The Geometry of Geometrical Optics

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2022

References:

Isaac Barrows Optical Lectures, 1667
Translated by H.C. Fay
Edited by A.G. Bennett
Publisher:
The Worshipful Company of Spectacle Makers
London, England; 1987
ISBN # 0-951-2217-0-1

Plane and Solid Geometry
G. A. Wentworth; 1899 revised edition

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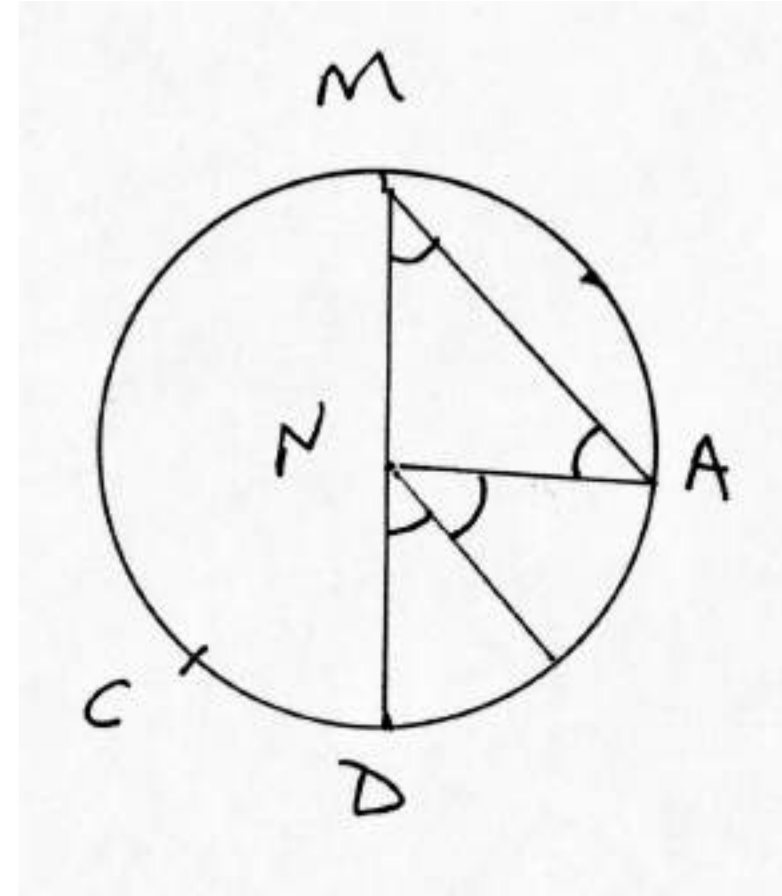
Introductory Geometry

$\angle DNA = 2\angle DMA$

$\angle DNC = 2\angle DMC$

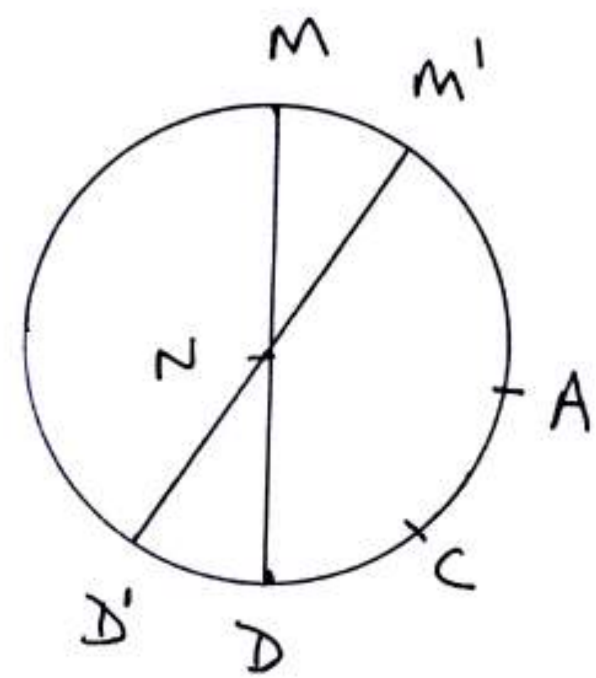


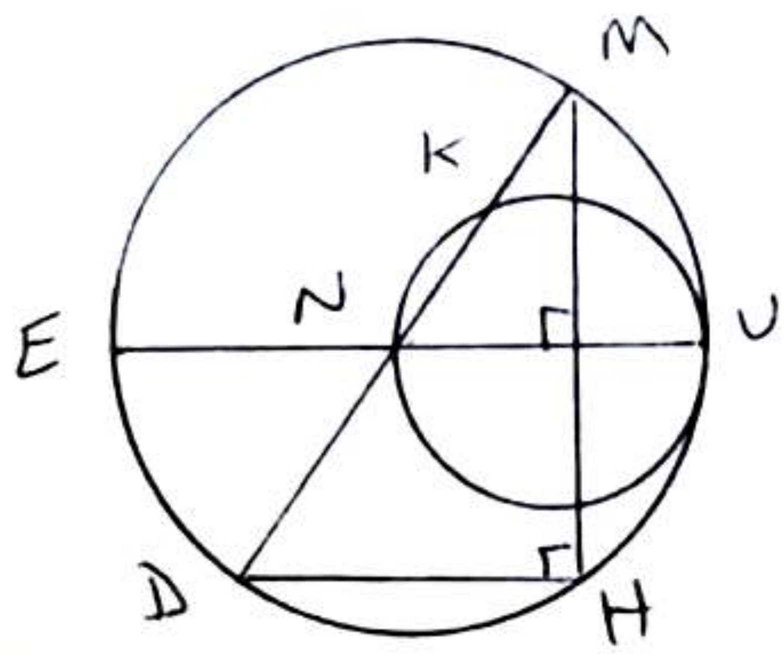
$\angle ANC$

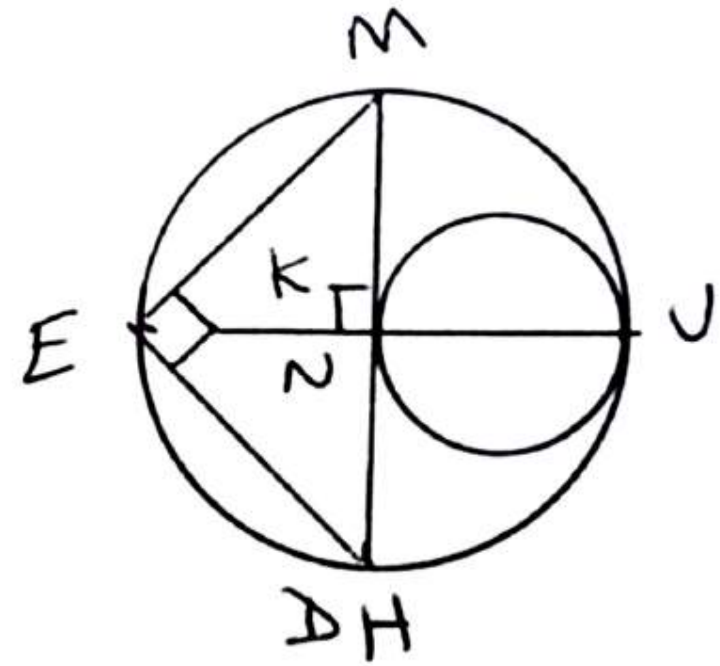
$= \angle DNA \; +/- \; \angle DNC$

$= 2(\angle DMA +/- \angle DMC)$

$= 2\angle AMC = 2\angle AM'C$

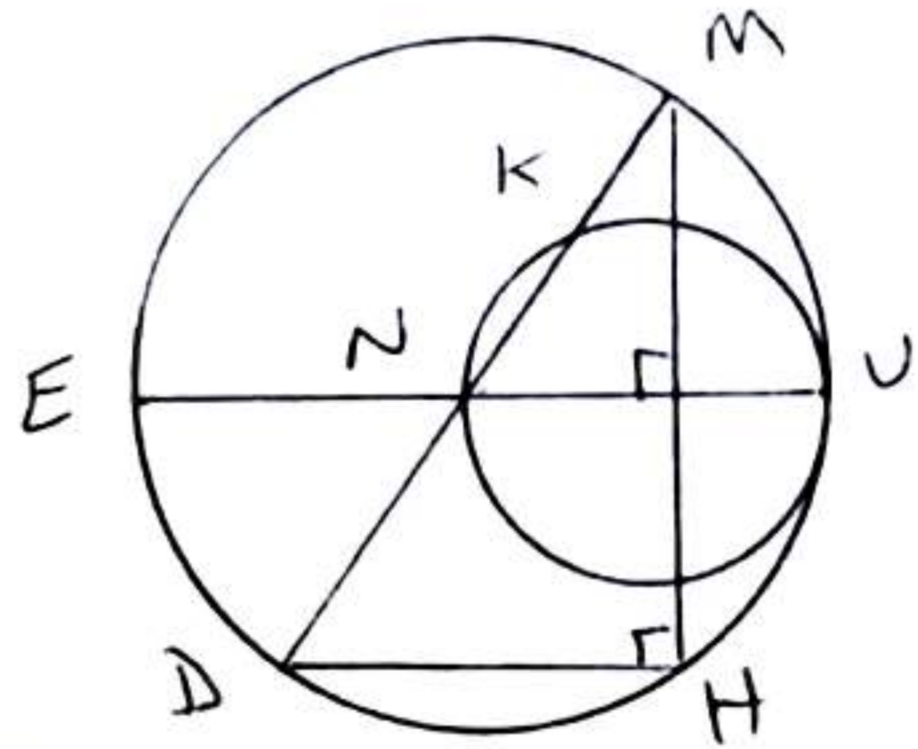


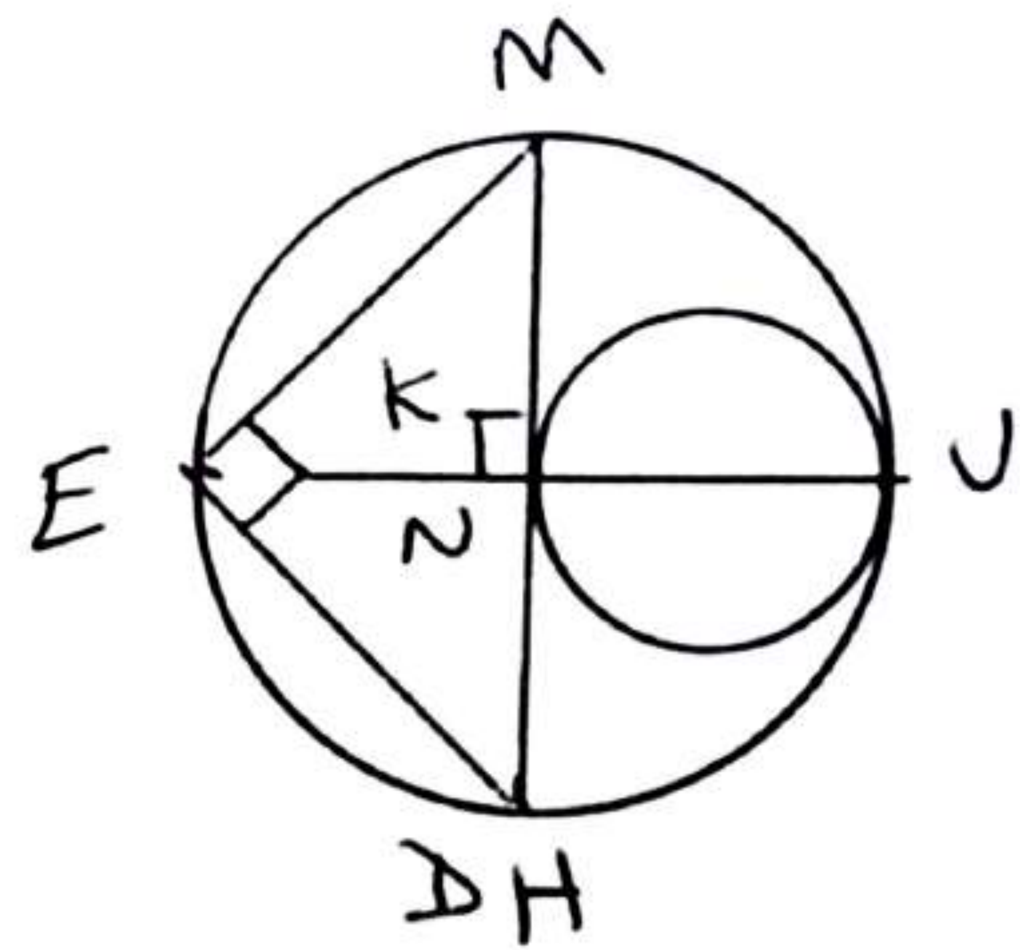




~UK/UN = ~MH/MD = 2~UM/UE = 2~UM/2UN

~UK = ~UM





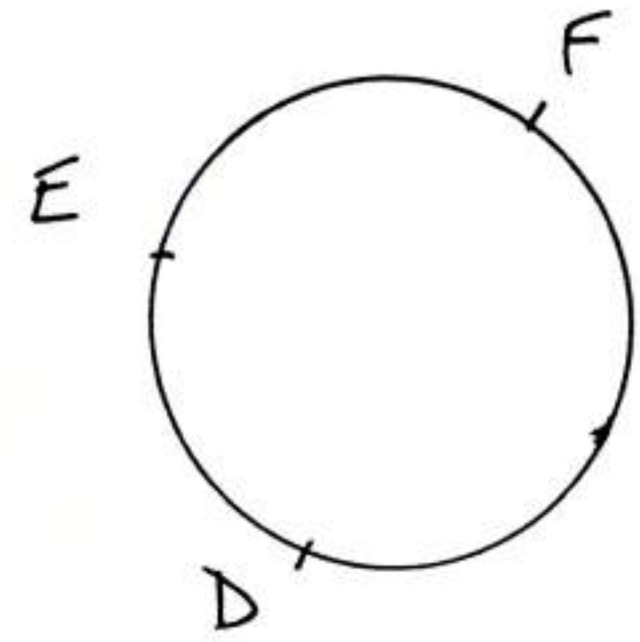
As K ⇒ N, and D ⇒ H:

2~KU/UN = 2∠MNU = ∠MNH ⇒ π

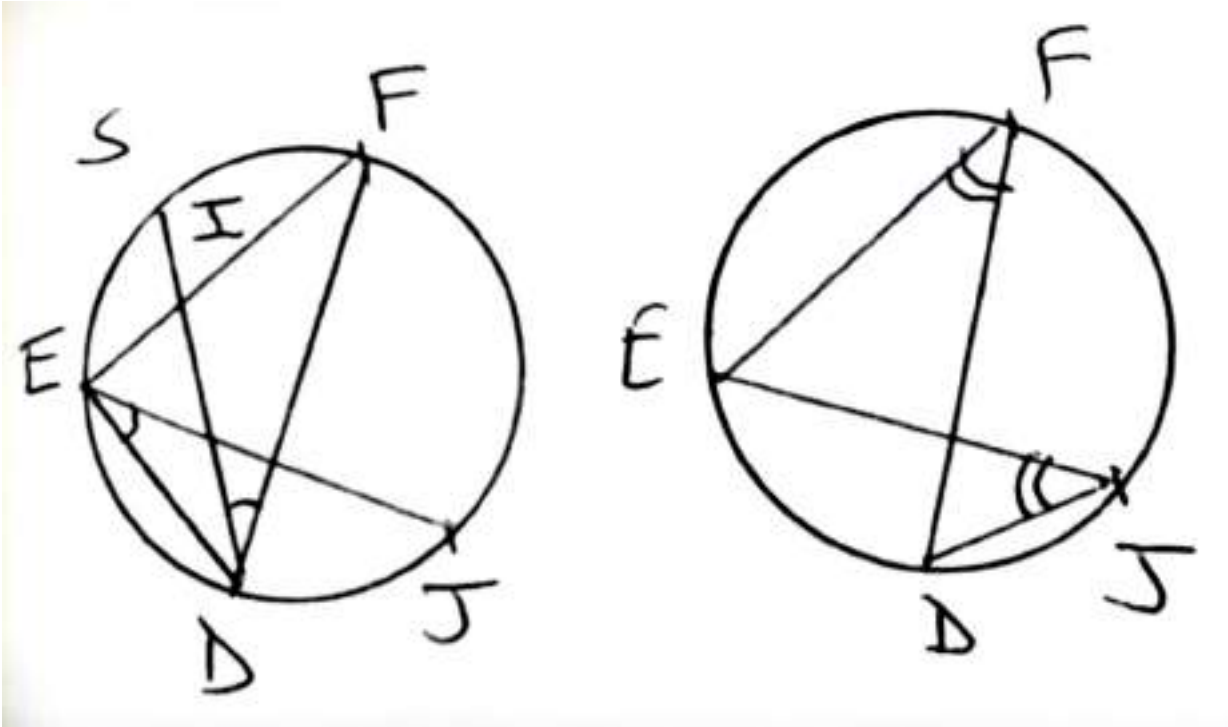
$\angle FDE = \sim EF/DM$

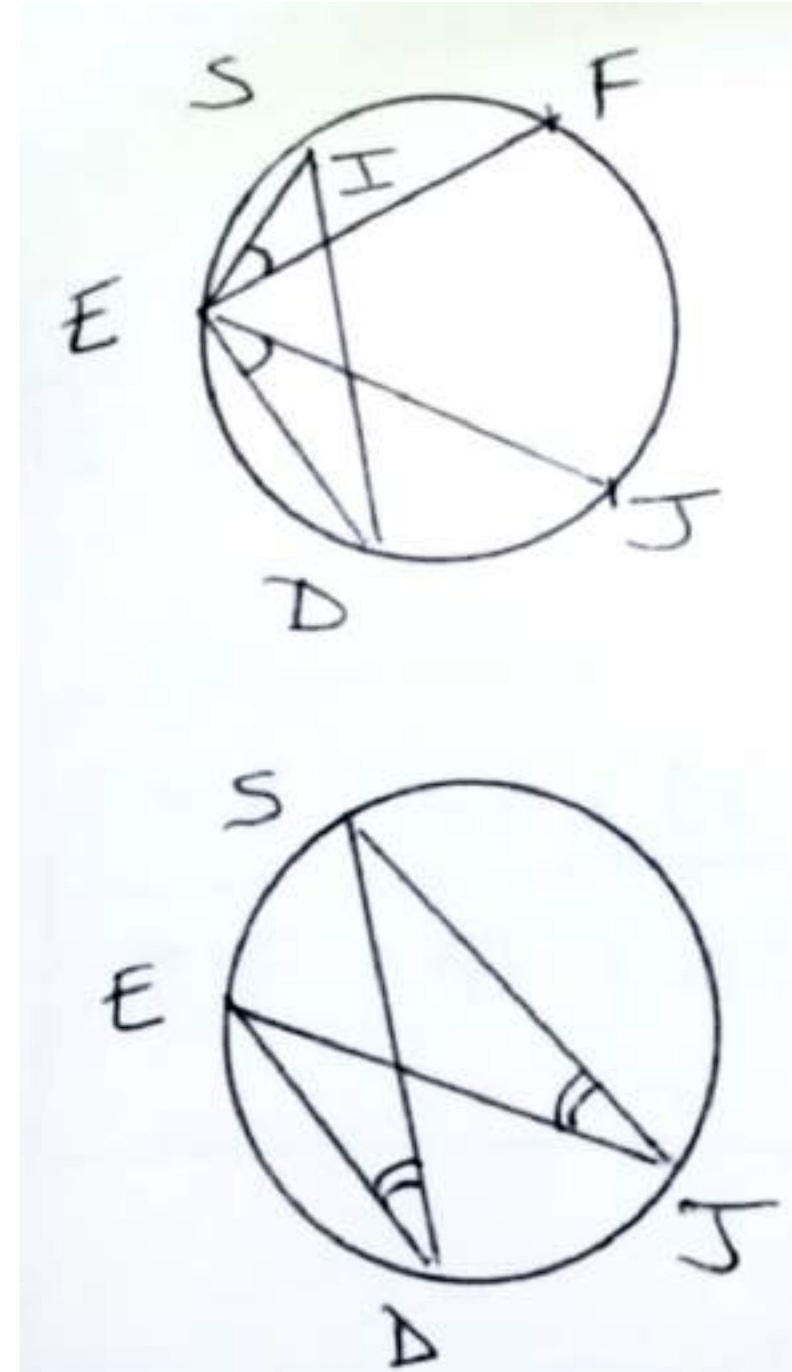
$\angle DEF = \sim DF/DM$

$\angle EFD = \sim DE/DM$



$\angle FDE + \angle DEF + \angle EFD = \pi$





$SD \parallel FJ$

$\Delta EJD \cong \Delta DFI, \ FD/FI = JE/JD$

$\Delta EJS \cong \Delta EDI, \ EI/ED = ES/EJ$

$(FD)(EI) / (FI)(ED)$

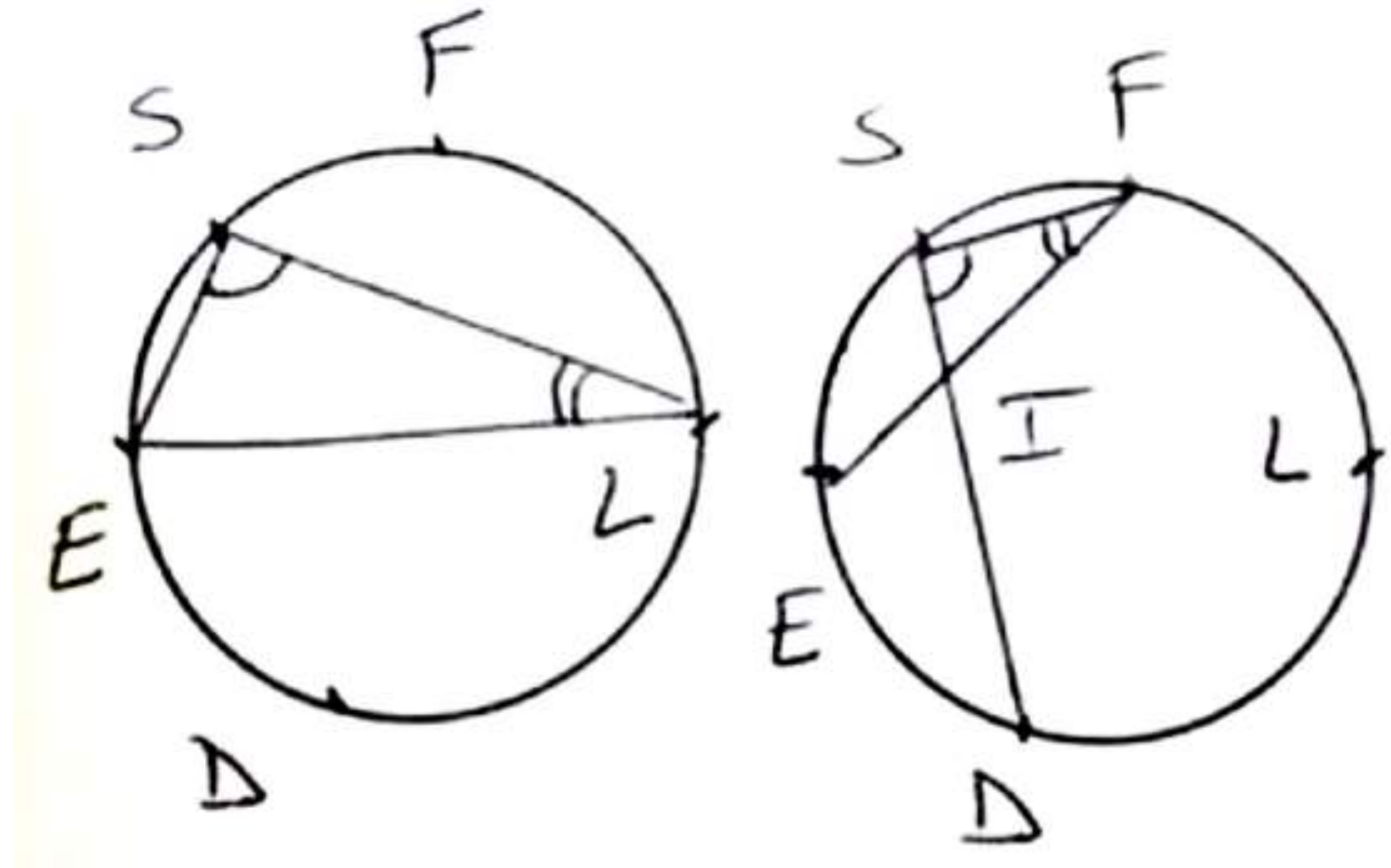
$= (JE)(ES) / (JD)(EJ) = SE/SF$

$IE/IF = (SE)(DE) / (SF)(DF)$

$LD \parallel FE$

$DE/DF = LF/LE$

IE/IF
$= (SE)(LF) / (SF)(LE)$

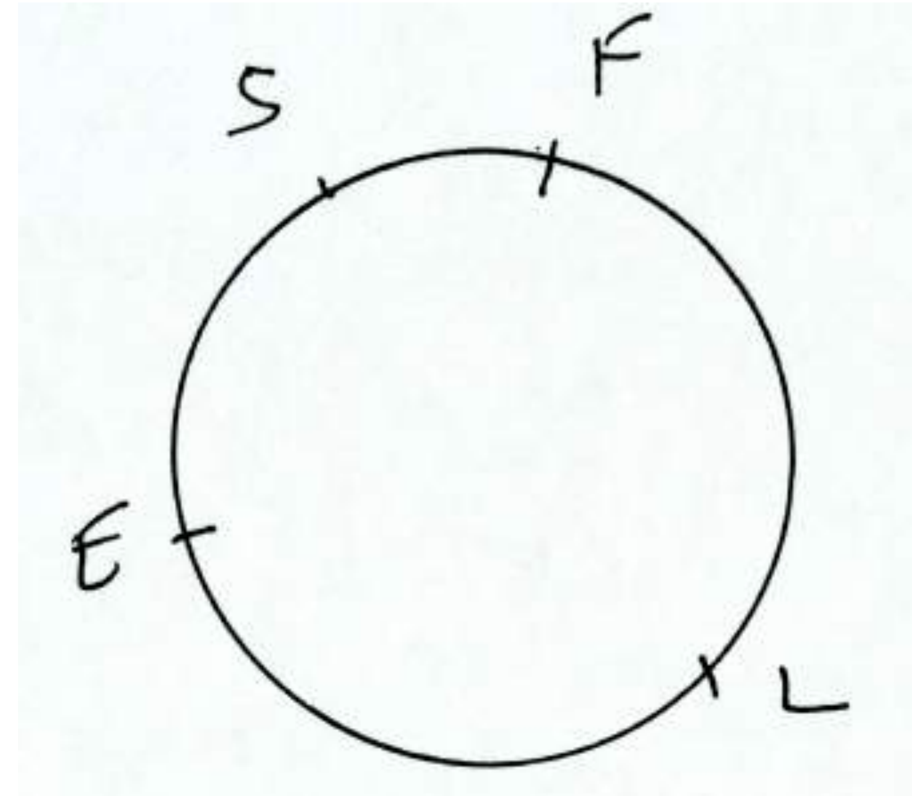


$$FE/FI = \{(SE)(LF) + (SF)(LE)\} / (SF)(LE)$$

$LD \parallel FE$, $\quad \sim EL = \sim FD$, $\quad \Delta LSE \cong \Delta FSI$

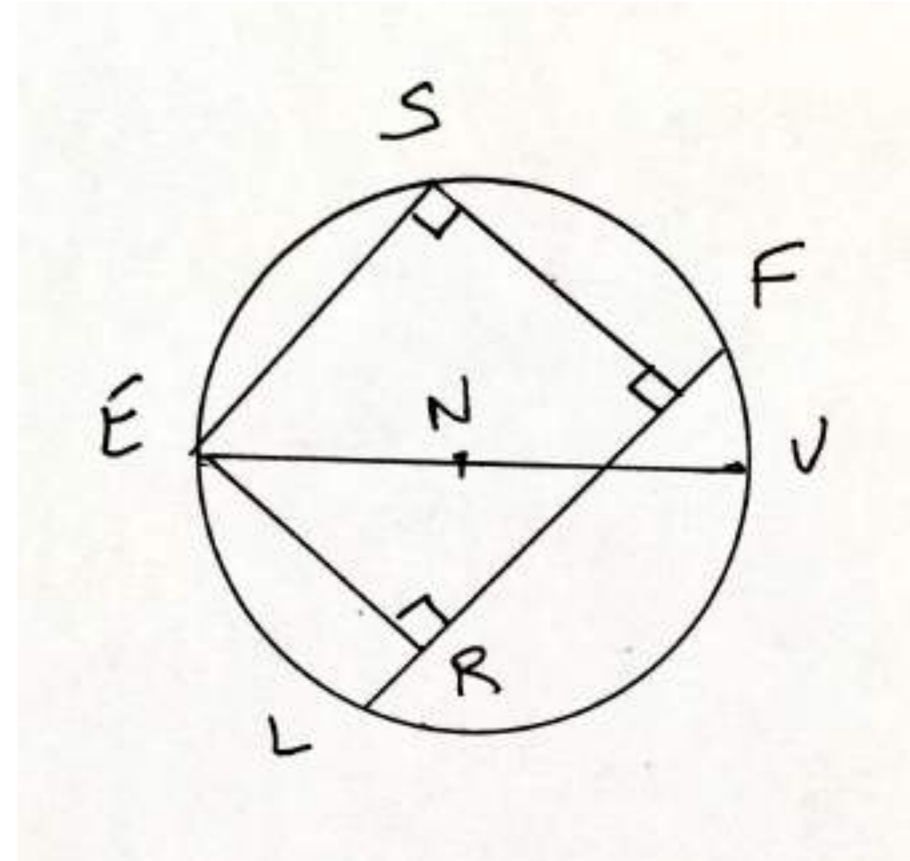
$LS/FS = LE/FI$, $\quad LS = FS(LE) / FI$

Ptolemy's Theorem:
(FE)(LS) = (SE)(LF) + (SF)(LE)



Pythagorean's Theorem can be shown when the cyclic quadrilateral SELF is a rectangle, and the law of cosines can be shown when it is a trapezoid.

