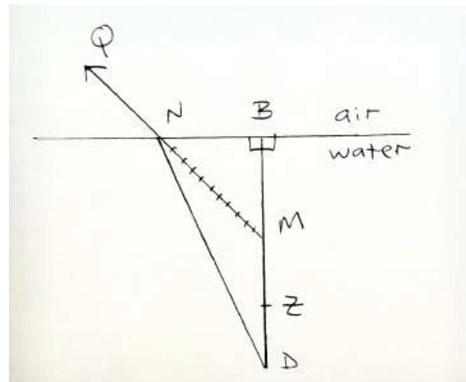
Reference:
Isaac Barrows Optical Lectures, 1667
Translated by H.C. Fay
Edited by A.G. Bennett
Publisher:
The Worshipful Company of Spectacle Makers
London, England; 1987
ISBN # 0-951-2217-0-1

Geometrical Optics - 2021

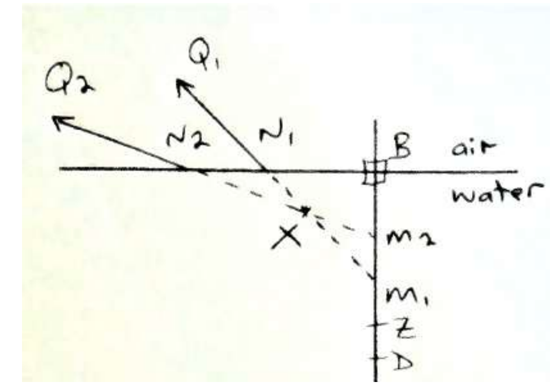
1). images seen through water

If an underwater object D is at a perpendicular distance BD from line BN along the water's surface, the image of the object seen directly above from air, (along BD), is at Z; and BD/BZ = 4/3.

Isaac Barrow showed that the image of object D, (when seen from Q *obliquely* along image ray MNQ), lies above the object, but also towards the observer relative to DB.

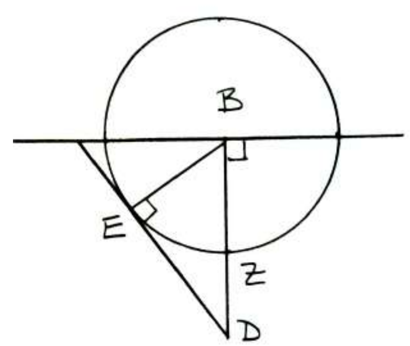


As the first step in finding an oblique image ray XNQ, along which the image of object D is seen at a designated point X, Isaac Barrow described a method of finding *all* possible oblique image rays through the designated point X, without knowing their points of refraction (N) along the surface of the water, or their intersections (M) with the perpendicular DB.



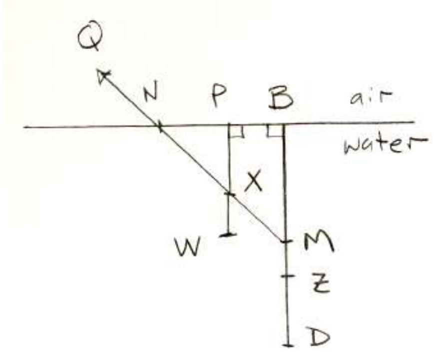
To do this, he first drew a *reference right triangle* created by drawing BE = BZ as shown, which created the following constant ratios for air/water refraction:

$BD/BZ = BD/BE = 4/3$
$DB/DE = 4/\sqrt{(16-9)} = 1.5$
$ED/EB = \sqrt{(16-9)}/3 = 0.87$



He showed that, given a designated desired clear image location X, if we draw PW as shown, where:

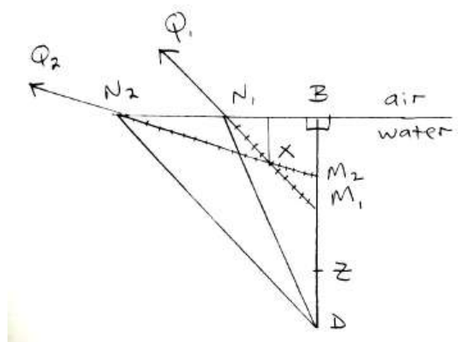
$PW/PX = DB/DE = 1.5$

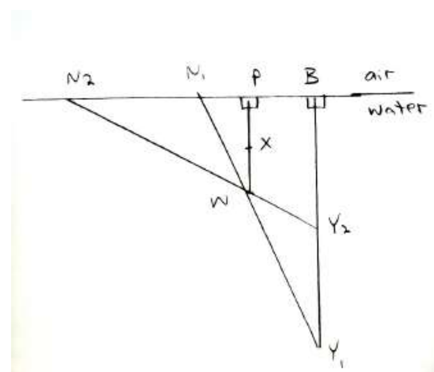


all possible image rays through X, (MXNQ) are found using:

DB/YN = ED/EB = 0.87

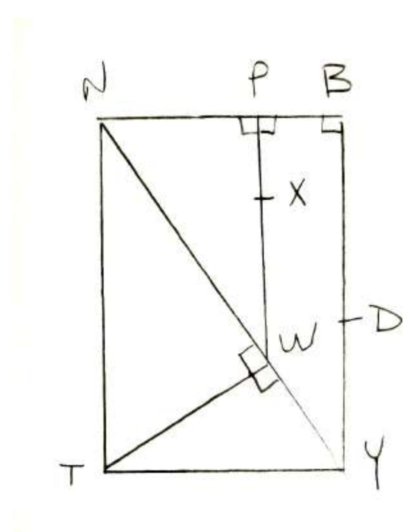
by drawing all possible reference lines of length YN = DB/0.87 through W, in order to locate the required positions of N.





He showed that there can be a maximum of *two* image rays through a designated point X, since only two reference line segments within the right angle $\angle$(Y)B(N), and equaling his calculated constant YN, can fit through point W. This is true since $Y_2N_2 = Y_1N_1$ means that the right triangle ΔY_2BN_2 must equal the right triangle ΔN_1BY_1.

Isaac Barrow showed that YN can be drawn as the shortest segment through W bounded by the right angle $\angle$(Y)B(N) when right triangles ΔYBN, ΔNWT, and ΔTWY are all drawn as similar.

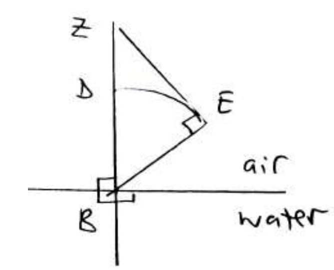


The *length* of YN through a designated W and bounded by the right angle $\angle$(Y)B(N) must be varied as it is rotated about W to find the position of its minimum length. Therefore, the position of N and Y must change to find N that corresponds to an image ray QNXM with its clear image at the designated (unchanging) point X. Furthermore, since:
PW/PX = DB/DE is constant,
ED/EB = DB/YN is also constant,
so DB varies with the length YN as a constant proportion.

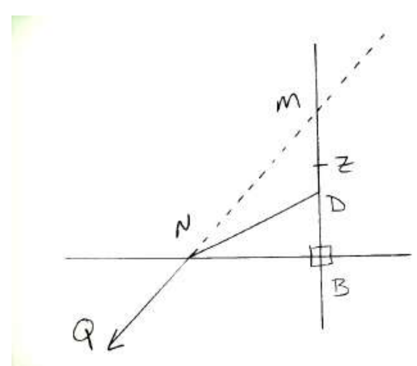
With an object underwater, Isaac Barrow's method does not allow for finding the location of the image ray on which a designated clear image is seen, while keeping both the image location *and the object position* constant. It does, however, allow for a geometric understanding of the conditions required to provide a clear image. As will be now demonstrated, with an object in air, Isaac Barrow's method actually *does* allow for finding the location of the image ray on which a designated clear image is seen, while keeping both the image location and the object position constant.

If an object D in air is at a perpendicular distance BD from line BN along the water's surface, the image of the object along that perpendicular when seen from underwater is at Z, and BZ/BD = 4/3.
A *reference right triangle* created by drawing BE = BD as shown, creates the following additional constant ratios:

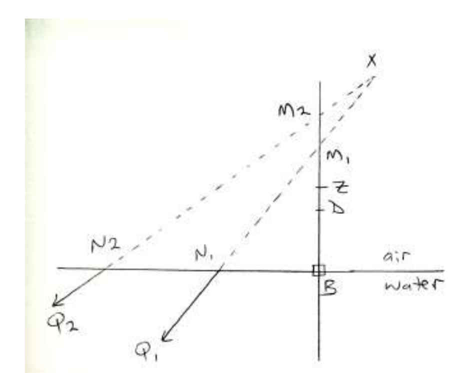
BZ/BE = 4/3
ZB/ZE = $4/\sqrt{(16-9)}$ = 1.5
EZ/EB = $\sqrt{(16-9)}/3$ = 0.87



Isaac Barrow showed that the image of object D, (when seen from Q *obliquely* along image ray MNQ), lies above the object, but also away from the observer relative to BD.

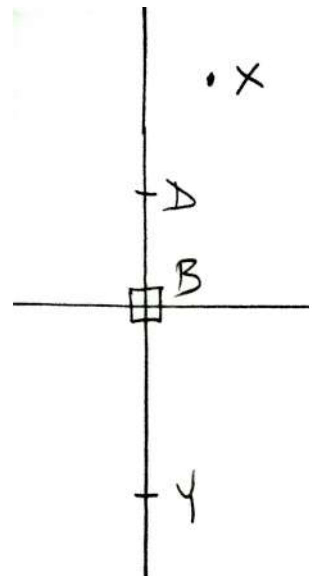


As the first step in finding an oblique image ray XMNQ, along which the image of object D is seen at a designated point X, Isaac Barrow described a method of finding *all* possible oblique image rays through point X, without knowing their points of refraction (N) along the surface of the water, or their intersections (M) with the perpendicular BD.

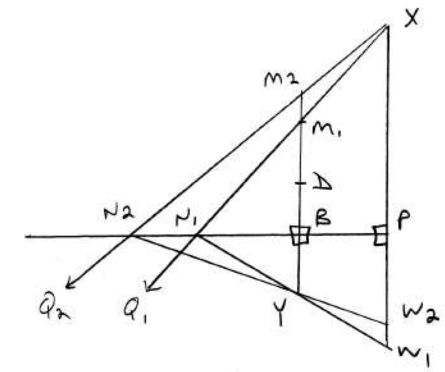


If we draw BY as shown, where:

BY/BD = ZB/ZE = 1.5

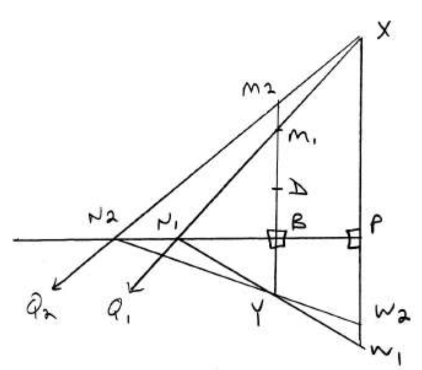


Isaac Barrow showed that all possible image rays through X, (XMNQ) are found using:

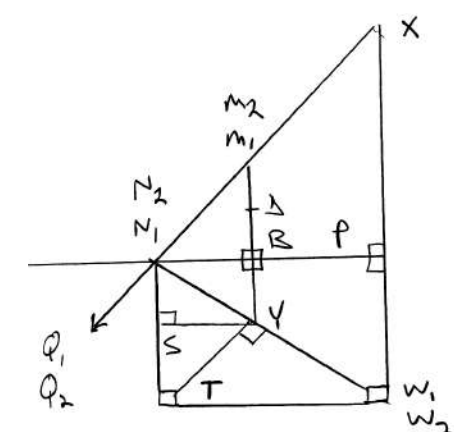


XP/WN = MB/YN = EZ/EB = 0.87
by drawing all possible reference lines of length WN = XP/0.87 through Y.

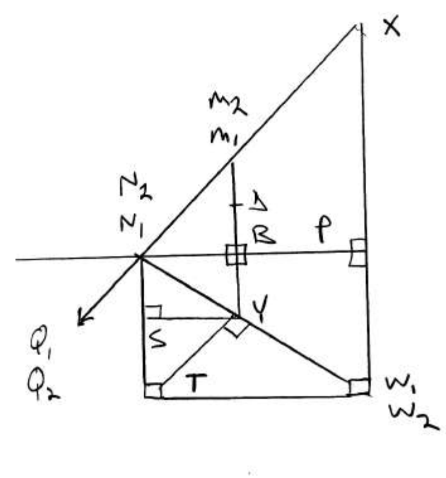
He showed that there can be a maximum of *two* image rays through any designated point X, since only two reference line segments within the right angle ∠(W)P(N), and equaling his calculated constant WN, can fit through point Y.



The point X that is the clear image of object D seen along a to-be-determined XMNQ is found using the *minimum* reference line segment length (W)Y(N) through Y, that is bounded by the right angle ∠(W)P(N).