When the cyclic quadrilateral SELF is a trapezoid, and:



LF > ES || LF

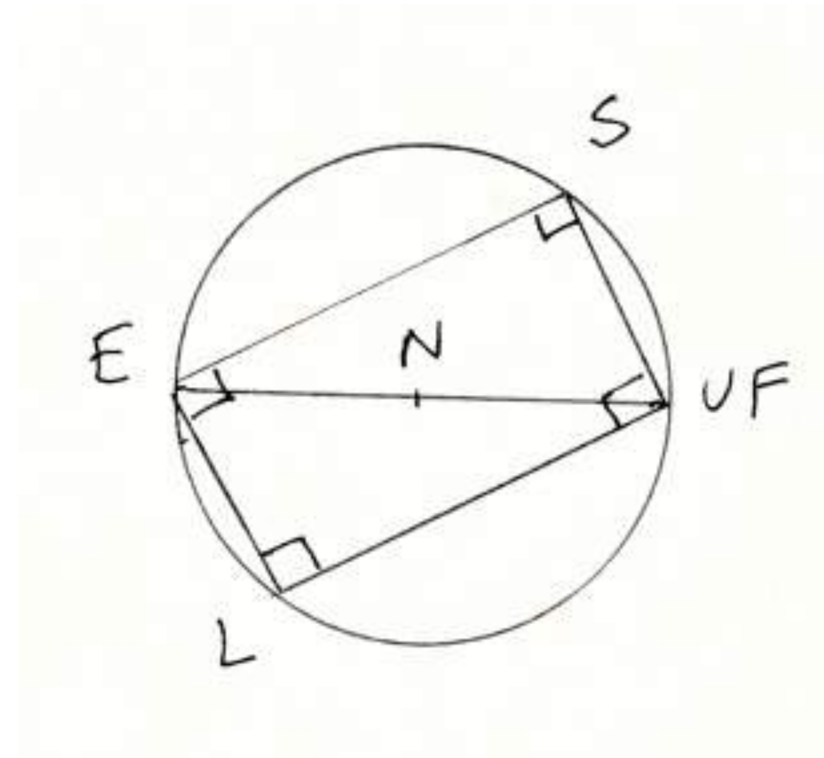
∠ELF = ~ESF/EU < ~EU/EU = π/2

$EF^2 = EL^2 + LF(ES)$

LF(ES) = LF[LF - 2(EL)(LR/LE)]

LR/LE = UF/UE = cosine ∠ELF

When the cyclic quadrilateral SELF is a rectangle, so:



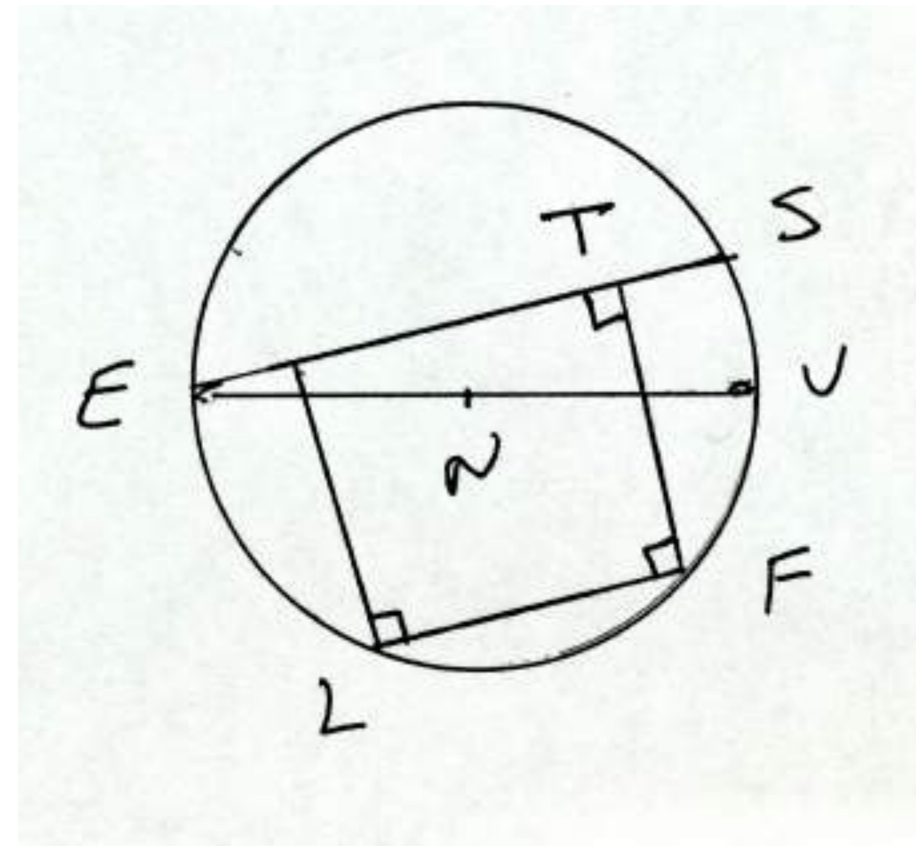
LF = ES || LF

∠ELF = ~ESF/EU = ~EU/EU = π/2

$EF^2 = EL^2 + LF(ES)$

$LF(ES) = LF^2$

When the cyclic quadrilateral SELF is a trapezoid, and:



LF < ES || LF

∠ELF = ~ESF/EU > ~EU/EU = π/2

$EF^2 = EL^2 + LF(ES)$

LF(ES) = LF[LF + 2(EL)(TS/SF)]

TS/SF = UF/UE = cosine ∠ELF

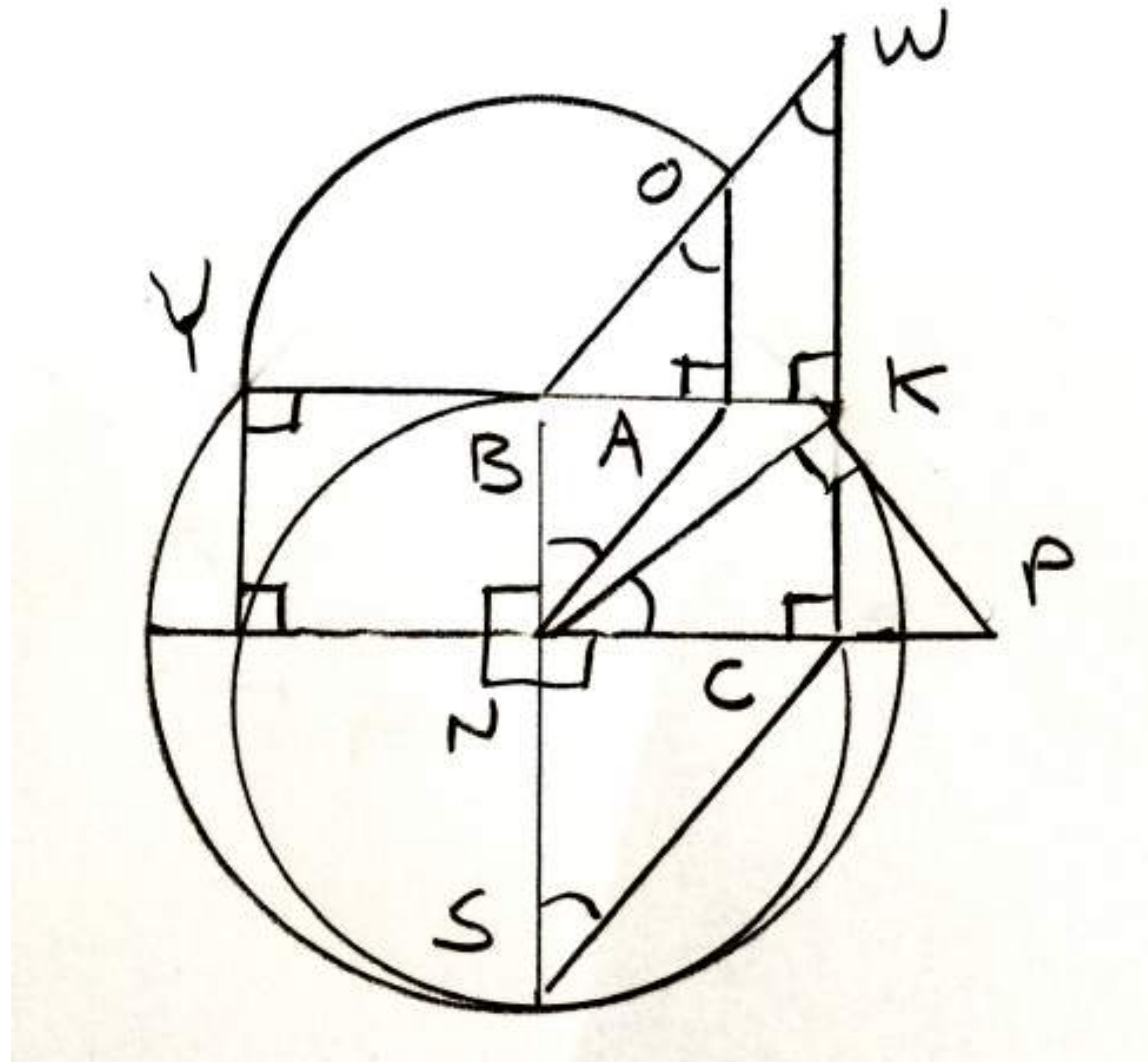
Refraction Along a Line

Let:
(NK/NC) = (CN/CK)

When:
ΔCKP ≅ ΔKNP
= ΔNSC = ΔKWB,

ΔCKP = ΔBNA = ΔAOB

and KW = YN

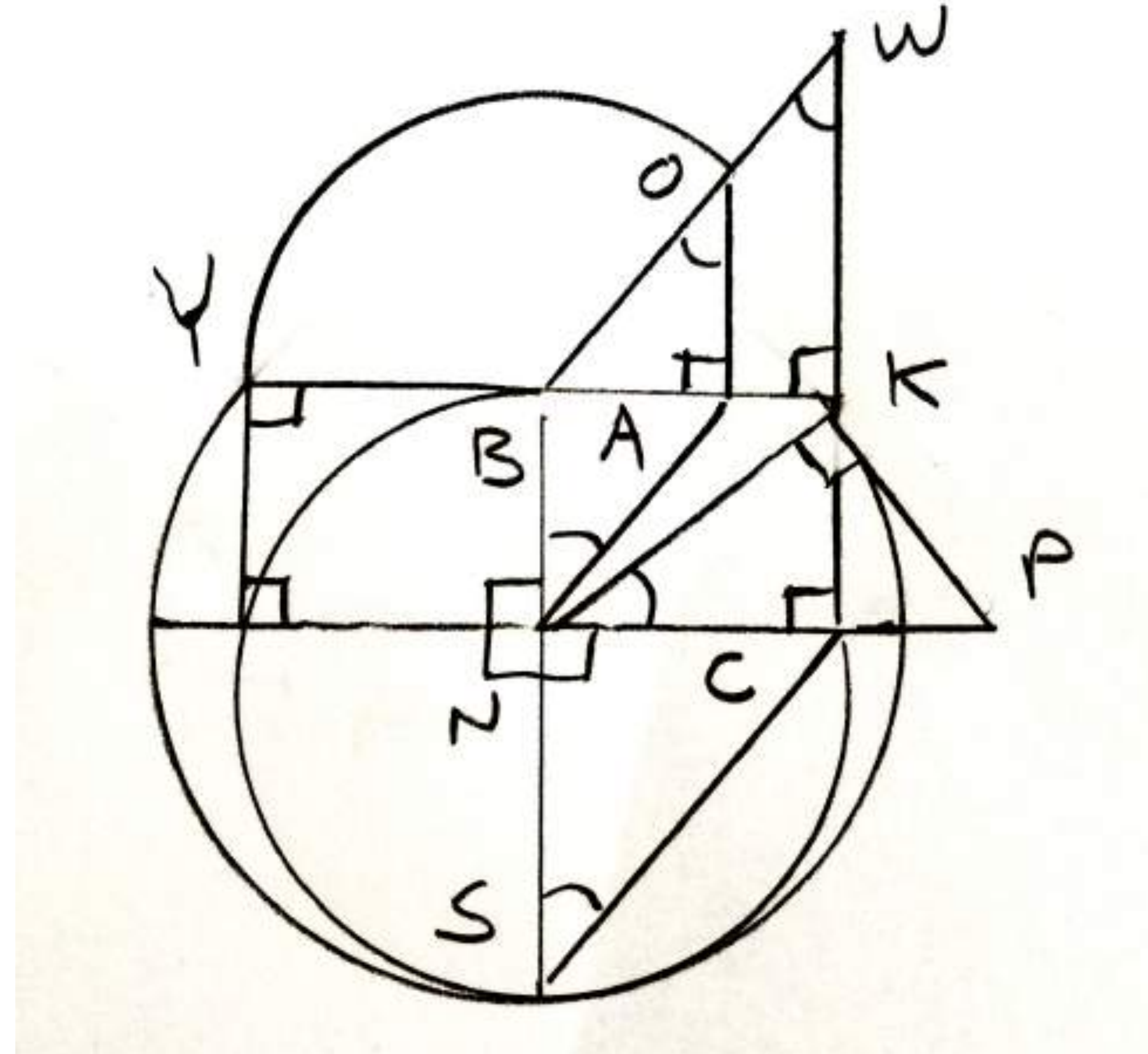


But also, ***whenever*:**

$KB^2 = KN^2 - BN^2$

$= KN^2 - (AN^2 - AB^2)$

$= (KN^2 - AN^2) + AB^2$



and:

$AN^2 - BN^2 = BO^2 - AO^2$

so:

$(AO^2 + AN^2)$

$= (BO^2 + BN^2) = YN^2$

if:

(KB/KW) = (AB/AO) = (CK/CN

so:

KB^2/KW^2

$= (AB^2 + CK^2)/(AO^2 + CN^2)$

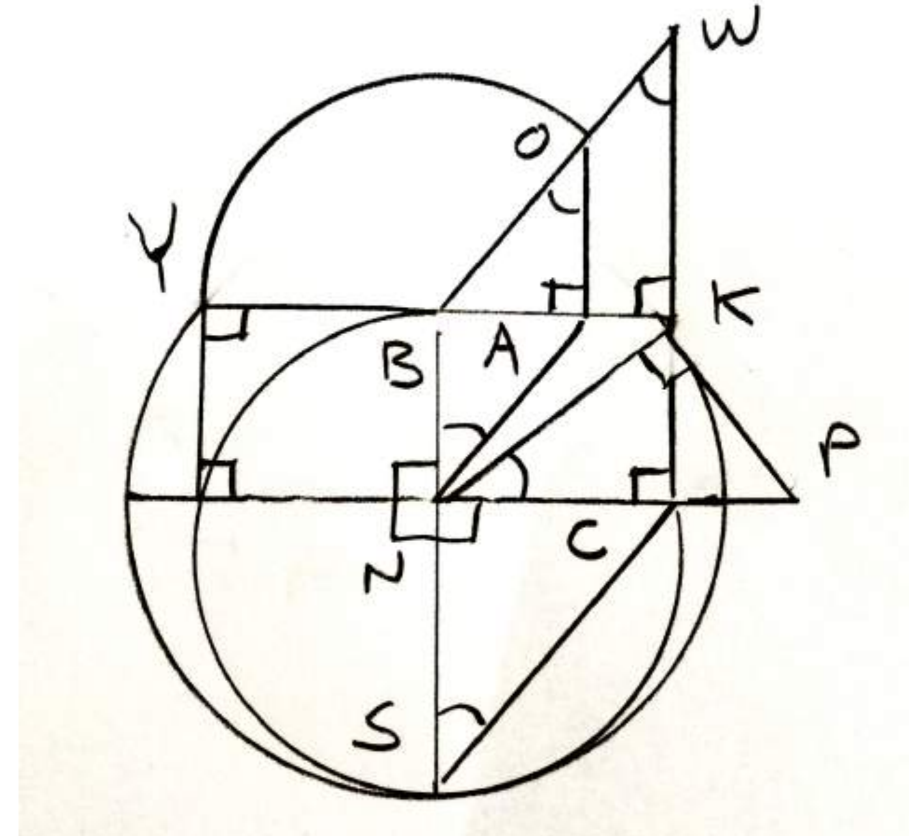
and if:

AN = CN,

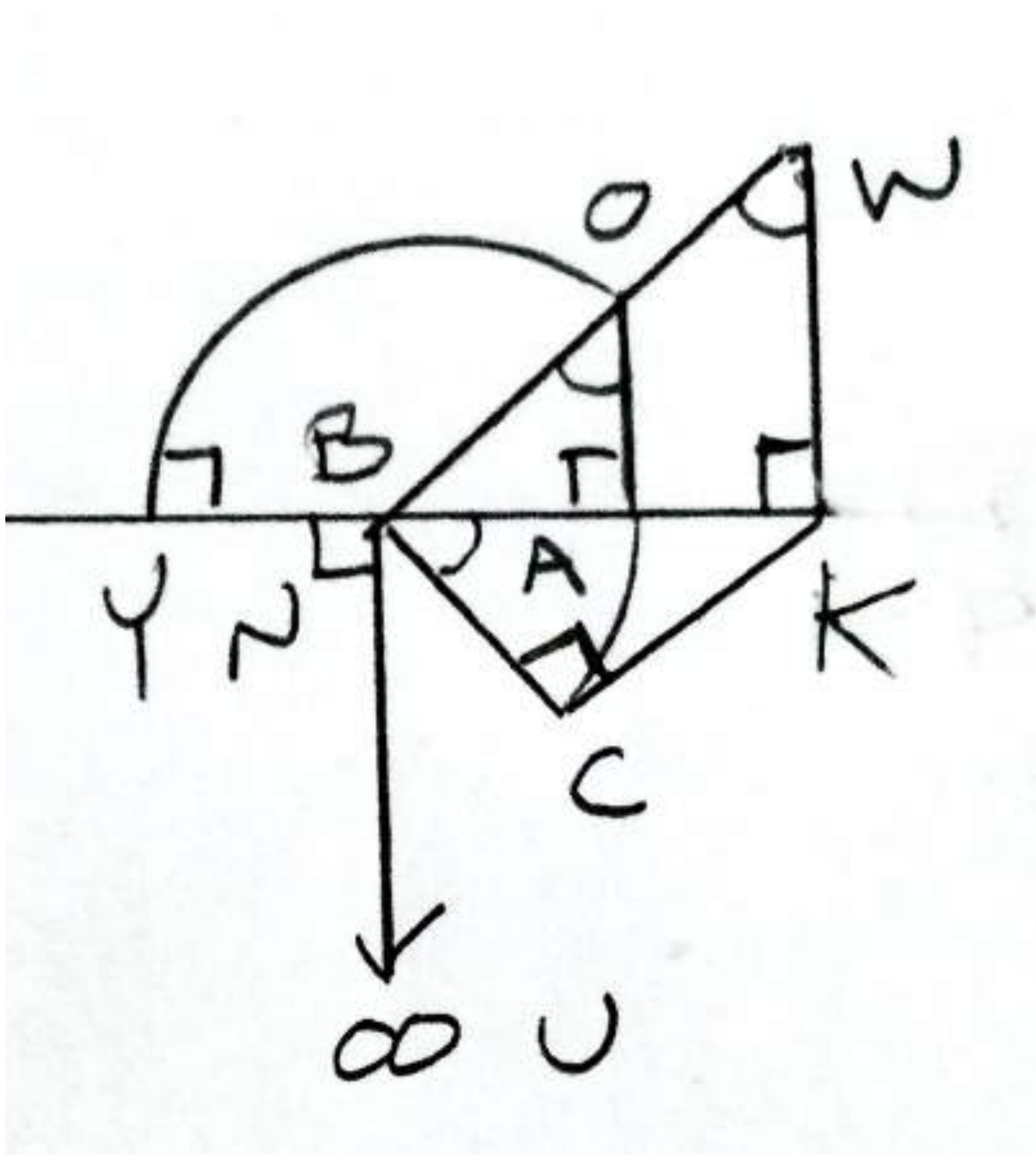
then:

$KW^2 = (AO^2 + CN^2) = YN^2$

KW = YN



Under these conditions, it can also be shown that:



As N ⇒ B, KW ⇒ YN

because:

KW/OA ⇒ NK/NA

= NK/NC

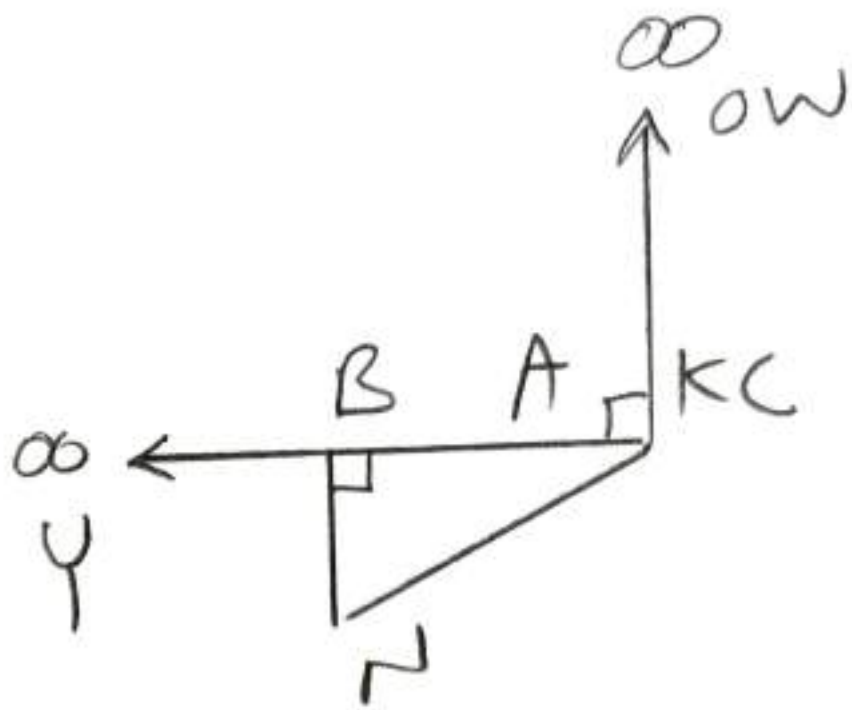
= OB/OA

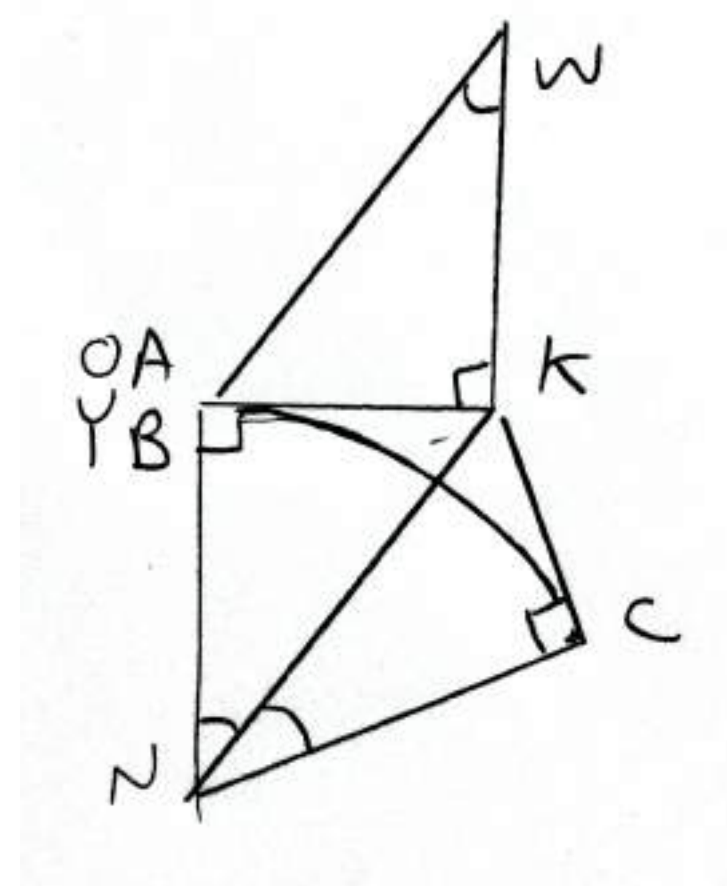
= WB/WK

so that:

KW ⇒ OB ⇒ YN

and both that:





As A ⇒ K,

KW ⇒ YN

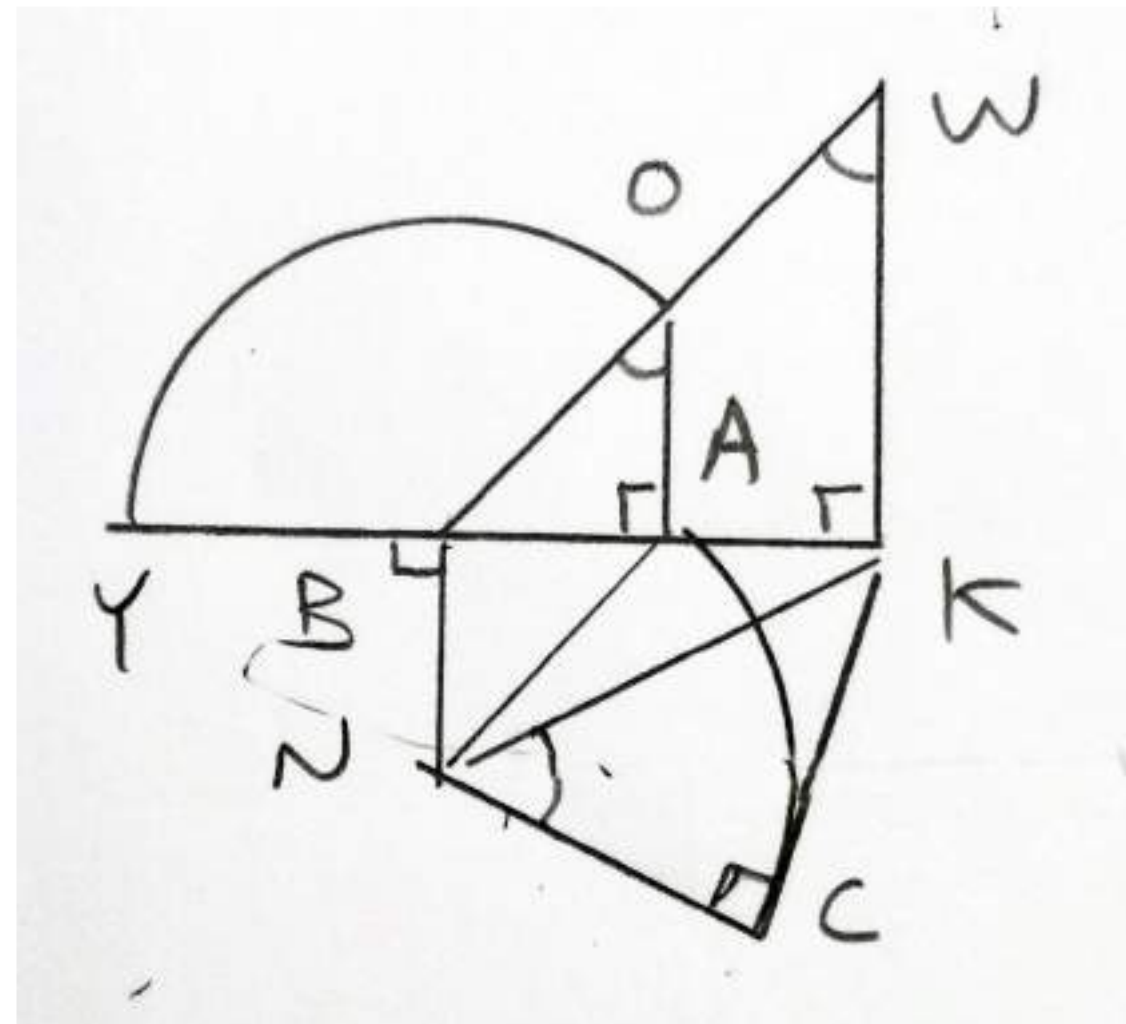
As A ⇒ B,

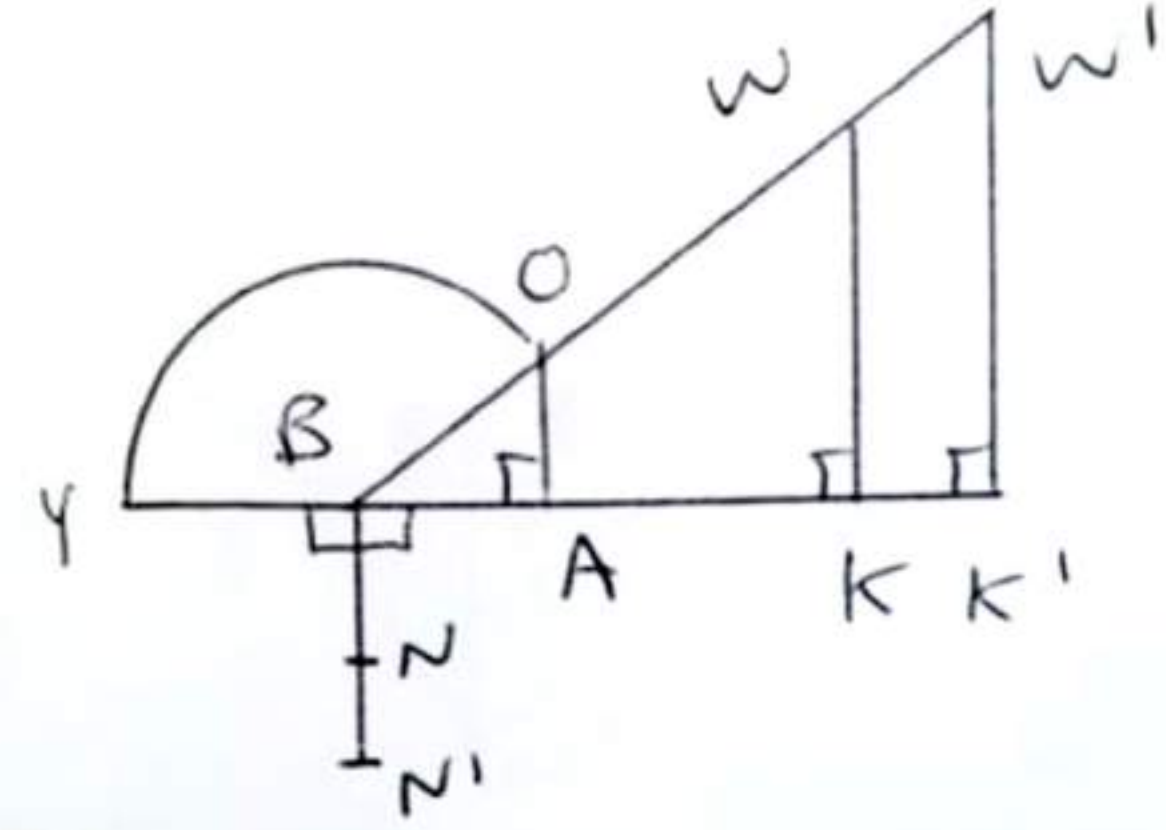
KW ⇒ YN

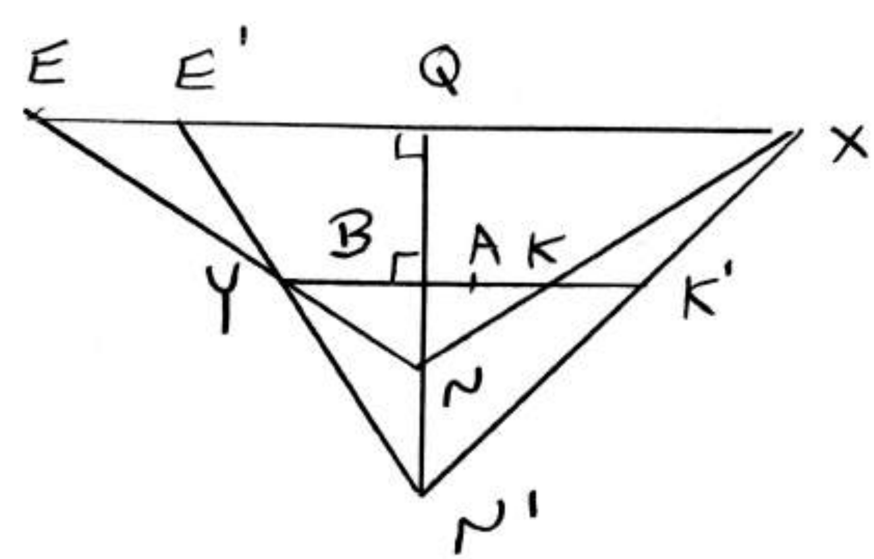
Therefore, whenever
A lies on KB
of right triangle ΔKBN,

if:
ΔCNK ≅ ΔAOB
≅ ΔKWB,
and NA = NC,

then KW = YN







QX/EN = KB/YN
= K’B/YN’ = QX/E’N’

OB/OA = NK/NA = N’K’/N’A

EN = E’N’

KW = YN
K’W’ = YN’

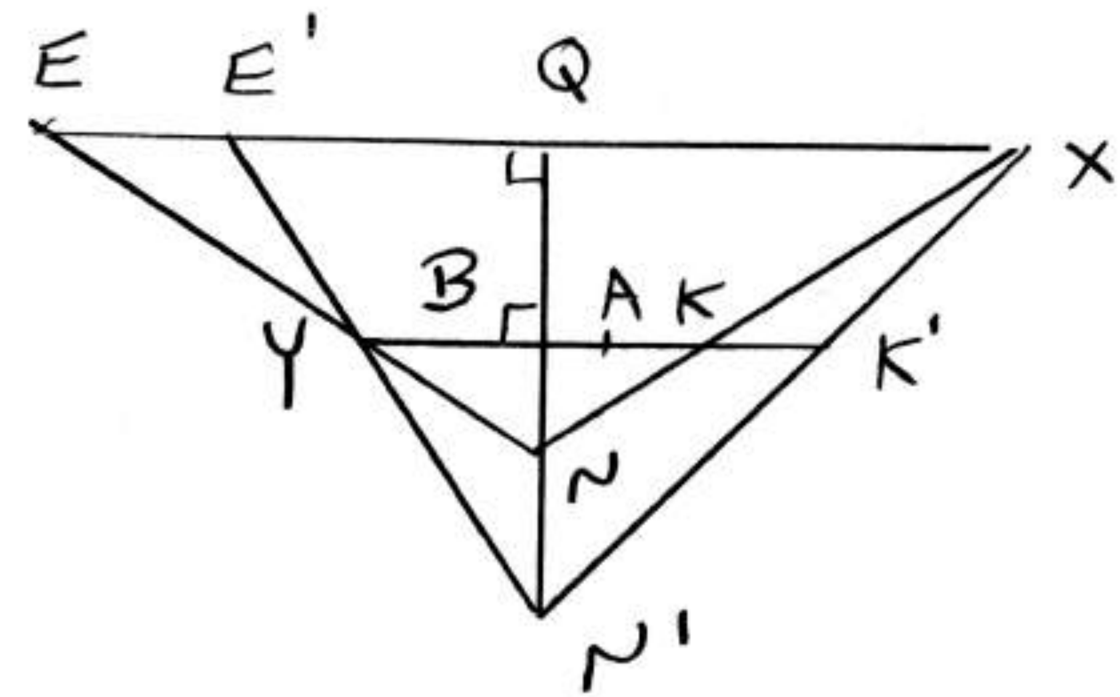
Only one N’K’X exists for NKX since only one E’N’ exists equal to EN.

KB/YN = K’B/YN’

When EN is changed to become the smallest segment through Y,
bound by the right angle EQN:

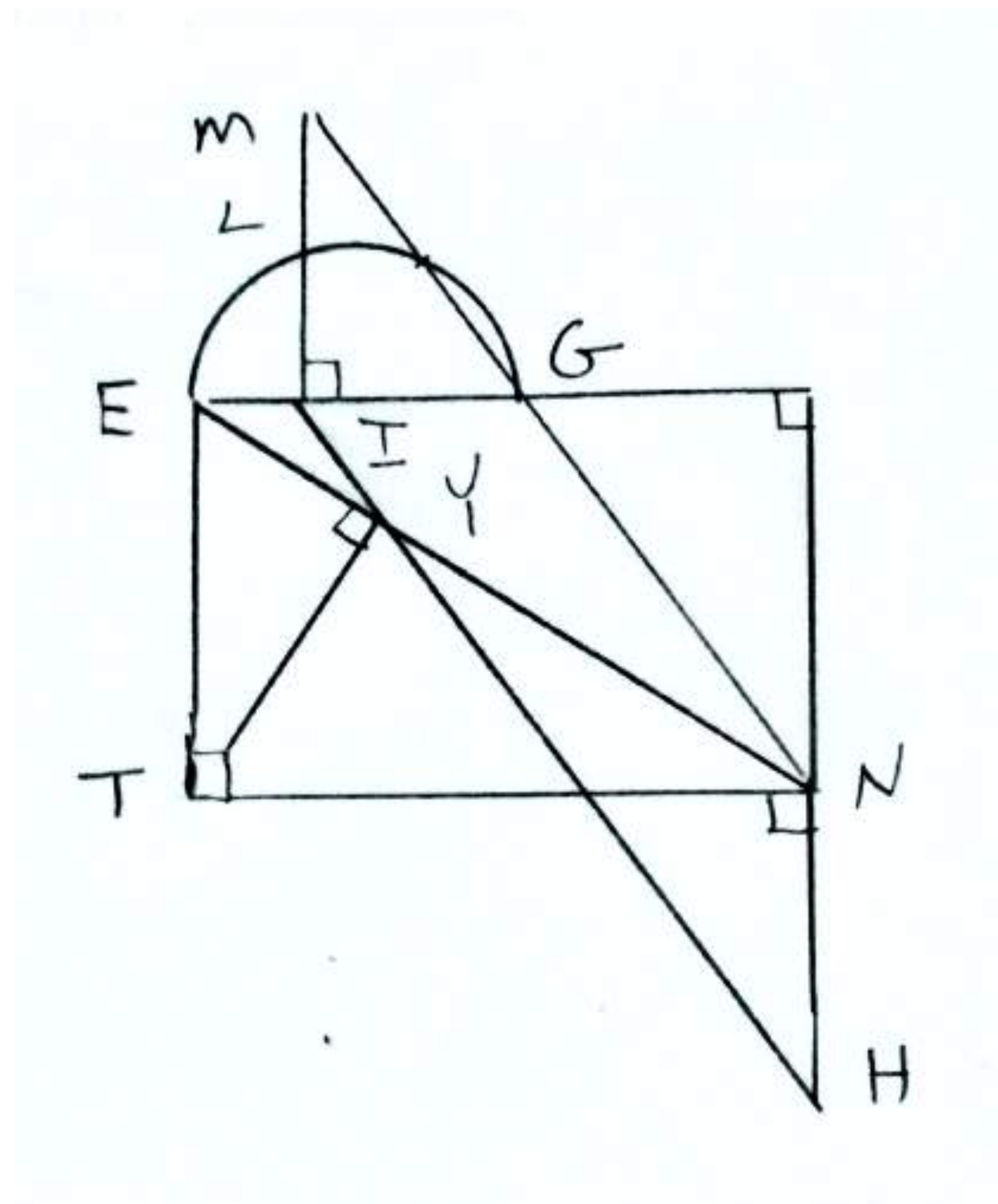
E’ lies at E, and
N’ lies at N.

Also, QX varies with EN because:
QX/EN = KB/YN
= KB/KW, which is a constant.



To specify EN as the shortest hypotenuse through Y:

NE || GL
TY || EL
HI || NM
HI = NM > NL

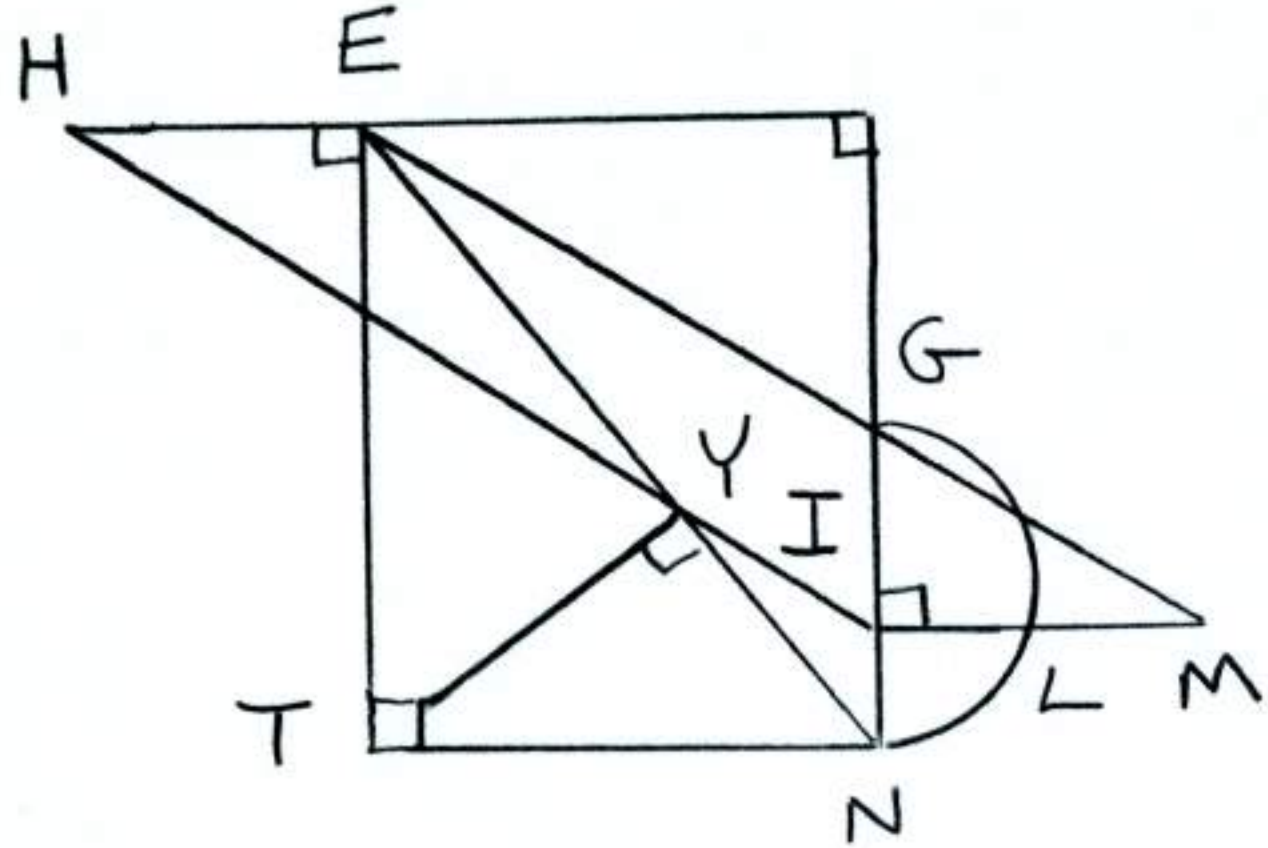


NL is the hypotenuse of right triangle NEL, so:

NL > NE
HI > NE

But also:

NE || GL
TY || NL
HI || EM
HI = EM > EL

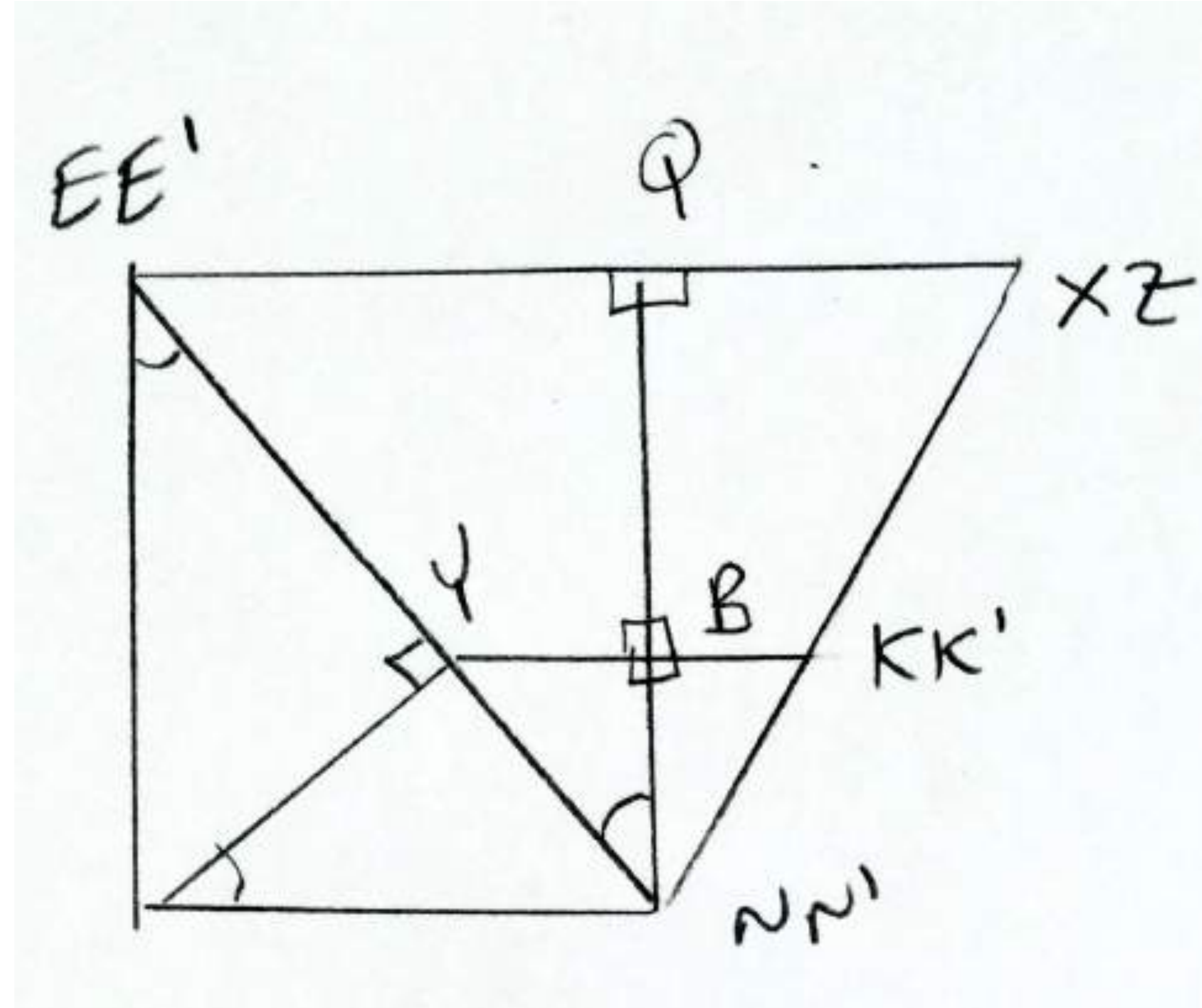


EL is the hypotenuse of right triangle ENL, so:

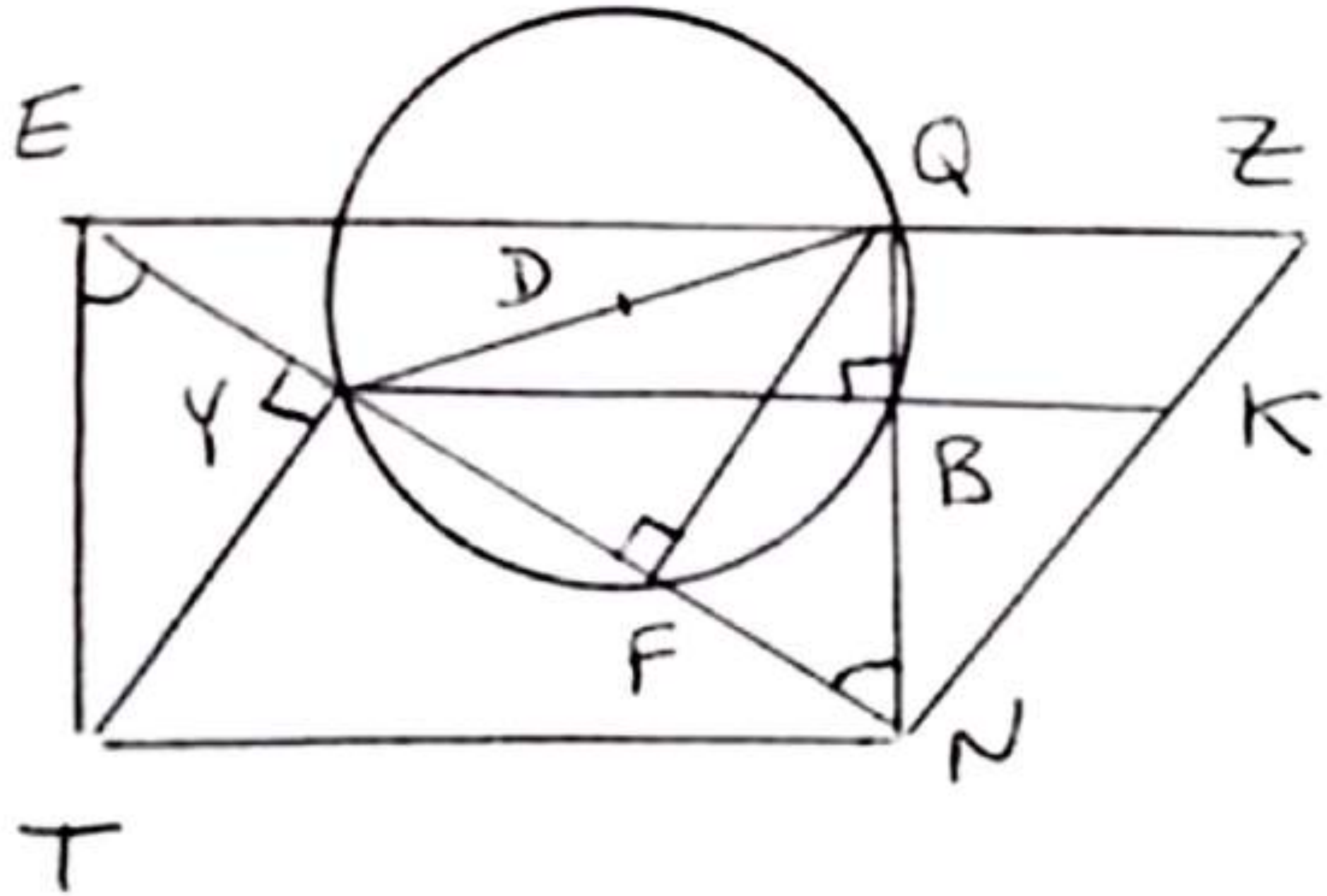
EL > EN
HI > EN

Let X = Z when EN is the shortest segment through Y included in right angle EQN.

In order to find Z given ΔYBN, we must find E = E’ using:

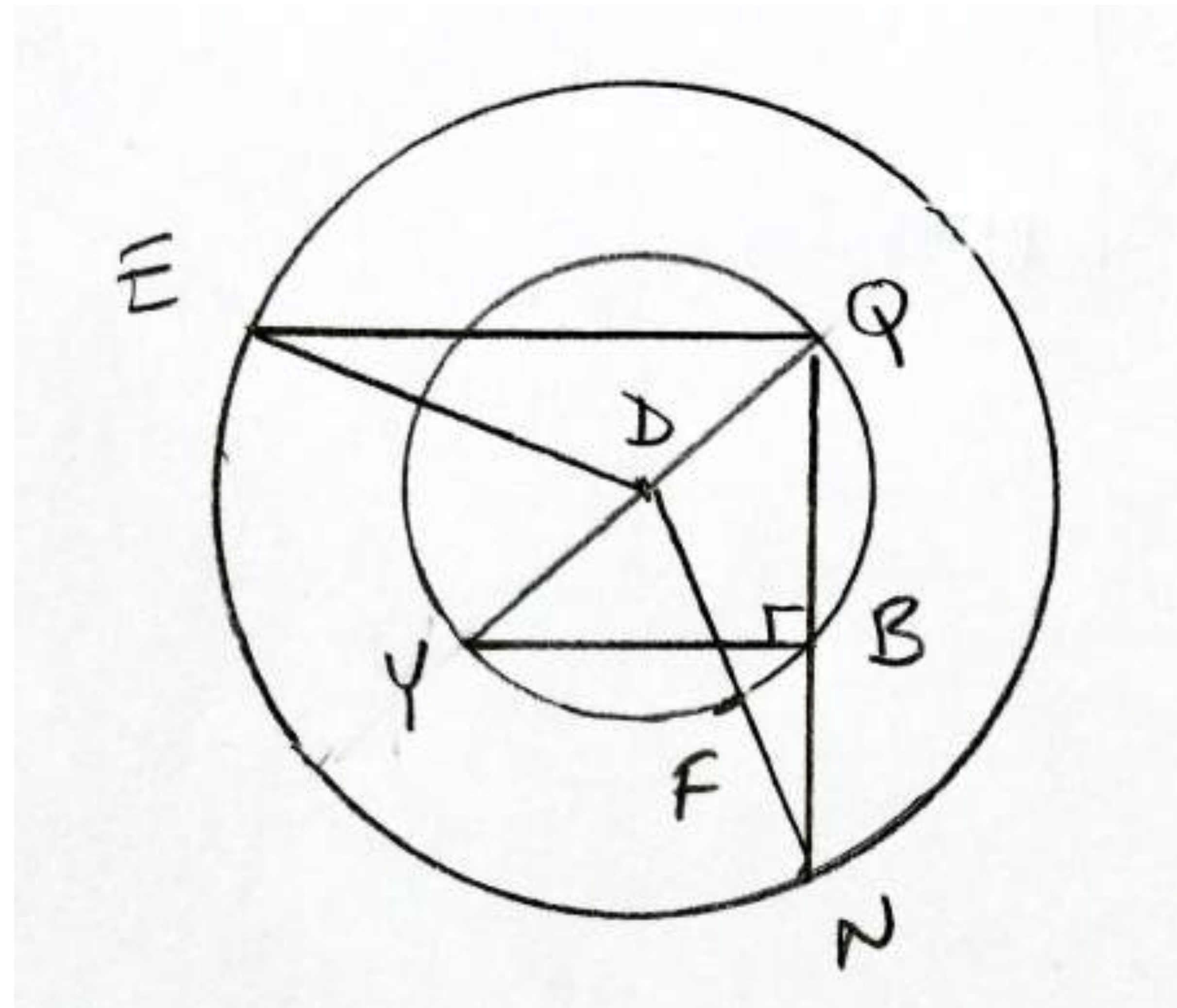


ΔYBN ≅ ΔNYT ≅ ΔNTE



In order to find Z given ΔYBQ, we must find EN = E’N’ by making ΔTYE a right triangle.

Draw a concentric circle around ⊙YBQ using its center at D, (the midpoint of hypotenuse YQ), containing an arc ~EN, so that YF lies on its chord EN. The arc intercepted by ∠DEN then equals that intercepted by ∠DNE.



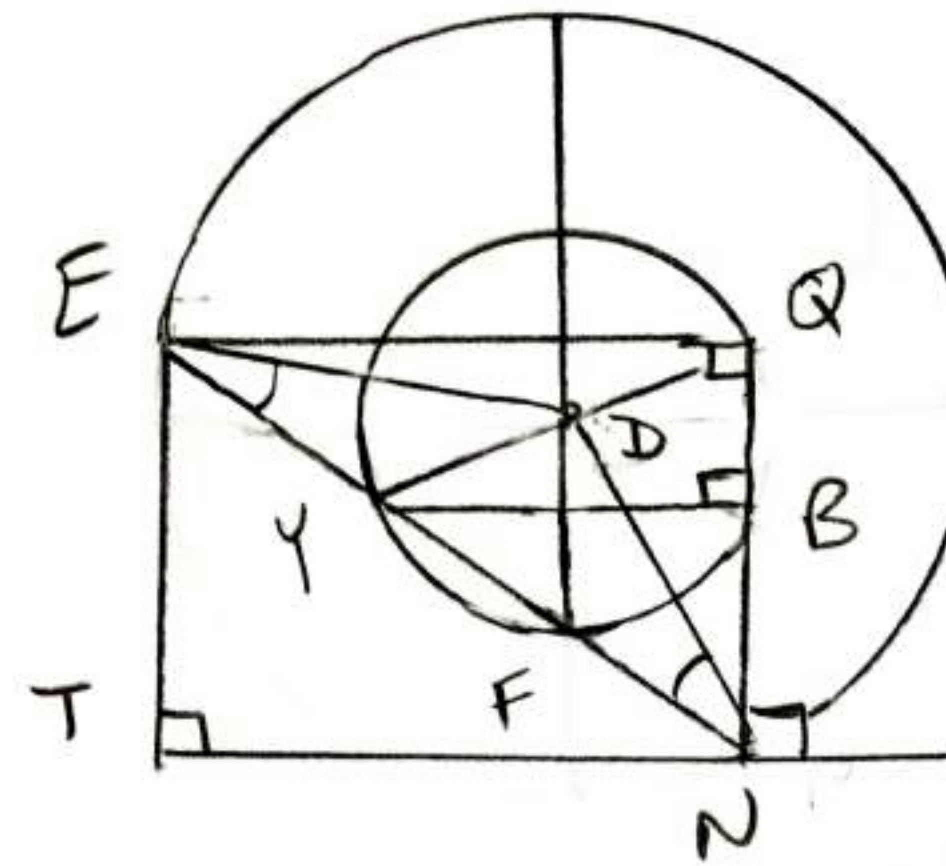
∠DEY = ∠DNF

DY = DF ; DE = DN

ΔEDY = ΔNDF

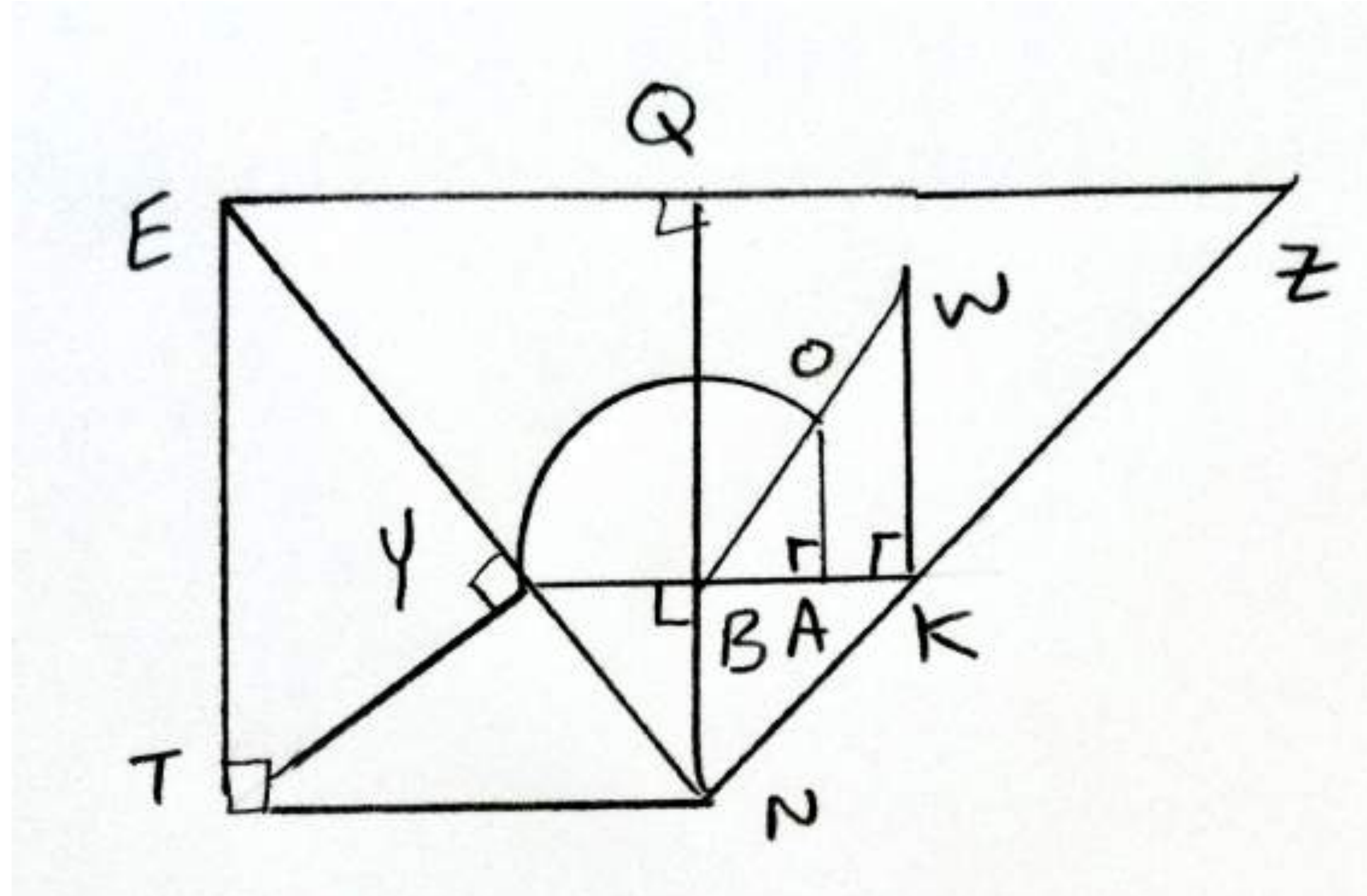
EY = NF

Since ΔQFN is a right triangle, so is ΔTYE.