Isaac Barrow showed that WN can be drawn as the shortest segment through Y bounded by the right angle ∠(W)P(N) when right triangles ΔWPN, ΔNYT, and ΔWYT are all drawn as similar.

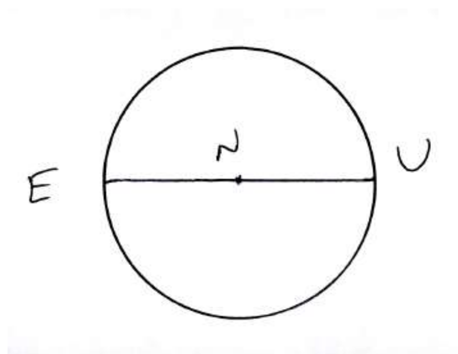


As any two equal segments W_1YN_1 and W_2YN_2 are rotated about Y in order to approach their single common minimum length, N_2 approaches N_1, and ΔN approaches zero. Both the positions of N_2 *and* N_1 must change during this process of finding the point N associated with a designated clear image X.

Since Y (not W) is the pivot point as segments W_1YN_1 and W_2YN_2 rotate, BY remains unchanged. Therefore, BD also remains unchanged because BY/BD = BZ/BE. Therefore, unlike when the object is in water, when the object is in air, this method can find an image ray XMNQ that will produce a designated clear X, while holding the object position constant.

2). prerequisite geometry

On a circle with diameter EU and center N:



Two equal arcs ~SE and ~JR can be shown to subtend equal angles by drawing any two parallel lines SD and JF. Since parallel lines intercept equal arcs across a circle,

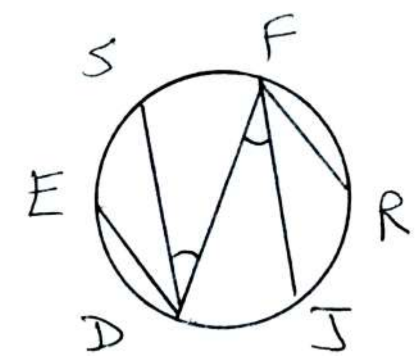
~SF = ~JD

~SE + ~SF = ~JR + ~JD

~EF = ~RD

ED || RF, and therefore:

∠SDE = ∠JFR



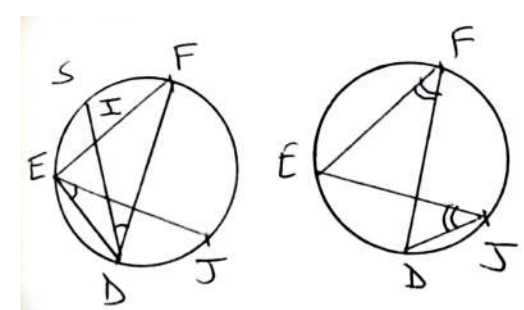
Since conversely, equal angles along a circle subtend equal arcs, any angle along any circle can be defined in terms of its subtended arc and the circle's diameter.

For example: ∠RFJ = ~RJ/EU

Triangles need only two equal angles to be the same shape, (or ≅).
Since equal arcs subtend equal angles along a circle:

ΔEJD ≅ ΔDFI

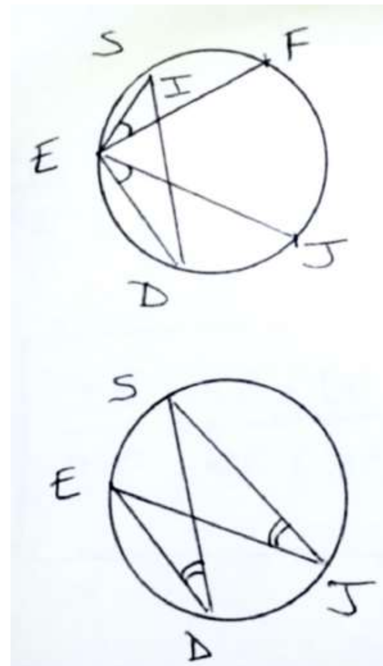
FD/FI = JE/JD



~SJ = ~FD

ΔEJS ≅ ΔEDI

EI/ED = ES/EJ



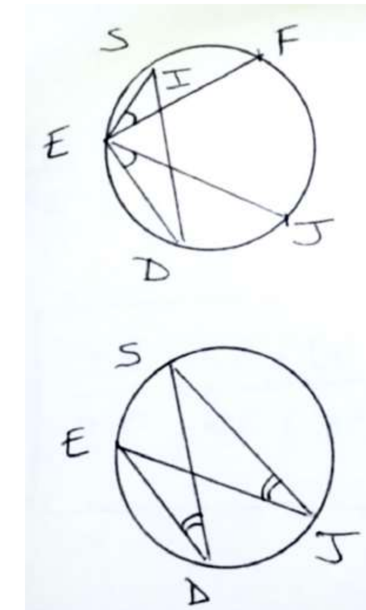
[(FD)(EI)]/[(FI)(ED)]

= [(JE)(ES)]/[(JD)(EJ)]

= SE/SF

IE/IF = [(SE)(DE)]/[(SF)(DF)]

which describes an important property of any cyclic quadrilateral SEDF



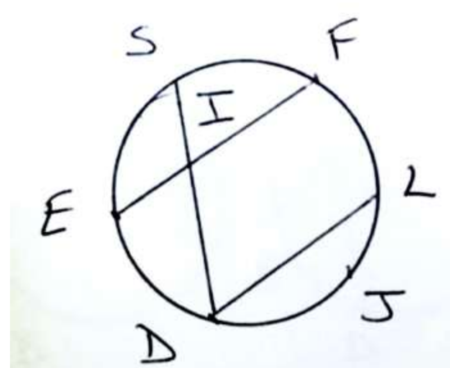
LD || FE

DE/DF = LF/LE

IE/IF = (SE)(LF)/(SF)(LE)

FE/FI

= {(SE)(LF) + (SF)(LE)}/(SF)(LE)

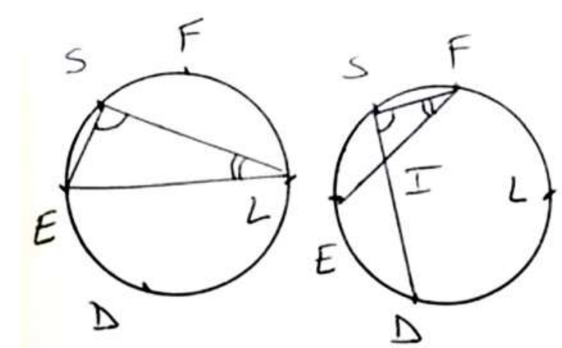


LD || FE

~EL = ~FD

ΔLSE ≅ ΔFSI

LS = {(FS)(LE)}/FI



(FE)(LS) = (SE)(LF) + (SF)(LE)

which describes an important property of any cyclic quadrilateral SELF

$\angle$KNU = $\angle$MDH

$\angle$MDH = ~MH/MD

= ~MH/UE

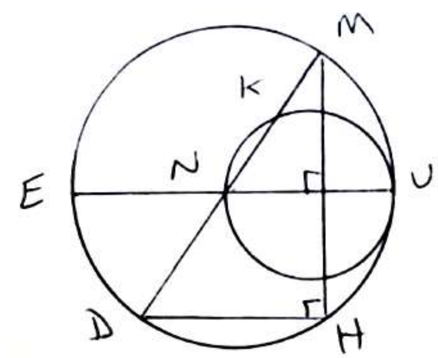
= 2(~UM)/UE

= 2$\angle$MEU

$\angle$KNU = ~UK/UN

= 2(~UM)/2(UN)

~UK = ~UM



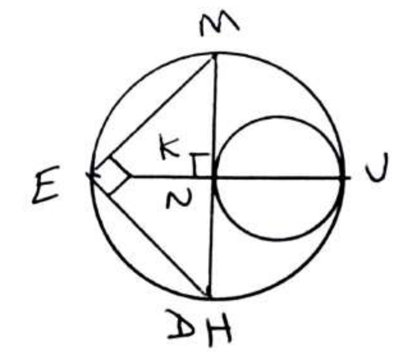
Let K ⇒ N and D ⇒ H:

~UK/UN = ~MH/MD

= ~MH/UE = $\angle$MEH

~UK/UN = $\angle$MNU

2(~UK)/UN = $\angle$MNH = π



NS/NC = NC/NB

NK/NC = CN/CK

ΔNSC = ΔKWB = ΔKNP

NC = KP

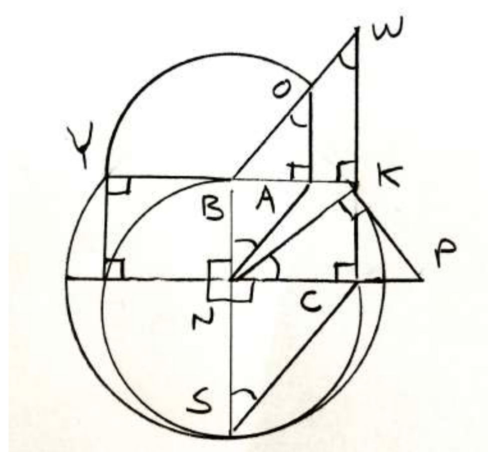
ΔCKP = ΔBNA = ΔAOB

NA = KP

NC = NA = OB

NC = KB = YB

WK = NS **= YN**



Keeping only:

NA = NC, and

ΔCNK ≅ ΔAOB ≅ ΔKWB:

As N ⇒ B, WK ⇒ YN

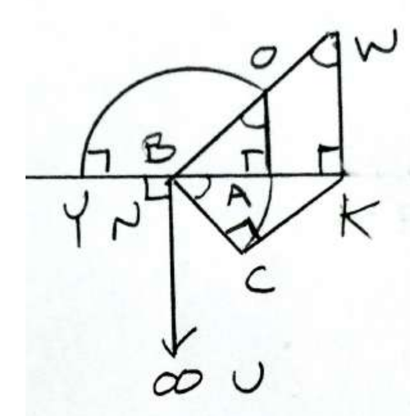
because:

WK/OA ⇒ NK/NA = NK/NC

= OB/OA = WB/WK

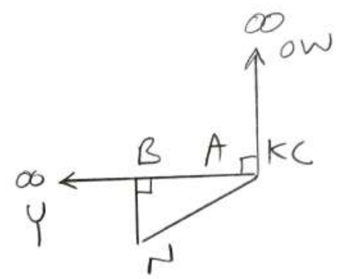
so that:

WK ⇒ OB ⇒ YN



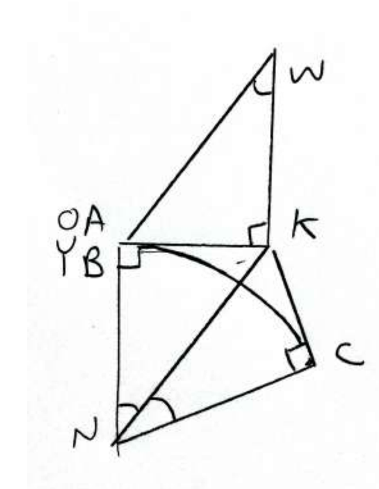
Keeping only:
NA = NC, and
ΔCNK ≅ ΔAOB ≅ ΔKWB:

As A ⇒ K, WK ⇒ YN



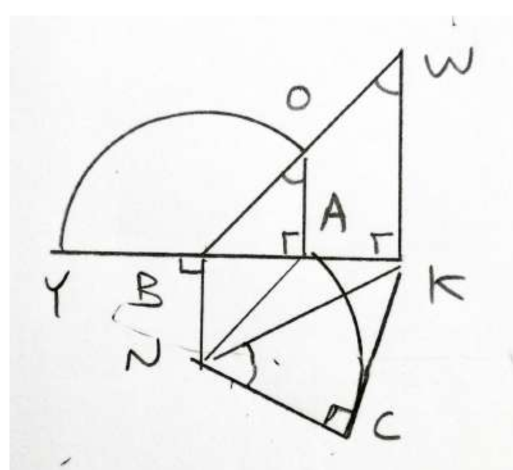
Keeping only:
NA = NC, and
ΔCNK ≅ ΔAOB ≅ ΔKWB:

As A ⇒ B, WK ⇒ YN



We can therefore assume that whenever A lies on BK, given right triangle ΔKBN, if NA = NC, and ΔCNK ≅ ΔAOB ≅ ΔKWB as shown, then:

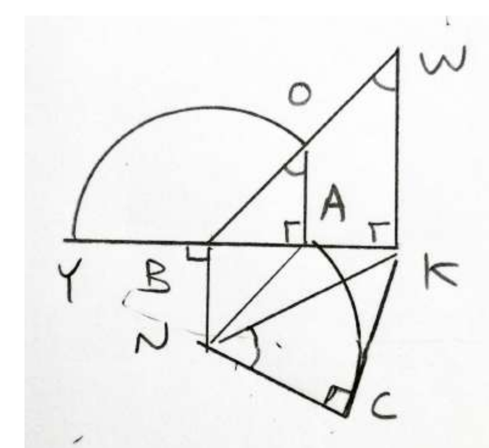
WK = YN



$(CK/CN)^2 = (AB/AO)^2$
$= (KB/KW)^2$
$= (CK^2 + AB^2)/(CN^2 + AO^2)$

Since $KB^2 = CK^2 + AB^2$
$WK^2 = CN^2 + AO^2$
$= AN^2 + AO^2$
$= BA^2 + BN^2 + BO^2 - BA^2$
$= YN^2$

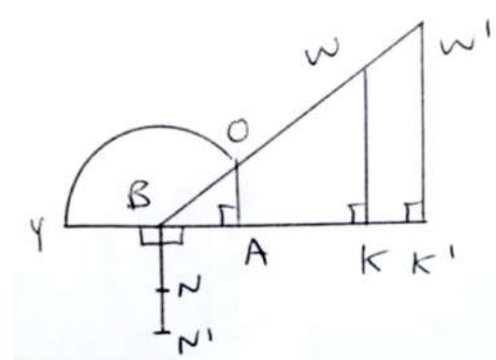
WK = YN



OB/OA = NK/NA
= N’K’/N’A

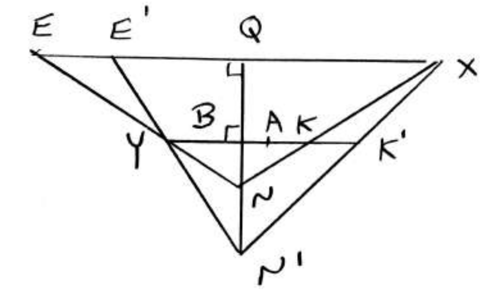
KW = YN
K’W’ = YN’

KB/YN = K’B/YN’



QX/EN = KB/YN
= K’B/YN’ = QX/E’N’

EN = E’N’

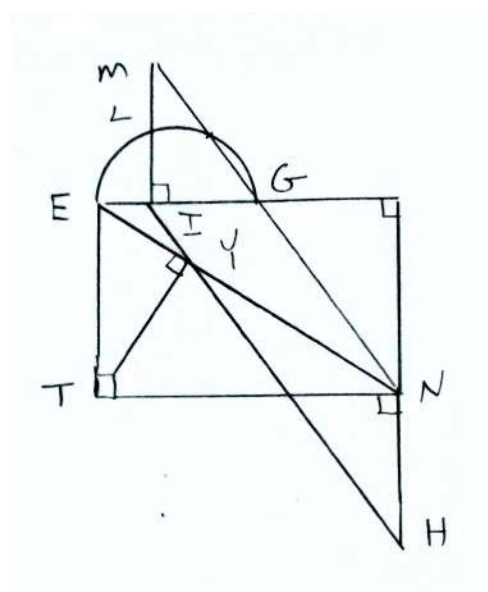


Only one N’K’X exists for NKX since only one E’N’ exists equal to EN. When EN is the smallest segment through Y included in the right angle EQN, E’ lies at E, and N’ lies at N.

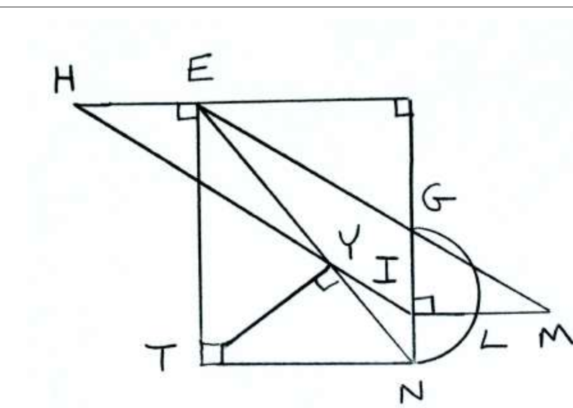
NE || GL
TY || EL
HI || NM
HI = NM
NM > NL

NL is the hypotenuse of right triangle NEL

NL > NE
HI > NE



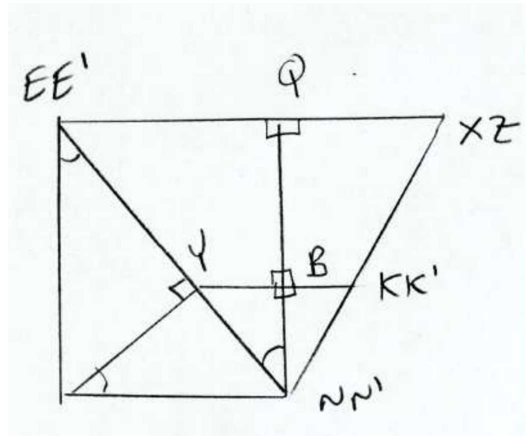
NE || GL
TY || NL
HI || EM
HI = EM
EM > EL



EL is the hypotenuse of right triangle ENL

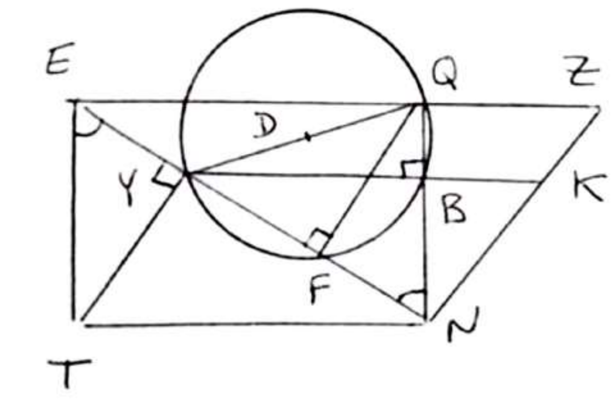
EL > EN
HI > EN

X = Z when EN is the shortest segment through Y included in right angle EQN

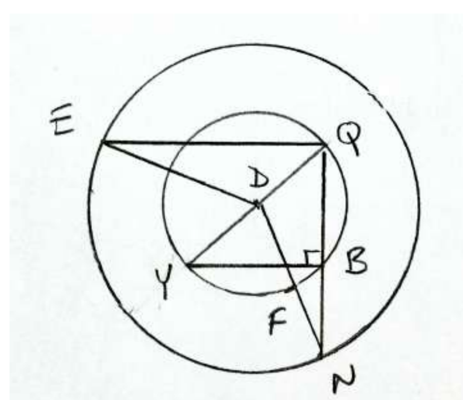


In order to find Z given ΔYBN and NK, we must find E using:

ΔYBN
≅ ΔNYT
≅ ΔNTE

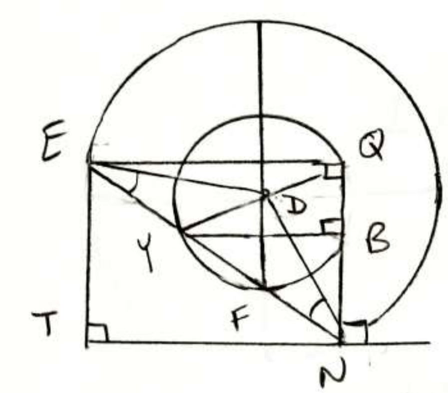


In order to find Z given ΔYBQ, we must find EN so that:
right triangle
ΔTYE = ΔQFN
by drawing a circle concentric with ⊙Y(F)BQ around its center D containing arc ~EN so that YF lies on chord EN.



Not only does:
DY = DF, but also:
ED = ND and therefore
ΔEDY = ΔNDF
so EY = NF

Since ΔQFN is a right triangle, so is ΔTYE. Once we have found EN, we must also find NK in order to find Z.

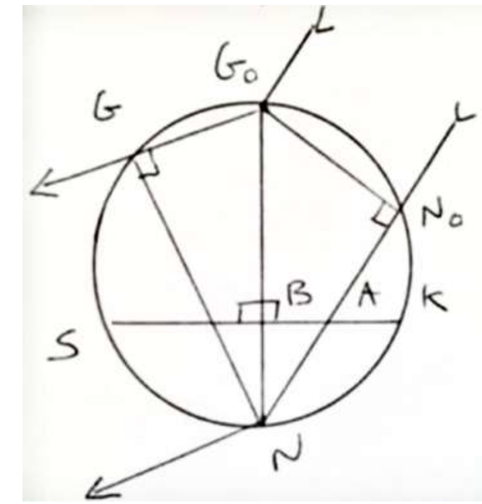


3). refraction along a line

$\Delta N_oNK \cong \Delta KNA$
because:
~NS = ~NK

Wavefront G_oN_o refracts into wavefront GN along G_oN, because it travels G_oG in the same time it travels N_oN.

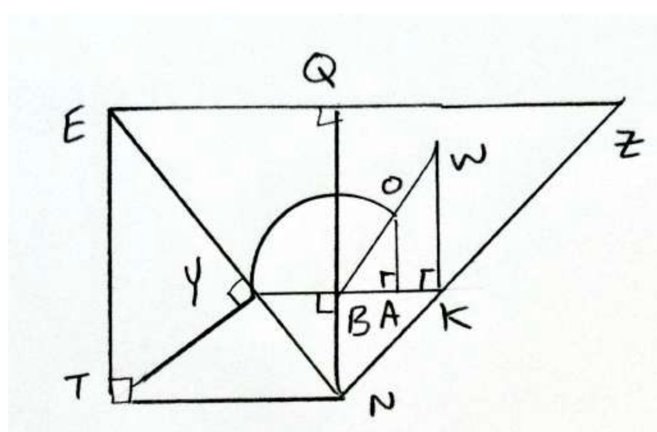
R = NN_o/GG_o
= NN_o/NK = NK/NA



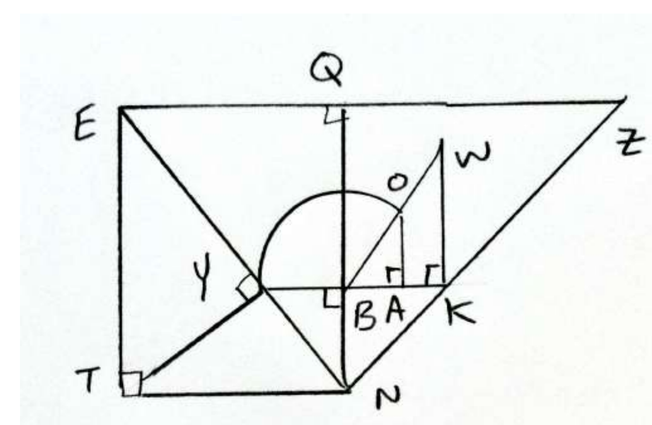
If ***R*** = OB/OA,

and KW = YN:

R = NK/NA



and Z is the clear image of object A refracted at N along BN

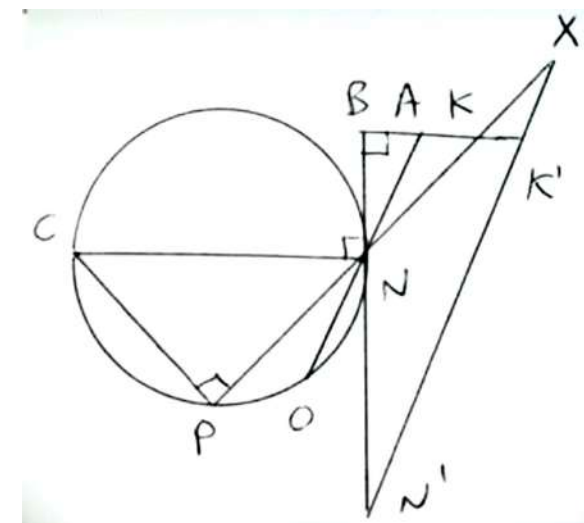


given ΔBAO:
use ΔBKW or ΔQBY to find ΔBNY
use ΔBNY to find ΔBKW or ΔQBY

4). refraction along a circle

ΔKNA ≅ ΔOCP

R = NK/NA
= N’K’/N’A
= CO/CP

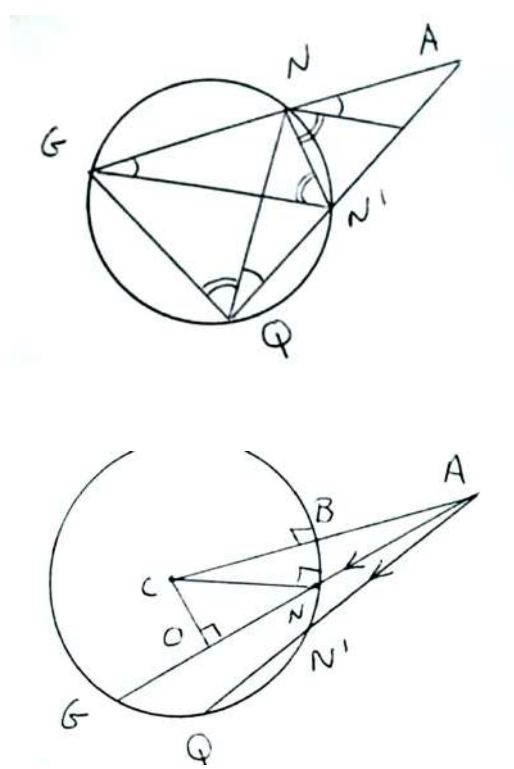


ΔANN’ ≅ ΔAQG

AG/AN’ = QG/NN’

(AG + AN’)/2AN’
= (QG + NN’)/2NN’

Real object A

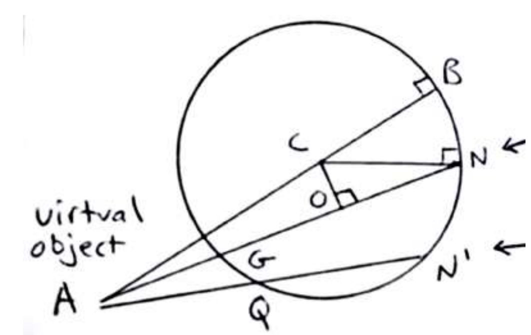


ΔANN’ ≅ ΔAQG

AG/AN’ = QG/NN’

(AG + AN’)/2AN’
= (QG + NN’)/2NN’

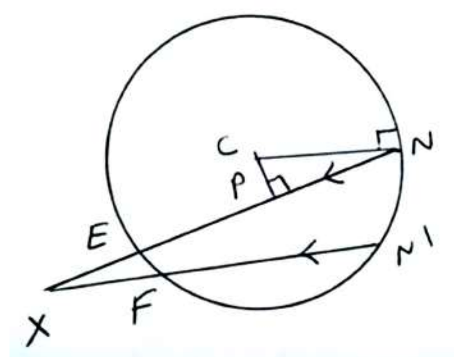
Virtual object A
can not be projected
on a screen due to
refraction at BN.



ΔXNN’ ≅ ΔXFE

XE/XN’ = EF/NN’

(XE + XN’)/2XN’
= (EF + NN’)/2NN’

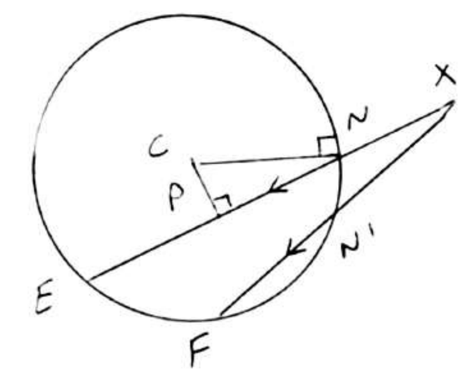


Real image at (X = Z) can be projected on a screen.

ΔXNN’ ≅ ΔXFE

XE/XN’ = EF/NN’

(XE + XN’)/2XN’
= (EF + NN’)/2NN’



Virtual image at (X = Z) can not be projected on a screen.

(AG + AN’)/2AN’ = (QG + NN’)/2NN’
(XE + XN’)/2XN’ = (EF + NN’)/2NN’

(QG + NN’)/(EF + NN’)
= [(AG + AN’)/2AN’][2XN’/(XE + XN’)]

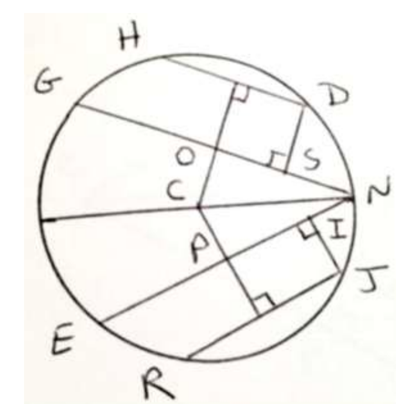
As N’ ⇒ N, X ⇒ Z, and:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ (QG + NN’)/(EF + NN’)

⇒ (AO/AN)(ZN/ZP)

Also, when HD = QN’
and RJ = FN’

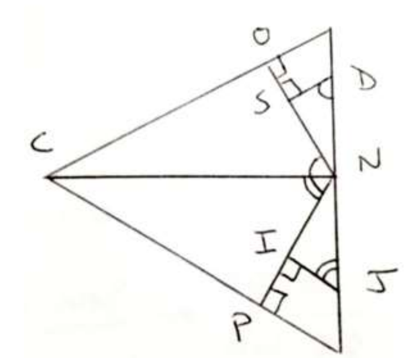
(~QG + ~NN’)/(~EF + ~NN’)
= 2(~ND)/2(~NJ) = ~ND/~NJ



As N’ ⇒ N, X ⇒ Z, and:

~DJ ⇒ line segment DJ, so:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ ND/NJ



$DS/JI = CO/CP$
$JI/JN = NP/NC$
$DN/DS = NC/NO$
$ND/NJ = (NP/NO)(CO/CP)$

As $N' \Rightarrow N$, $X \Rightarrow Z$, and:

$(\sim QG + \sim NN')/(\sim EF + \sim NN')$
$\Rightarrow (NP/NO)(CO/CP)$

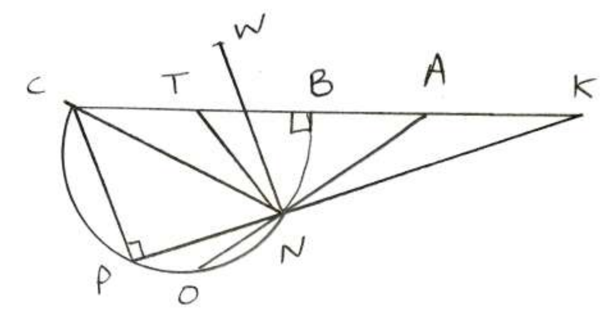
and therefore:
$(AO/AN)(ZN/ZP) \Rightarrow (NP/NO)(CO/CP)$

Thus **R** = CO/CP, and Z, (along both NP and CW), is the clear image of A refracted along ~BN, when:

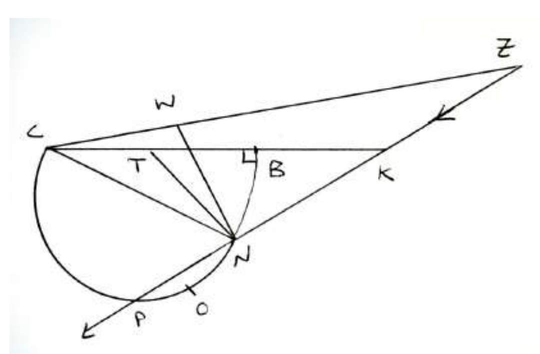
$NT \| CO$, so:
$AO/AN = CO/NT$ and:

$NW \| CP$, so:
$ZN/ZP = NW/CP$
and:

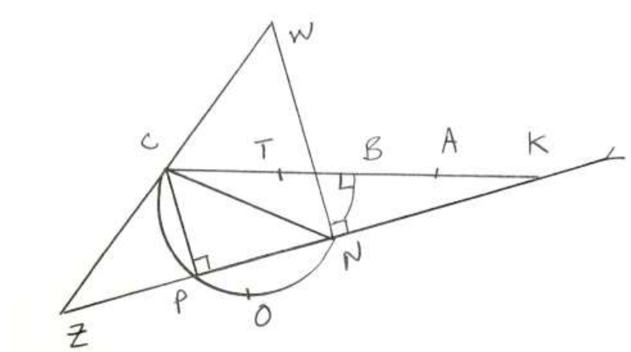
$NW/NT = NP/NO$
$(\Delta WNT \cong \Delta PNO)$



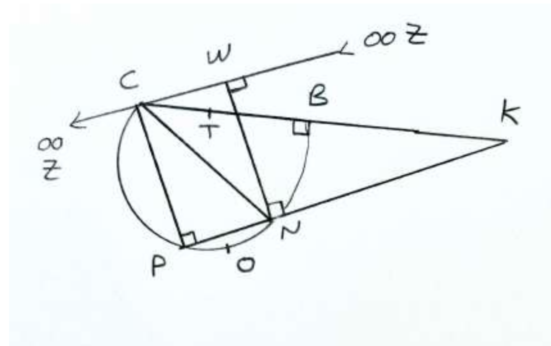
The off-axis rays from any on-axis object A, (real or virtual), can not form a virtual on-axis image at Z because NW must be less than CP for Z to be virtual; but NW must also be greater than NT.



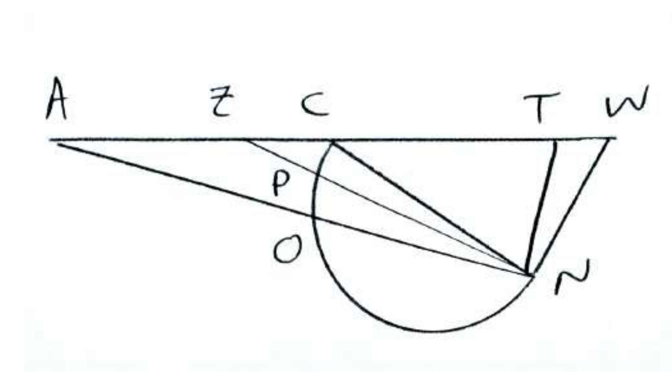
The off-axis rays from any real on-axis object A can not form a real on-axis image at Z because NW must be greater than (or equal to) CP for Z to be real; but NW must also be greater than NT.



The off-axis rays from any real on-axis object A can not form a real on-axis image at Z because NW must be greater than (or equal to, as shown here) CP for Z to be real; but NW must also be greater than NT.



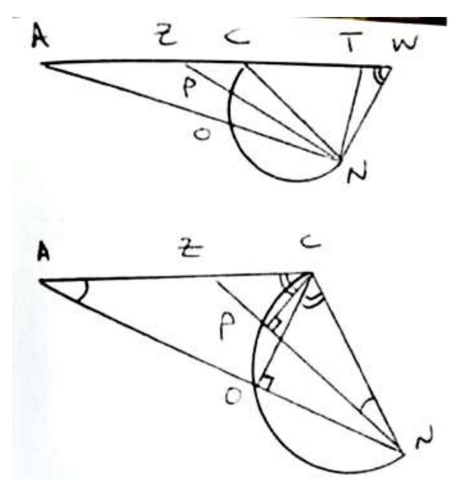
The off-axis rays from a virtual on-axis object A ***can*** form a real on-axis image at Z, if NW is greater than CP, and WT lies along the axis.



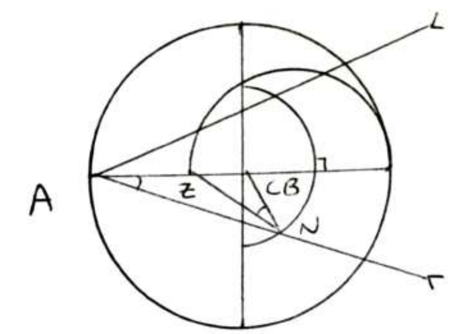
Since:
∠NWT = ∠NPO = ∠NCO
and NW||CP

WT lies along the axis when:

ΔNCO ≅ ΔZCP



When off-axis rays from a virtual on-axis object A form a real on-axis image Z, this occurs at all points N because:



ΔACN ≅ ΔNCZ for all N

5). refraction through a circle’s center

Keeping:

R = (CO/CP) =
(NO/NP)(AO/AN)(ZN/ZP)

constant as:
N ⇒ B:

(BC/BC)(AC/AB)(ZB/ZC) ⇒ ***R***

Refraction through a circle’s center occurs when N lies at B, so that an object’s ray from A to N lies along ABC, and an image ray lies along BCZ. The locations of the object A and image Z along the optic axis BC are described by the equation:

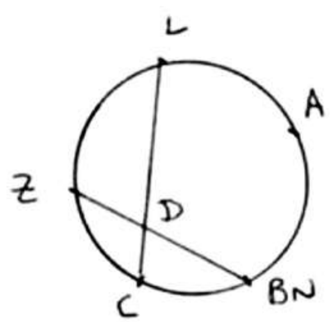
R = CO/CP = (AC/AB)(ZB/ZC)

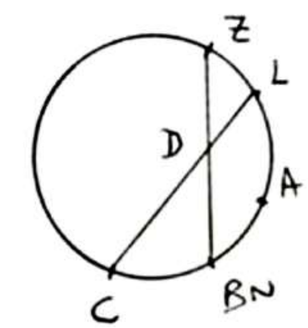
If we draw A and Z along the optic axis BC ***as if*** it were a circle, and draw CDL so that AL || ZB:

ΔACB ≅ ΔZCD, and:

(AC/AB)(ZB/ZC) =
(ZC/ZD)(ZB/ZC) =
(ZB/ZD)
so as the reference circle’s radius ⇒ ∞

(ZB/ZD) ⇒ ***R***



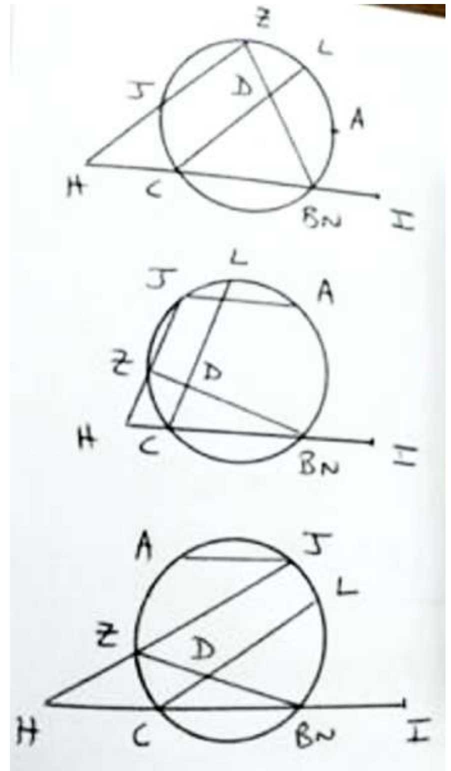


AL ll ZB
AZ = BL
~AZ = ~BL

HZ ll CL
ZC = LJ
~ZC = ~LJ

~AZ + ~ZC = ~AZC
~BL + ~LJ = ~BLJ

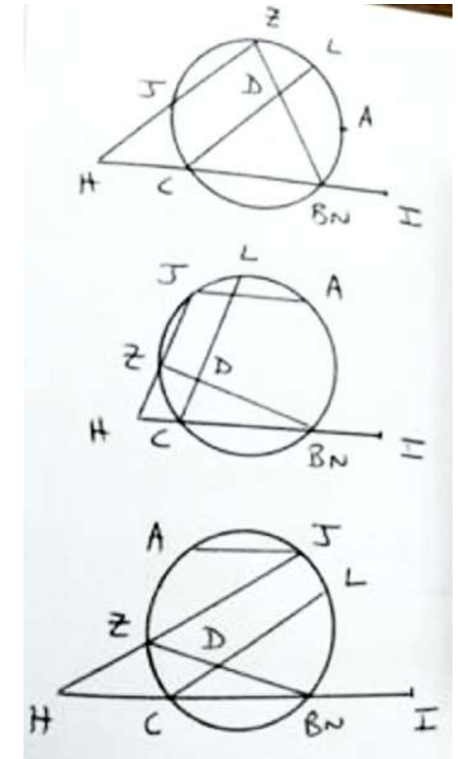
~AZC = ~BLJ
AJ ll CB



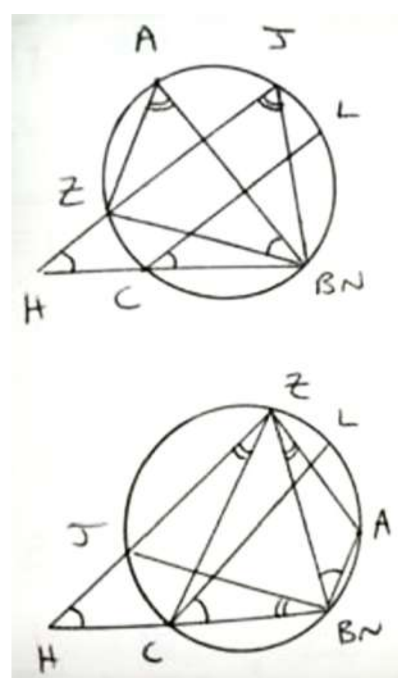
HZ ll CL
ZB/ZD = HB/HC
ΔHBZ ≅ ΔHJC
when ΔHJC = ΔIAB:
HC = IB, and:
IB/IA = HZ/HB

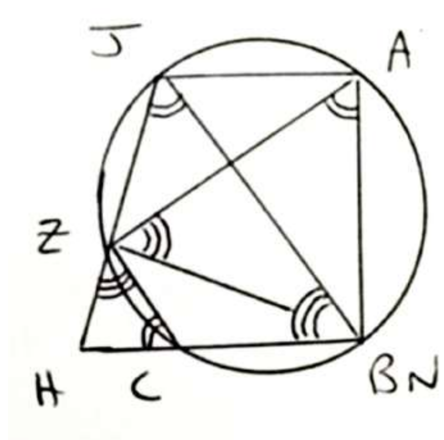
This results in
Newton's Equation
as the reference circle's
radius ⇒ ∞:

(AI)(ZH) = (BI)(BH)



ΔHCZ ≅ ΔHJB ≅ ΔBAZ
(HC/HZ) = (BA/BZ)
[1/(HZ)(BA)] = [1/(HC)(BZ)]





as the reference circle's radius ⇒ ∞:

[1/(HZ)(BA)] = [1/(HC)(BZ)] ⇒ ***R***/(HB)(BZ)

and the resulting possible sums occur:

HZ= HB + BZ
HB = HZ + BZ
BZ = HZ + HB

which, when multiplied by the above three factors, form the **conjugate foci equations**.

The conjugate foci equations allow for the effect of axial refraction at a circle to be expressed as the term:

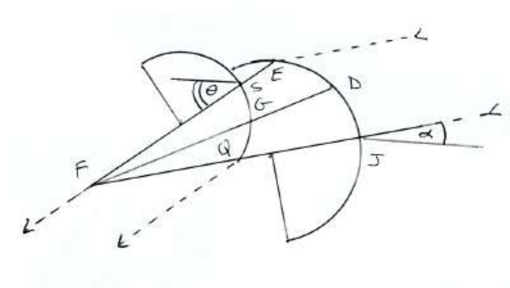
(1/HC) = (***R***/HB)

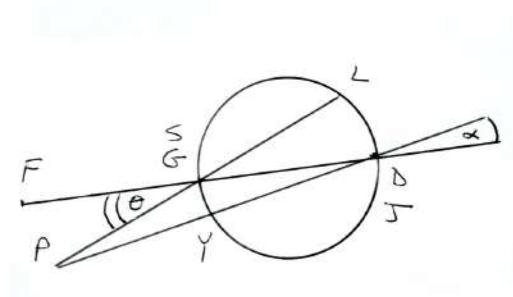
which is then additive with object vergence, defined as (1/BA); or image vergence, defined as (***R***/BZ).

6). afocal angular magnification/ minification

Afocal Angular Magnification

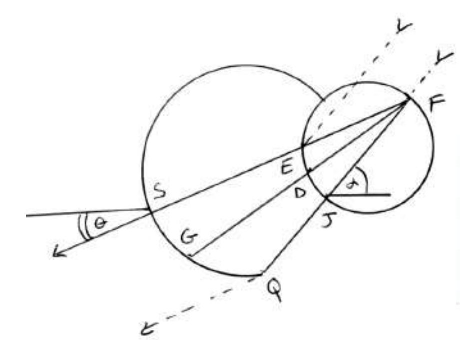
When distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis DGF as shown;
as ∠JFD = ∠SFG,
and both approach zero:

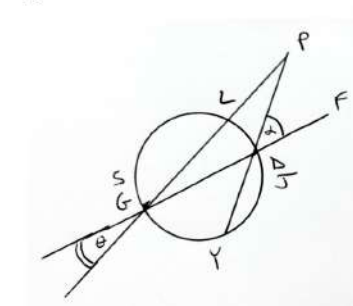




Afocal Angular Minification

Or when distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis FDG, as shown;
as ∠JFD = ∠SFG,
and both approach zero:





$\theta/\alpha \Rightarrow (\sim LD/GD)/(\sim YG/GD)$ as $P \Rightarrow F$

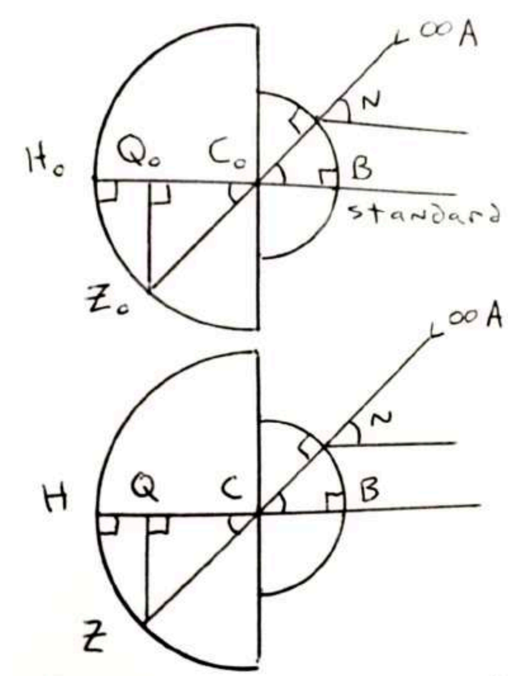
$\theta/\alpha \Rightarrow (FD/FG)$ as $P \Rightarrow F$

so that **afocal *__axial__* angular magnification/minification** equals:

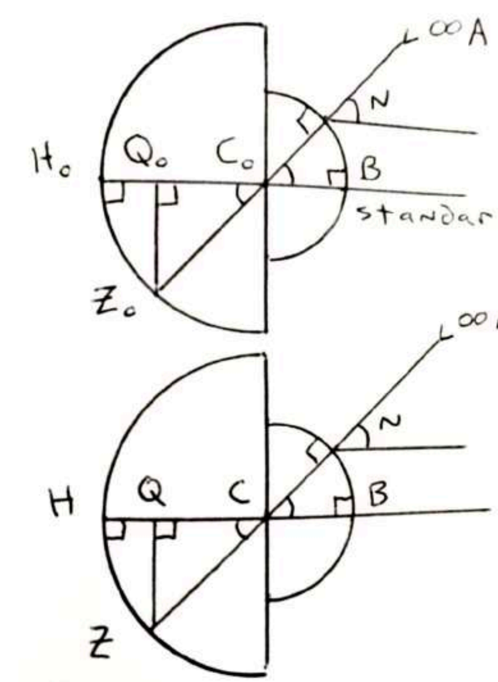
FD/FG

7). retinal image size magnification

The top diagram illustrates a standard single-surfaced eye with a distant object A, and resulting retinal image size H_oZ_o.



The bottom diagram illustrates any single-surfaced eye with a distant object A, and resulting retinal image size HZ.



As N ⇒ B, the retinal image size

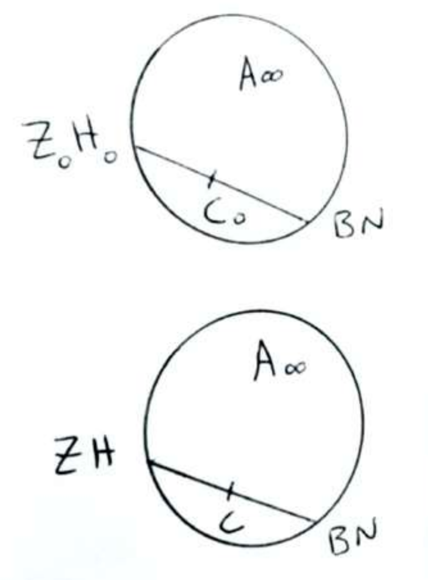
magnification, ZH/Z_oH_o, (relative to an arbitrary standard which factors out with subsequent comparisons), then approaches its <u>*axial*</u> value:

$ZQ/Z_oQ_o = ZC/Z_oC_o = HC/H_oC_o$

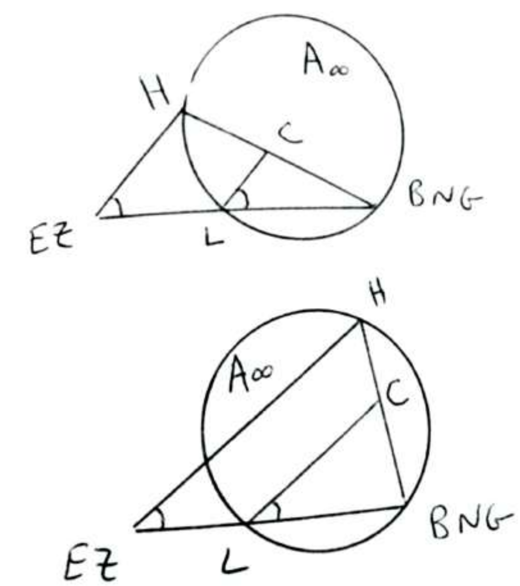
$= (BH/\boldsymbol{R})/(BH_o/\boldsymbol{R}) = BH/BH_o$

8). axial magnification of distance correction

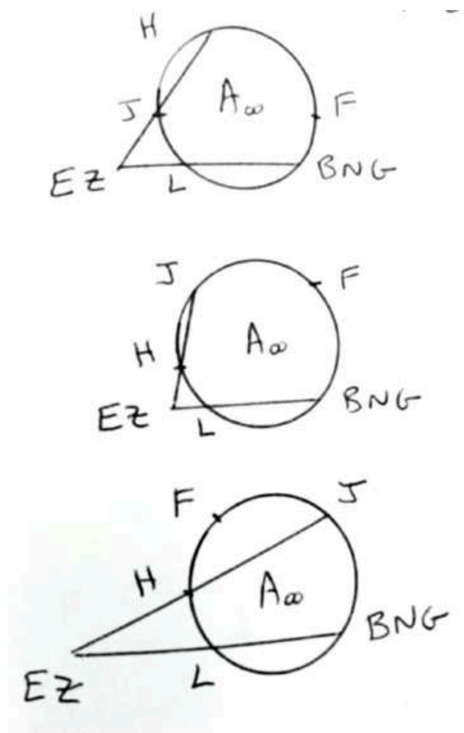
Once again representing the optic axis BCZ as a circle of infinite radius, the distant object A is focused by the curve of radius BC towards the axial object Z, (which lies at the retina H when there is no distance refractive error).



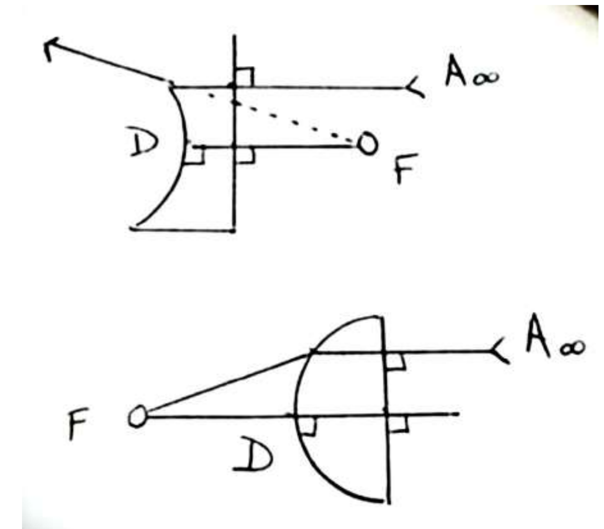
additional refraction at G (at B) will create distance refractive error and a combined single refractive surface of radius BL.



A distance correction must focus the distant object A towards the focal point F of the refractive error G, so that JF || BE, in order to move Z back to H.

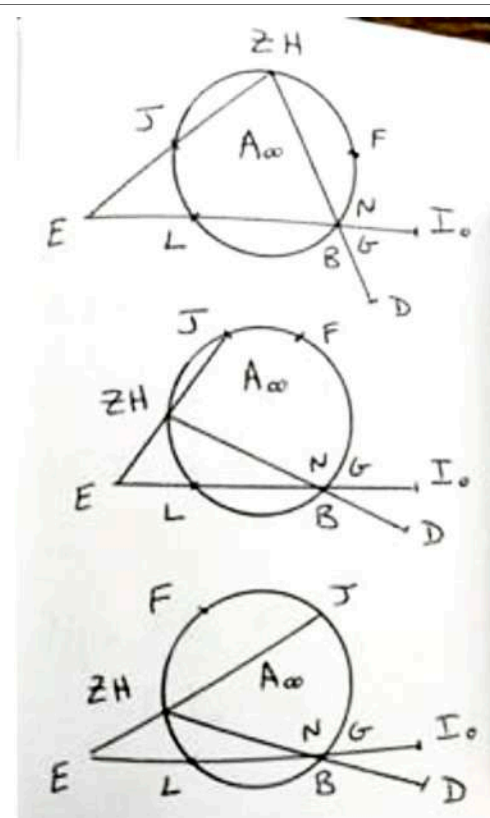


The distance correction at D:



Since the distance correction at D moves Z to H, rays leaving G after this correction must be afocal, resulting in afocal axial angular magnification equaling:

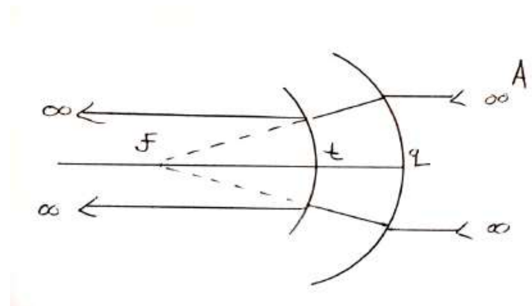
FD/FG (= FD/FB)



The (total) axial magnification of distance correction equals:

$M = (BH/BH_o)(FD/FB)$

When the front surface of a spectacle lens that corrects distance refractive error is not flat, it is convex; and adds an additional “shape” factor, (fq/ft), to the afocal axial magnification of distance correction. (Point “t” lies at D, and FD/FB remains the “power” factor of the afocal axial magnification of distance correction).



$\Delta EBH \cong \Delta EJL$

If E is at H_o, the distance refractive error is completely due to an axial length that is not standard.

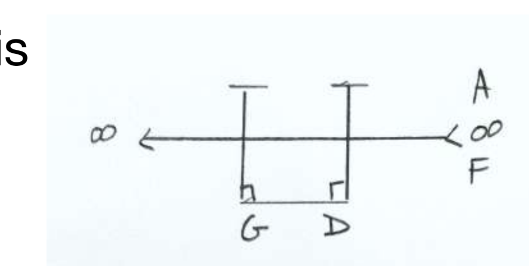
If $\Delta EJL \cong \Delta I_oFB$, then:

$M = (FB/FI_o)(FD/FB) = FD/FI_o$

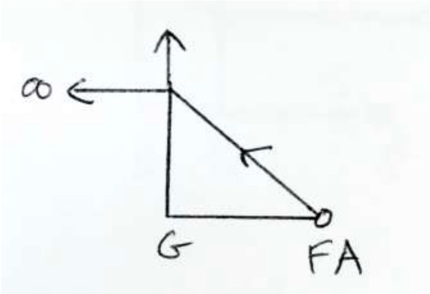
There is then no (total) axial magnification of distance correction if the correction D lies at I_o, the front focal point of the standard eye.

9). axial magnification of near correction

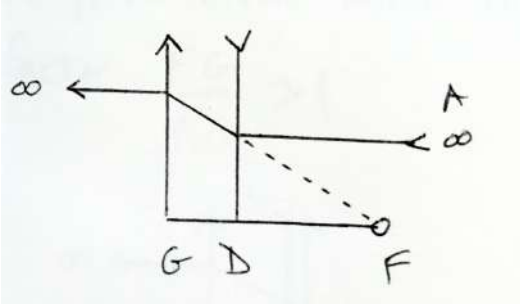
There is no afocal axial angular magnification FD/FB when object A is at distance with an emetropic eye.
(The refractive error at G, (at B), is zero; and the focal point F of that refractive error lies at infinity).



There is also no afocal axial angular magnification when object A is at the front focal point of an uncorrected myopic eye. (The system is not afocal, and involves only one refracting element).



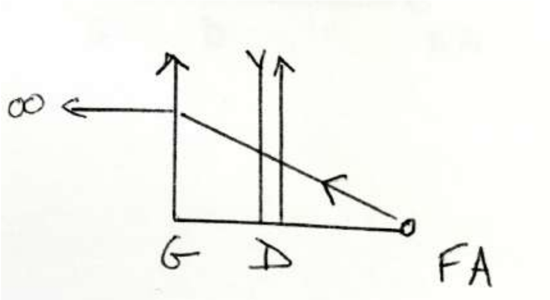
As discussed, a distance myopic correction at D creates afocal axial angular minification:



FD/FG < 1

and this is relative to either the myopic eye with object A at its front focal point F, or the emetropic eye with object A at distance.

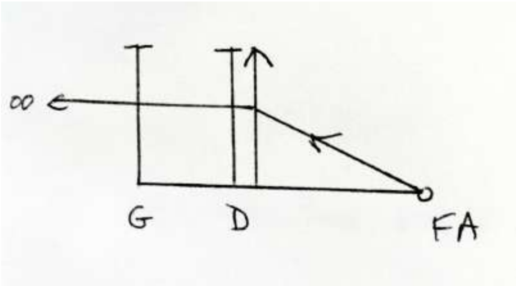
Removing the myopic distance correction at D with a converging lens at D removes this afocal axial angular magnification with the factor:



FG/FD > 1

and this magnification of near correction is relative to the distance corrected myope.

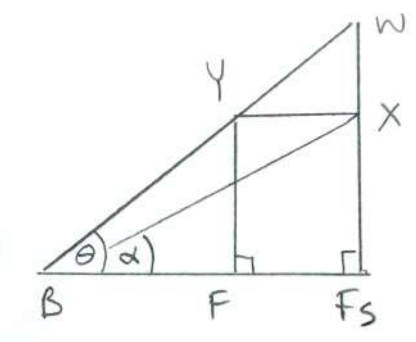
If additional converging power is added to the converging lens so that the near focal point is in focus for an *emetropic* eye, which we then consider to be the reference eye, the magnification of near correction is still that which is removed with the factor:



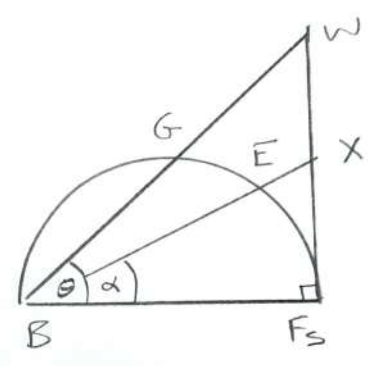
FG/FD > 1

10). object angular subtense magnification

When an object at a standard distance Fs is moved to F:



The object angular subtense magnification equals:



θ/α = (~GFs/BFs)/(~EFs/BFs)

as XFs ⇒ 0

the object angular subtense magnification approaches its axial value:

θ/α ⇒ WFs/XFs = WFs/YF = BFs/BF

which equals the *axial* object angular subtense magnification.

The ratio describing axial object angular subtense magnification:

BFs/BF

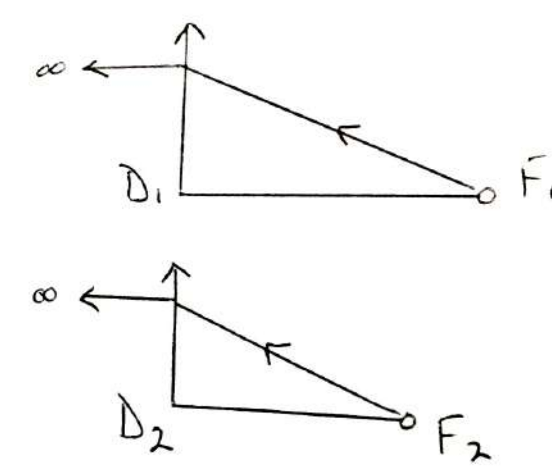
when multiplied by the ratio describing near magnification due to a single converging lens producing parallel light for an emmetropic eye:

FB/FD

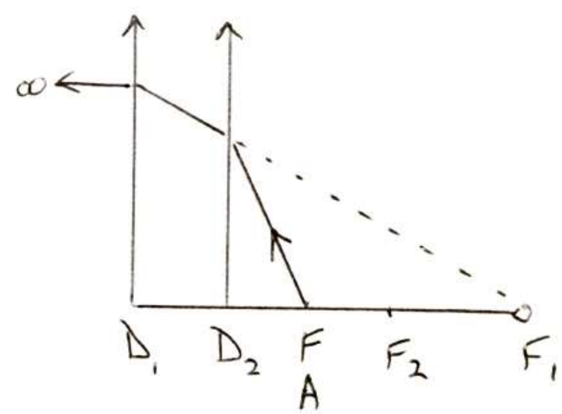
produces a ratio which factors out the object's actual distance to the eye, confirming that when a converging lens is used with its front focal point at the object, so parallel light leaves the converging lens from the object, the image size is the same regardless of the object-to-eye distance.

11). stand magnifier magnification

When the converging lens at D is split into two converging lenses:



with the same combined focus F:



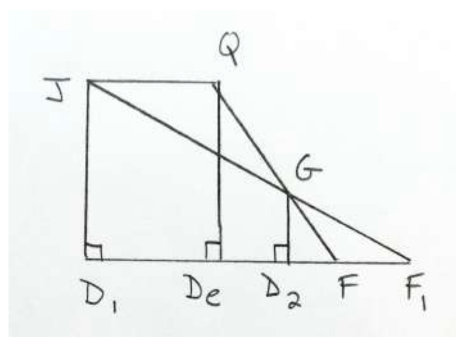
the ratio describing axial near magnification due to a single converging lens producing parallel light for an emmetropic eye:

FB/FD

must be expressed ***as if*** all convergence occurred at a single unknown axial point De:

FB/FDe

De can be located using triangles.



$D_2G/D_2F = DeQ/DeF$

$D_2G/D_2F_1 = D_1J/D_1F_1$

$D_2F(DeQ/DeF) = D_2F_1(D_1J/D_1F_1)$

$DeQ/DeF = (D_2F_1/D_2F)(D_1J/D_1F_1)$

$1/DeF = (D_2F_1/D_2F)(1/D_1F_1)$

$FB/FDe = (D_2F_1/D_2F)(FB/D_1F_1)$

Multiplying the axial object subtense magnification by the axial magnification of near correction (relative to the same eye without refractive error) produces:

$BFs/FDe = (D_2F_1/D_2F)(BFs/D_1F_1)$

The converging lens D_2 creates a virtual image F_1 of an object at F. When considering a stand magnifier with lens D_2, constant stand height D_2F, and reading spectacle add or ocular accommodation D_1, the stand magnifier's (constant) enlargement of the object at F equals:

$$E = D_2F_1/D_2F$$

The stand magnifier's axial magnification is its (constant) enlargement factor E, multiplied by what would be produced by D_1 alone, if the object A were at F_1.

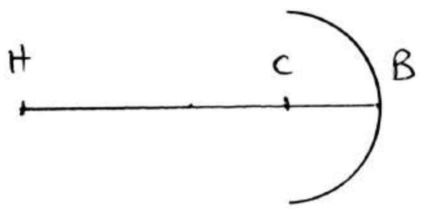
12). crossed cylinders

It is useful to know the meridian of maximum axial refraction when combining the effects of two cylindrical refracting surfaces at an oblique axis. To do this, we need to first describe how their axial radii of curvature change with various meridional cross sections. Meridional cross sections of cylindrical surfaces are ellipses until they become parallel lines along the cylinder axis.

However, assuming a cylinder is parabolic rather than spherical, and that meridional cross sections are parabolic until they rotate into a single line parallel to the cylinder axis, allows for a much simpler approximation of the axial radii of curvature of these meridional cross sections. When these axial radii of curvature are expressed in forms that are additive in terms of refraction, we can then find the maximum sum of those expressions in terms of the meridional axis.

With any axial radius of curvature CB, and index of refraction ***R***, the axial image of a distant object lies at H when:

R = HB/HC

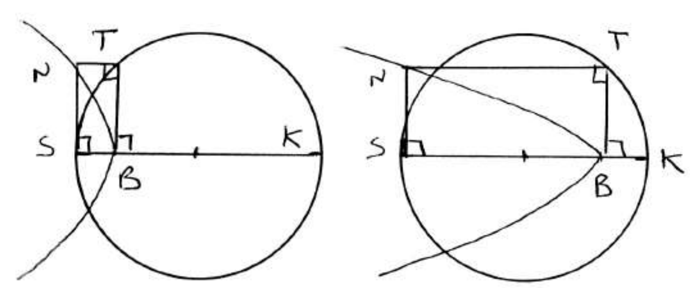


The axial refractive effects of compound refractive surfaces at B are additive only as their refractive "powers," which equal:

R/HB = 1/HC = [(HB - HC)/HC]/CB = (***R***- 1)/CB

All parabolas have the same shape, in the same way that all circles have the same shape. However, while circles have a single (internal) determining constant, the radius of curvature, parabolas have both a determining constant internal and external to the curve, and can be defined by either.

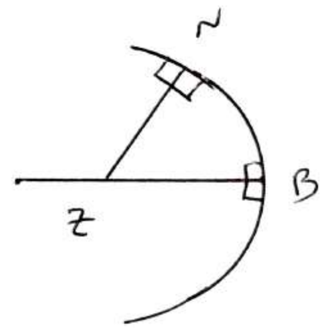
For example, a parabola's external determining constant equals BK when:



$$\frac{SB}{BT} = \frac{BT}{BK}$$

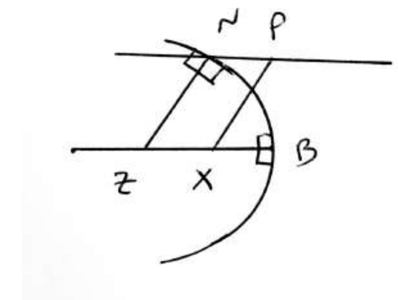
[2(SN) equals the sagitta corresponding to the sagittal depth SB].

We can set up the necessary off-axis conditions to determine a parabola's axial center of curvature in terms of its internal determining constant XB, by involving ZN in the geometric solution for XB.



In order to keep the determining geometrical relationships axial as N ⇒ B, they should also

depend on line NP being parallel to the axis, and XP being parallel to ZN.



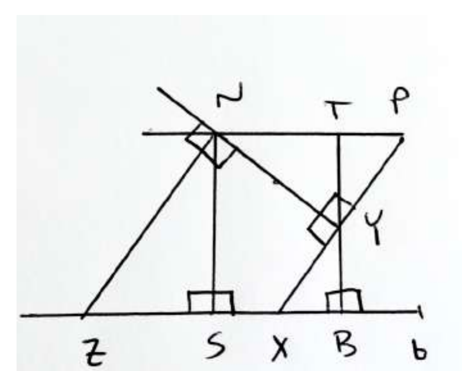
We know X lies between Z and B, since parabolas flatten in their periphery.

Since as N ⇒ B, Z ⇒ C by definition, and

since XP = ZN, P will remain external to the curve, and X can therefore not be its axial center of curvature, but must instead lie somewhere along CB.

In order to maintain ZN perpendicular to the parabola at N as N ⇒ B, the same geometrical

relationships must exist that allow for that when N lies at B.



In other words:

YP = YX and
Bb = BX so
CB = 2(XB)

Since:

$$\frac{TN}{TB} = \frac{TN}{2(TY)} = \frac{YB}{2(XB)} = \frac{YB}{CB} = \frac{TB}{2(CB)}$$

We know the external determining constant BK equals 2(CB), and the internal determining constant XB equals (CB)/2.

Axial refracting power equals (***R***- 1)/CB

Since for a parabola:

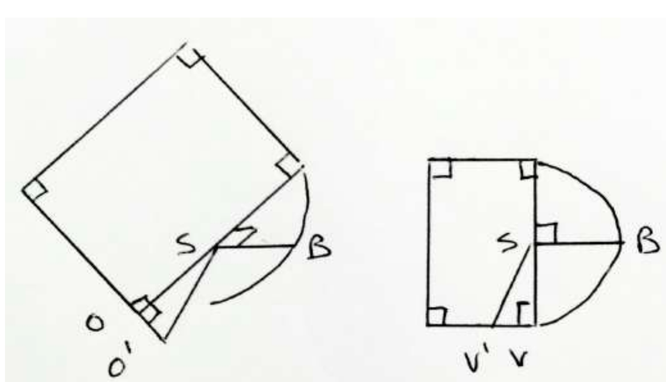
SB/SN = SB/TB = TB/[2(CB)]

If ***R*** = 1.5

The axial refracting power of a parabola equals:

$1/[2(CB)] = SB/SN^2 = 1/BK$

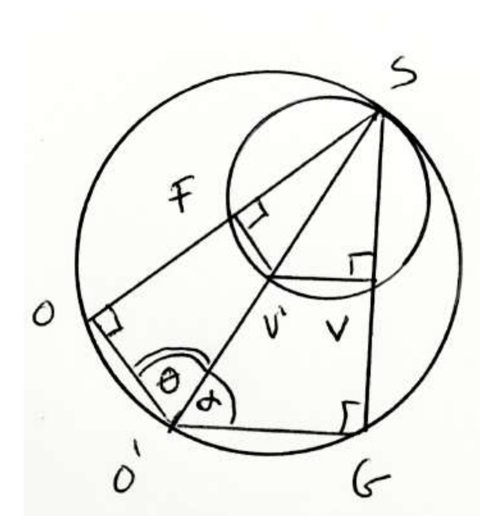
When 2(SO) equals the minimum sagitta of an oblique parabolic cylinder, and when with equal sagittal depth SB, 2(SV) equals the minimum sagitta of a more highly curved parabolic cylinder with a horizontal axis:



Keeping ΔOSV constant, as we rotate circle SOG with variable diameter SV'O' around point S:

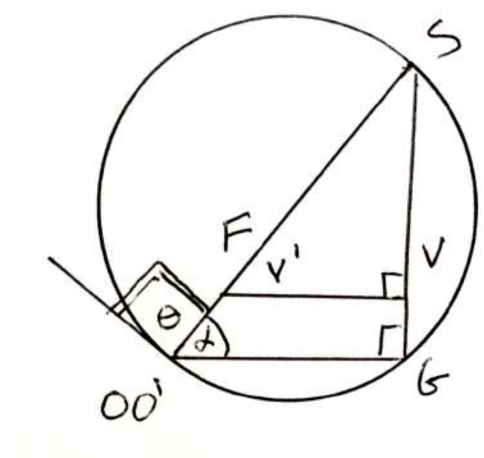
∠OO'G is constant because ∠OSG is constant,

so $\Delta\theta = -\Delta\alpha$



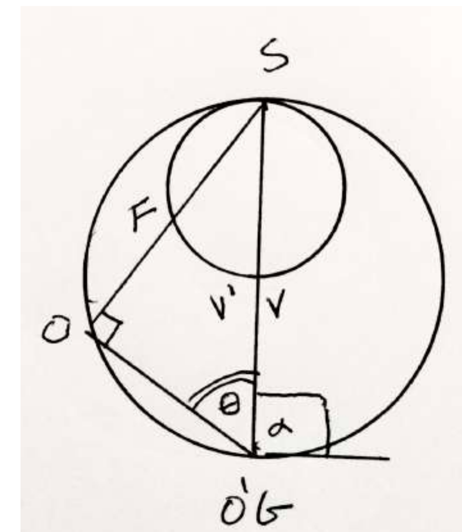
As O' ⇒ O

SV' increases more than SO' decreases



As V' ⇒ V

SO' increases more than SV' decreases



Since the sum (SO' + SV') increases when either:

O' ⇒ O,   or V' ⇒ V

there must be a specific SV'O' within ΔOSV producing a minimum sum (SO' + SV'),
which must be near where small rotations produce only minimal changes in (SO' + SV').

Since as when one term of the sum (SO' + SV') increases, the other always decreases, this process can be taken to its limits to determine the meridian with minimum (SO' + SV') using:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta(SO') = \lim_{\Delta\alpha \Rightarrow 0} \Delta(SV')$$

However, the combined effects of refraction are additive only as refractive powers, which, when ***R*** = 1.5, equal:

$SB/(SO')^2$ and $SB/(SV')^2$

Therefore, the meridian with the maximum combined effects of this refraction can be found using:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta \, [SB/(SO')^2] = \lim_{\Delta\alpha \Rightarrow 0} \Delta \, [SB/(SV')^2]$$

To solve this equation, all variables must be expressed in terms of the variables approaching zero, so:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta\{[SB(SO/SO')^2]/SO^2\} = \lim_{\Delta\alpha \Rightarrow 0} \Delta\{[SB(SV/SV')^2]/SV^2\}$$

$$\lim_{\Delta\theta \Rightarrow 0} \Delta\{[(SB)\sin^2\theta]/SO^2\} = \lim_{\Delta\alpha \Rightarrow 0} \Delta\{[(SB)\sin^2\alpha]/SV^2\}$$

$$(SB/SO^2) \lim_{\Delta\theta \Rightarrow 0} \{\Delta\sin^2\theta\} = (SB/SV^2) \lim_{\Delta\alpha \Rightarrow 0} \{\Delta\sin^2\alpha\}$$

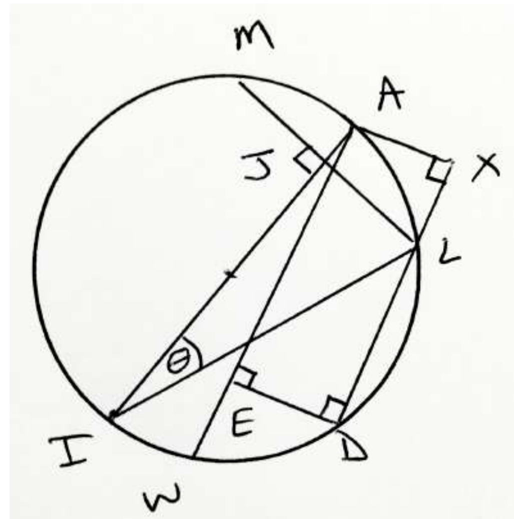
$$\{\text{Limit as } \Delta\theta \Rightarrow 0 \text{ of } [\Delta\sin^2\theta]\}/\{\text{Limit as } \Delta\alpha \Rightarrow 0 \text{ of } [\Delta\sin^2\alpha]\}$$

$$= [SO^2/SV^2]$$

Solve for

Limit $\quad \Delta \sin^2 \theta$
$\Delta\theta \Rightarrow 0$

on the reference circle:



AW ≥ LD || AW

∠ALD = ~AID/AI

≥ ~AI/AI = π

Establish the necessary functions of θ in terms of line segments and chords.

$\theta = \sim\frac{AL}{AI}$; $\sin^2 \theta = \frac{AL^2}{AI}$

$\Delta \theta = \sim\frac{LD}{AI}$; $\sin^2 \Delta \theta = \frac{LD^2}{AI}$

$(\theta + \Delta \theta) = \sim\frac{ALD}{AI}$; $\sin^2 (\theta + \Delta \theta) = \frac{AD^2}{AI}$

$\cos \theta = \frac{IL}{AI}$; $\cos (\theta + \Delta \theta) = \frac{DI}{AI}$

$\sin \theta = \frac{AL}{AI} = \frac{JL}{IL}$; $\sin \theta \cos \theta = \frac{JL}{IL} \frac{IL}{AI}$

$2 (\sin \theta \cos \theta) = \frac{ML}{AI} = \sin 2\theta$

Then consider the following property of the cyclic quadrilateral circle ALDW: AD(LW) = AL(DW) + LD(AW)

$\Delta DIA \cong \Delta EWD = \Delta XLA$; $AD^2 = AL^2 + LD(AW)$

$AW = LD + 2(AL) \frac{LX}{LA}$; $AW = LD + 2(AL) \frac{ID}{IA}$

$AD^2 - AL^2 = LD^2 + 2(LD)(AL) \frac{ID}{IA}$

$AI [\sin^2(\theta + \Delta\theta) - \sin^2\theta] =$

$AI [\sin^2\Delta\theta] + 2(LD)(AL)\cos(\theta + \Delta\theta) =$

$AI [\sin^2\Delta\theta] + 2(LD) [(AI)\sin\theta] \cos(\theta + \Delta\theta)$

Divide both sides by AI:

$\sin^2(\theta + \Delta\theta) - \sin^2\theta = \sin^2\Delta\theta + 2(LD) \sin\theta \cos(\theta + \Delta\theta)$

$\lim_{\Delta\theta \Rightarrow 0} \frac{\Delta(\sin^2 \theta)}{\sim LD} = 2 \sin\theta (\cos \theta) = \sin 2\theta$

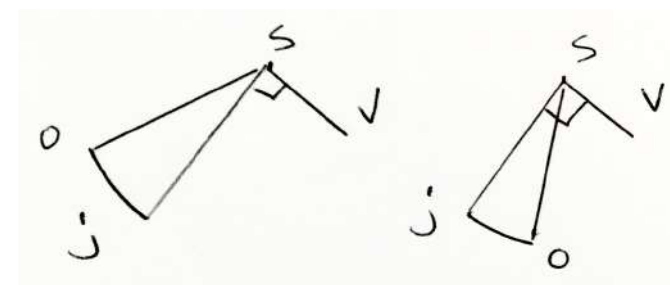
Therefore, the meridian with the maximum combined effects of refraction can be found using:

$$\frac{\sin 2\theta}{\sin 2\alpha} = \frac{SO^2}{SV^2}$$

The first step to solve this problem is to divide SV into SaV so that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

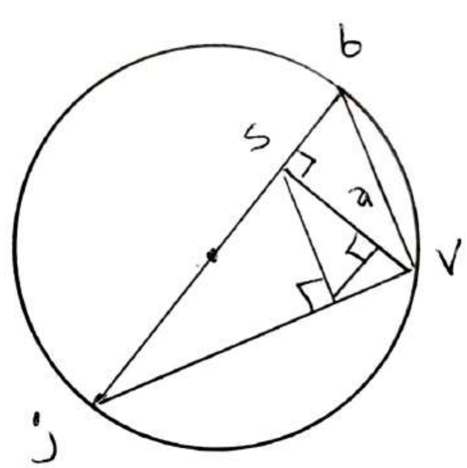
Make SO = Sj ⊥ SV to construct:



$$\frac{Sj}{SV} = \frac{SV}{Sb} \quad ; \quad \frac{Sj^2}{SV^2} = \frac{Sj}{Sb} = \frac{SO^2}{SV^2}$$

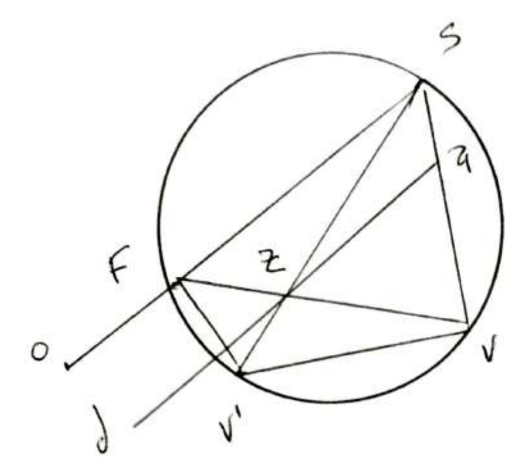
Similar triangles show that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

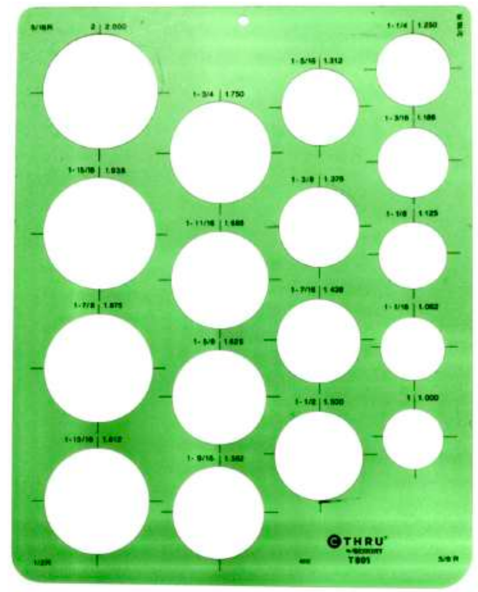


Draw ad || SO

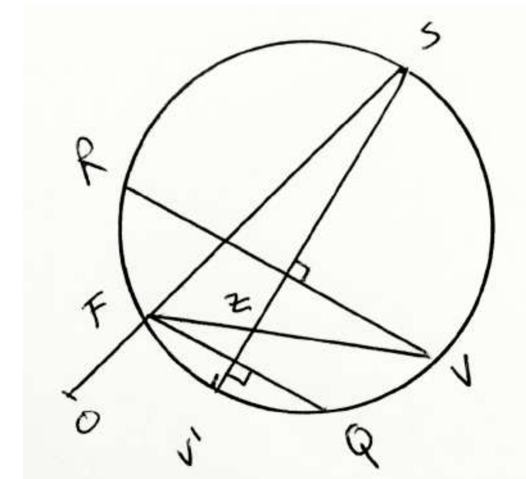
Choose a circle through S and V with a variable diameter SV' so that FZV lies on a common chord.



The easiest way to do this involves a template of various circles, each with the location of their diameters already marked.



SV' is the meridian with the maximum combined effects of refraction because:



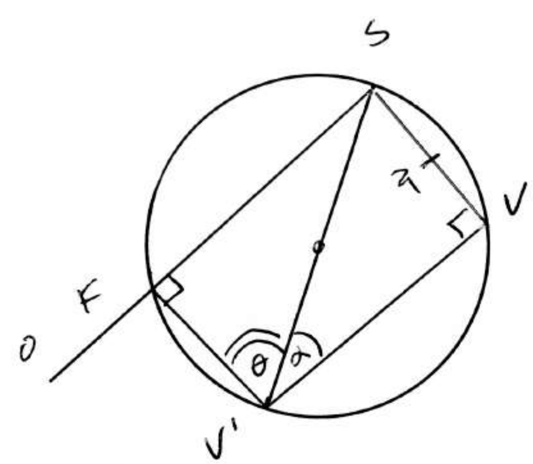
$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{FZ}{ZV} = \frac{FQ/2}{RV/2} = \frac{FQ}{RV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

Double-angle Method:

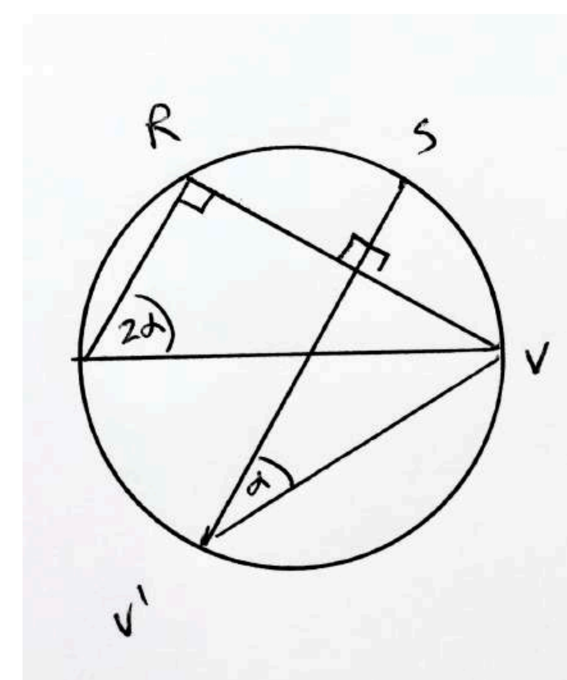
Given constant ΔOSV:
$\angle FSV$ is constant
$\angle FSV + (\theta + \alpha) = \pi$
$(\theta + \alpha)$ Is constant

We have already shown how to find single angles $\theta + \alpha$ so that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

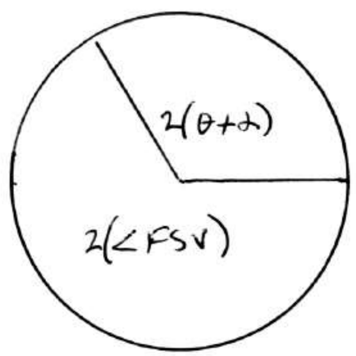


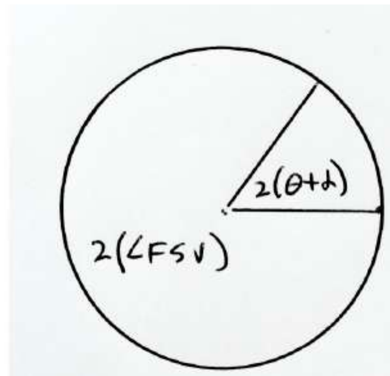
An angle on a circle equals its inscribed arc, divided by the arc's diameter. Since the sum of all angles measured on a circle's circumference add to π, when measured from a circle's center they add to 2π.



Therefore:

$$2(\angle FSV) + 2(\theta + \alpha) = 2\pi$$

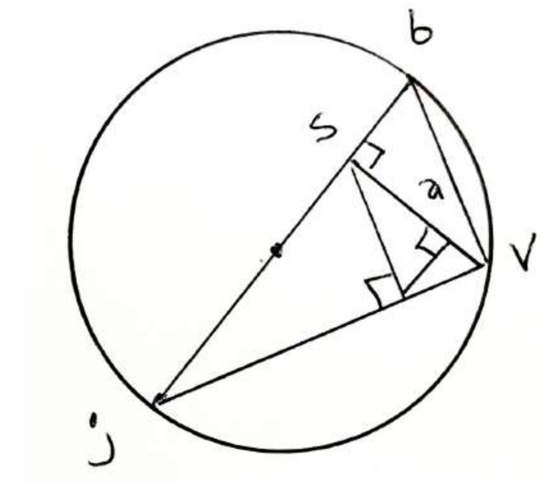




When:

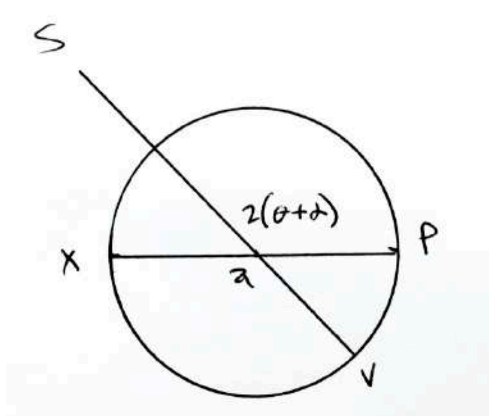
$$\frac{SO^2}{SV^2} = \frac{Sj^2}{SV^2} = \frac{aS}{aV}$$

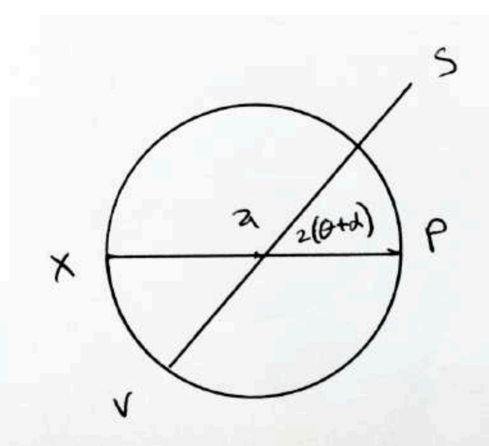
as drawn:



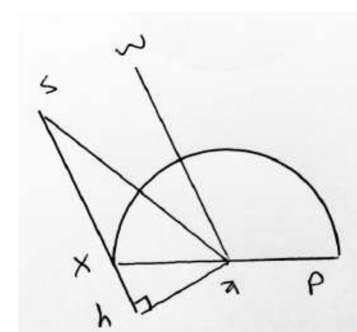
If we draw diameter XaP so:

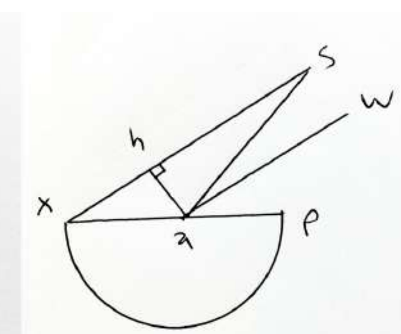
$aX = aV$, and $\angle SaP = 2(\theta + \alpha)$





$$\frac{SO^2}{SV^2} = \frac{aS}{aX} = \frac{ah/aX}{ah/aS} = \frac{\sin 2\theta}{\sin 2\alpha}$$





When aw || sX, we have divided the doubled angle
$2(\theta + \alpha) = \angle SaP$
into $2\theta = \angle WaP$, and $2\alpha = \angle WaS$.