WK = YN

Given ΔBAO:



use ΔBNY to find ΔBKW and ΔQBY,

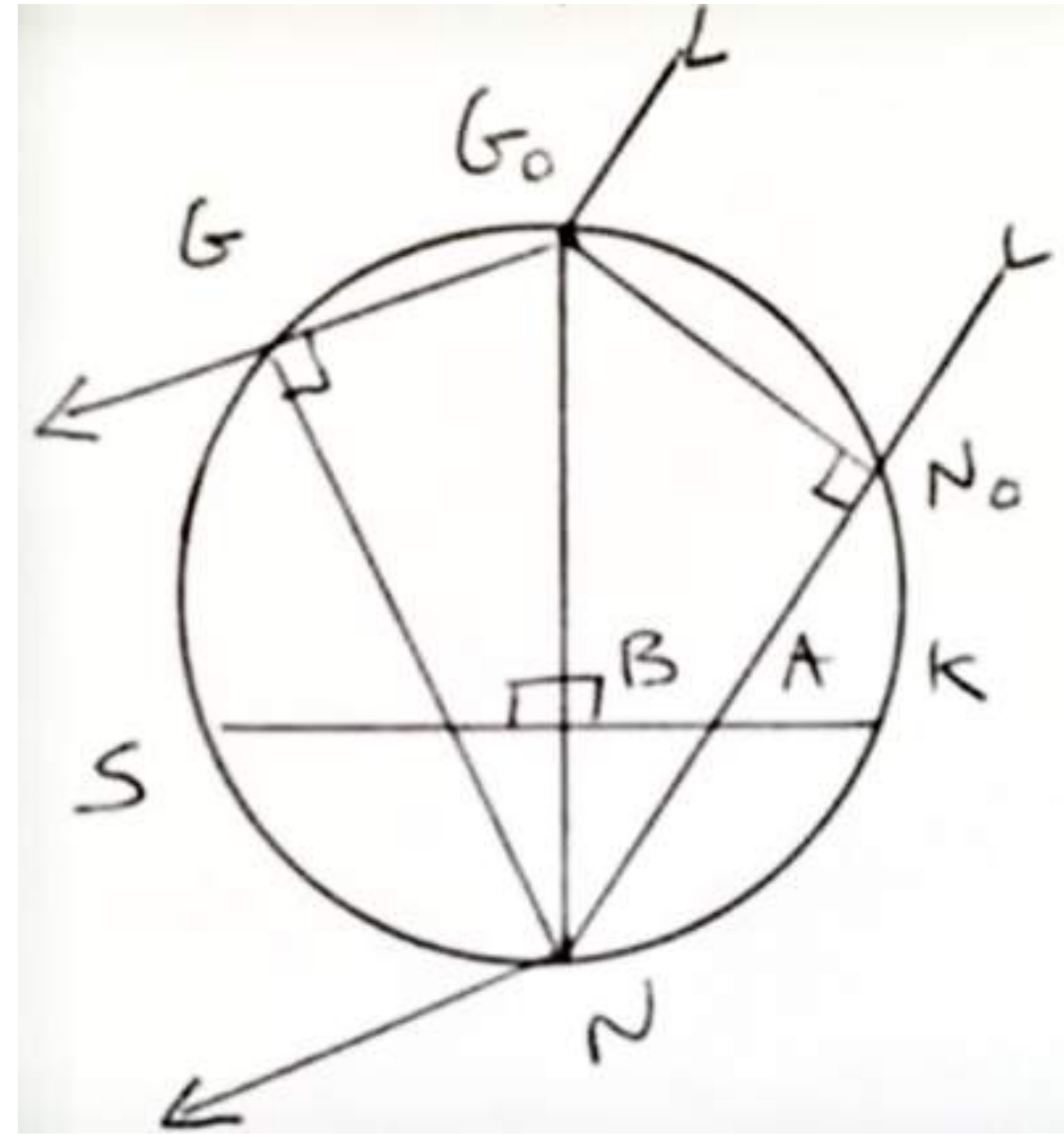
use ΔQBY or ΔBKW to find ΔBNY.

$\Delta N_o NK \cong \Delta KNA$

because:

~NS = ~NK

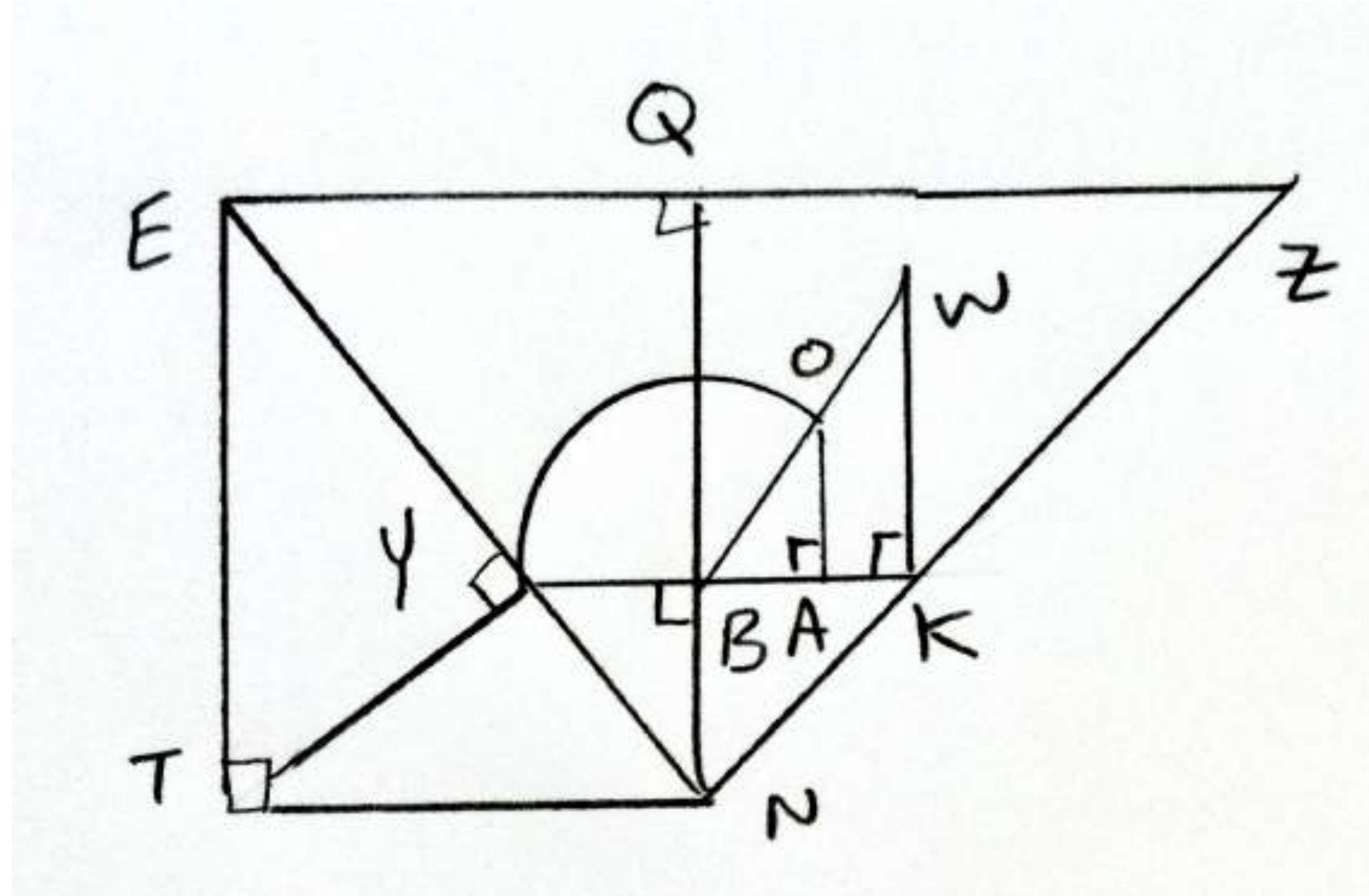
across diameter G_oN.



Wavefront G_oN_o refracts into wavefront GN along G_oN, since it travels G_oG in the same time it travels N_oN.

$$\mathbb{R} = NN_o/GG_o = NN_o/NK = NK/NA$$

Therefore, if
ℝ = OB/OA,
and WK = YN;
then,
ℝ = NK/NA



and Z is the clear image of object A refracted at N (= N’), along BN, because the two possible refracted rays through Z coincide at N.

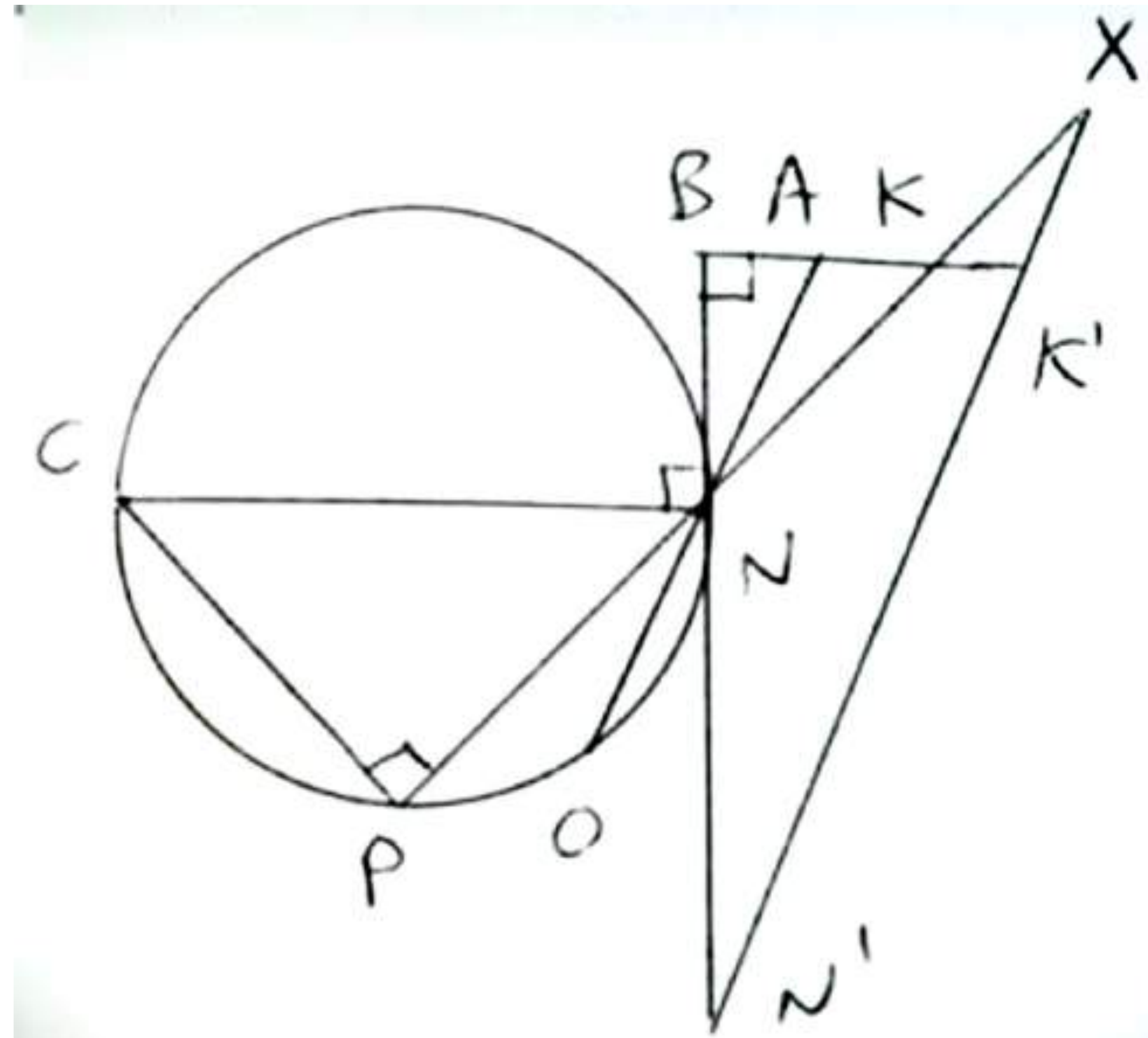
Refraction Along a Circle

$\Delta KNA \cong \Delta OCP$

$\mathbb{R} = NK/NA$

$= N'K'/N'A$

$= CO/CP$



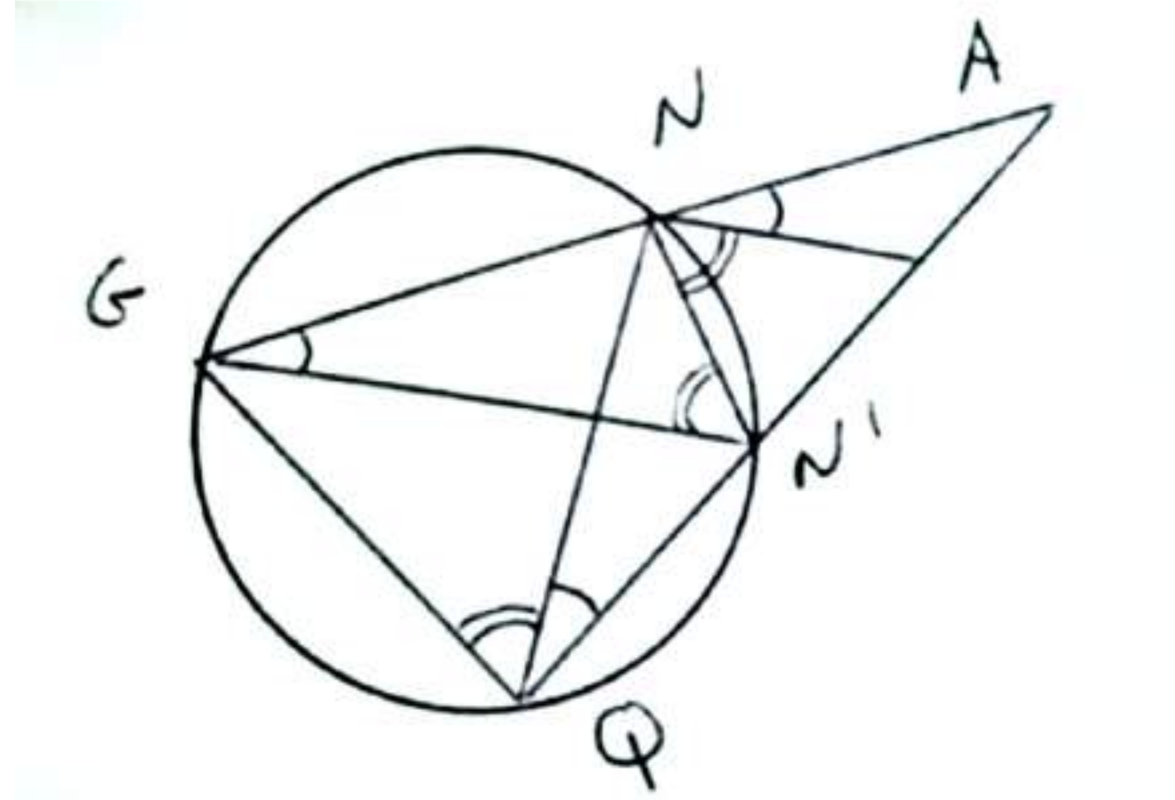
Real object A:

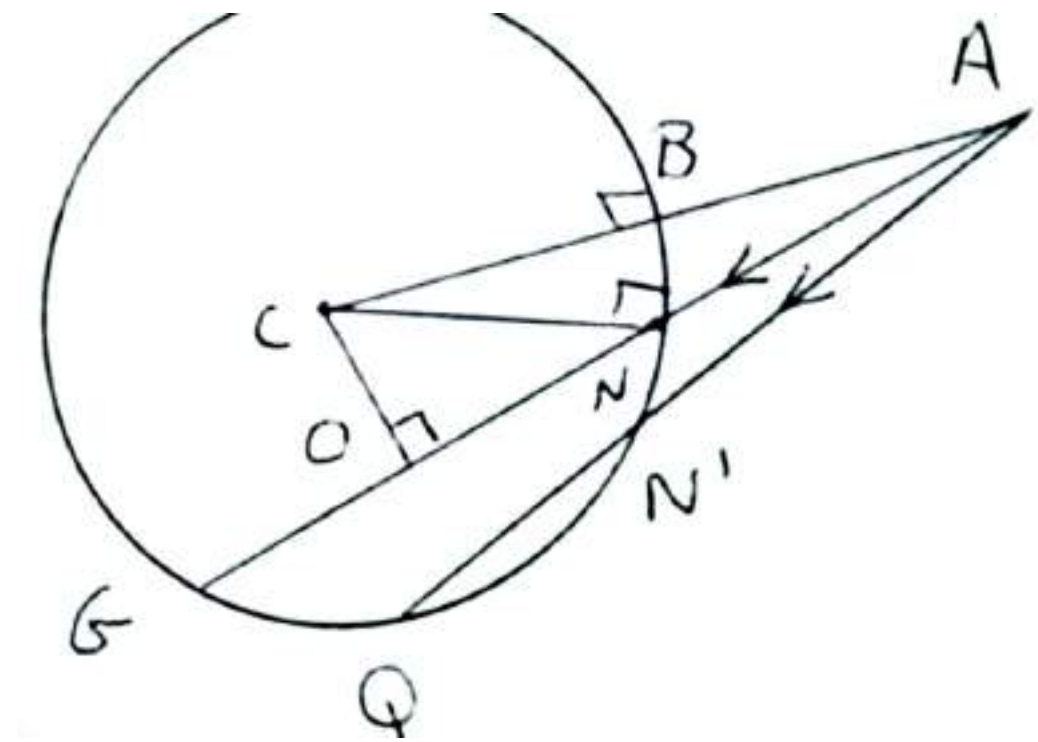
ΔANN’ ≅ ΔAQG

AG/AN’ = QG/NN’

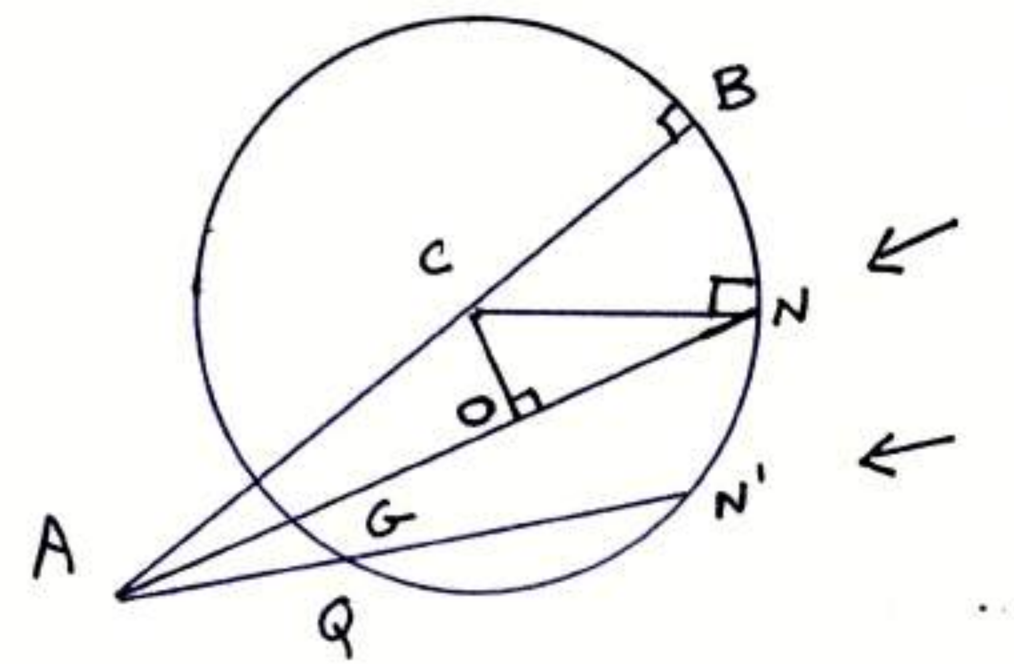
(AG + AN’)/2AN’

= (QG + NN’)/2NN’





Virtual object A, which can not be projected on a screen due to refraction at BN:



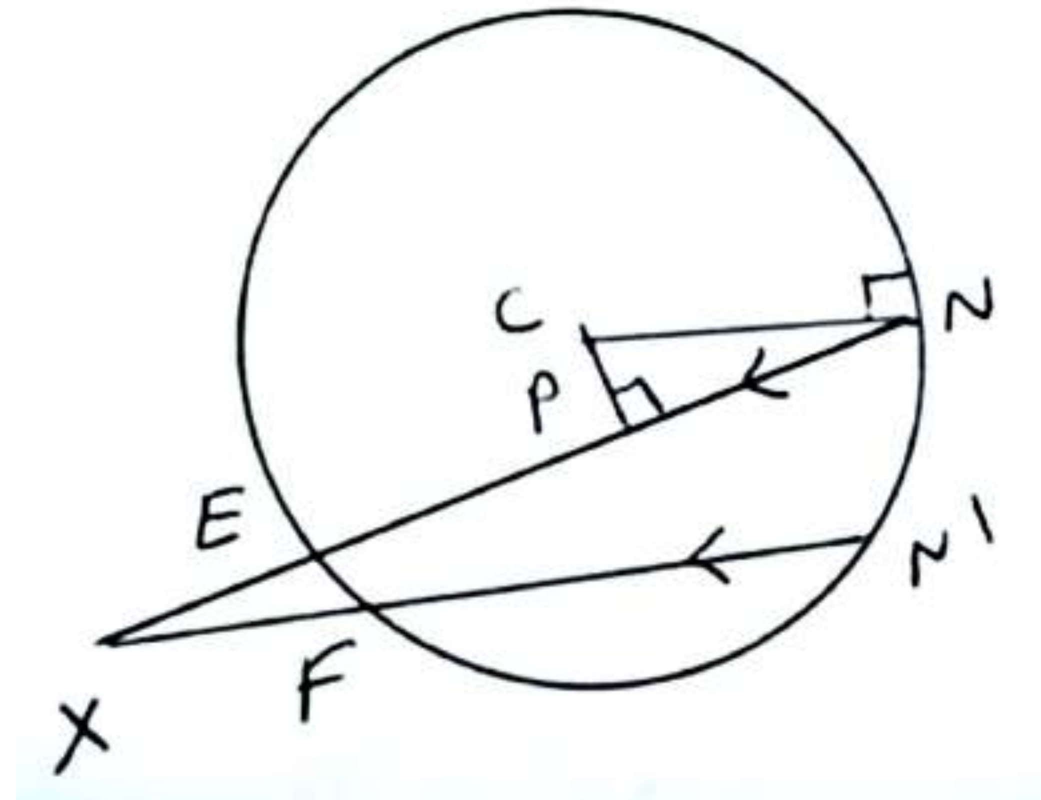
$\Delta ANN' \cong \Delta AQG$

$AG/AN' = QG/NN'$

$(AG + AN')/2AN'$
$= (QG + NN')/2NN'$

Real image at X,
(will be defined as clear as N’ $\Rightarrow$ N, and X $\Rightarrow$ Z),

can be projected on a screen:



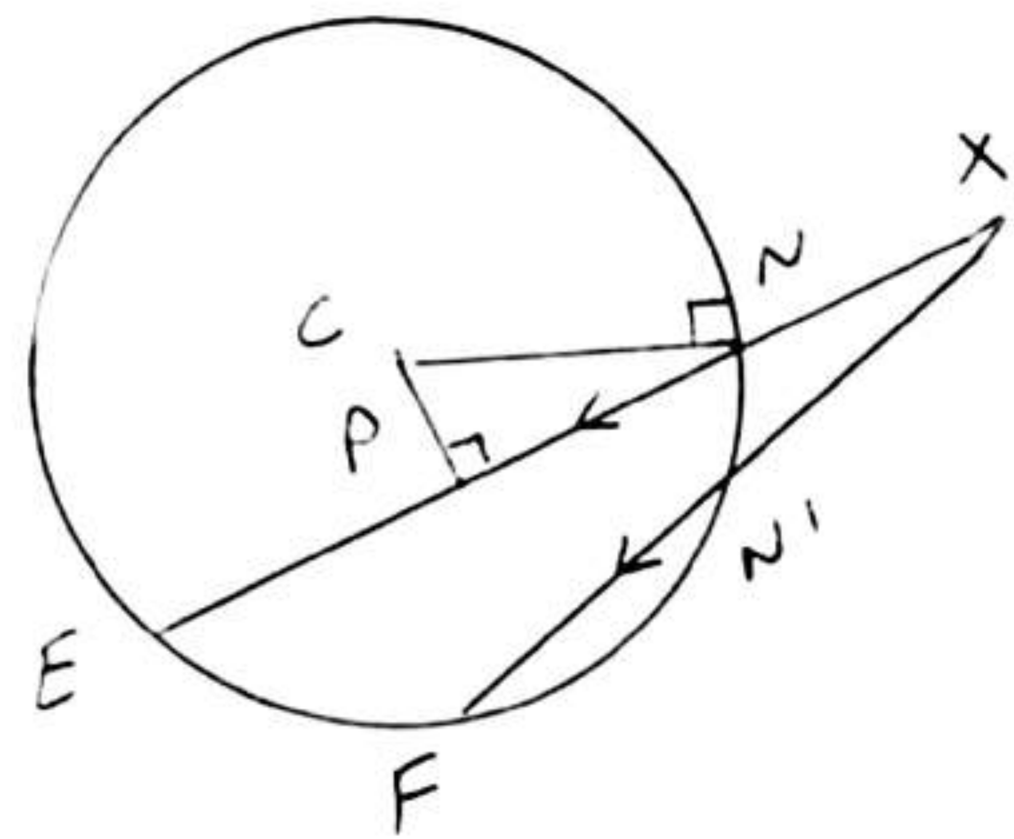
ΔXNN’ ≅ ΔXFE

XE/XN’ = EF/NN’

(XE + XN’)/2XN’
= (EF + NN’)/2NN’

Virtual image at X,
(will be defined as clear as N’ ⇒ N, and X ⇒ Z),

can not be projected on a screen:



ΔXNN’ ≅ ΔXFE

XE/XN’ = EF/NN’

(XE + XN’)/2XN’
= (EF + NN’)/2NN’

$$(AG + AN')/2AN' = (QG + NN')/2NN'$$
$$(XE + XN')/2XN' = (EF + NN')/2NN'$$

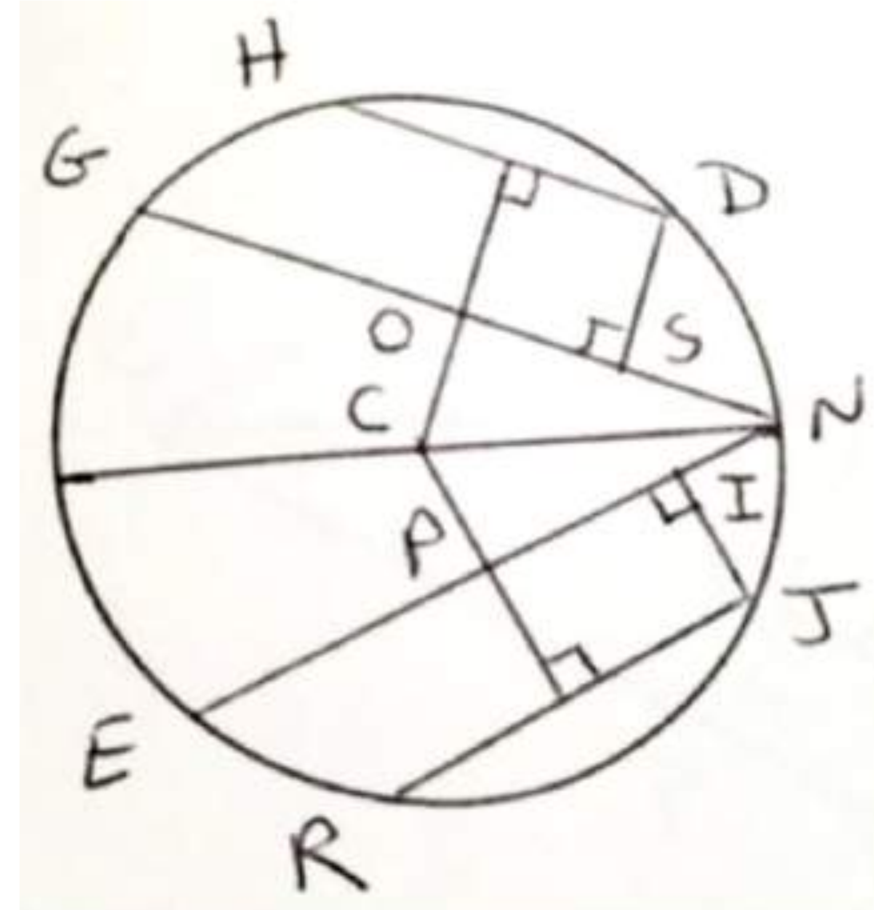
$$(QG + NN')/(EF + NN')$$
$$= [(AG + AN')/2AN'][2XN'/(XE + XN')]$$

As $N' \Rightarrow N$, $X \Rightarrow Z$, and:

$$(\sim QG + \sim NN')/(\sim EF + \sim NN')$$
$$\Rightarrow (QG + NN')/(EF + NN')$$

$$\Rightarrow (AO/AN)(ZN/ZP)$$

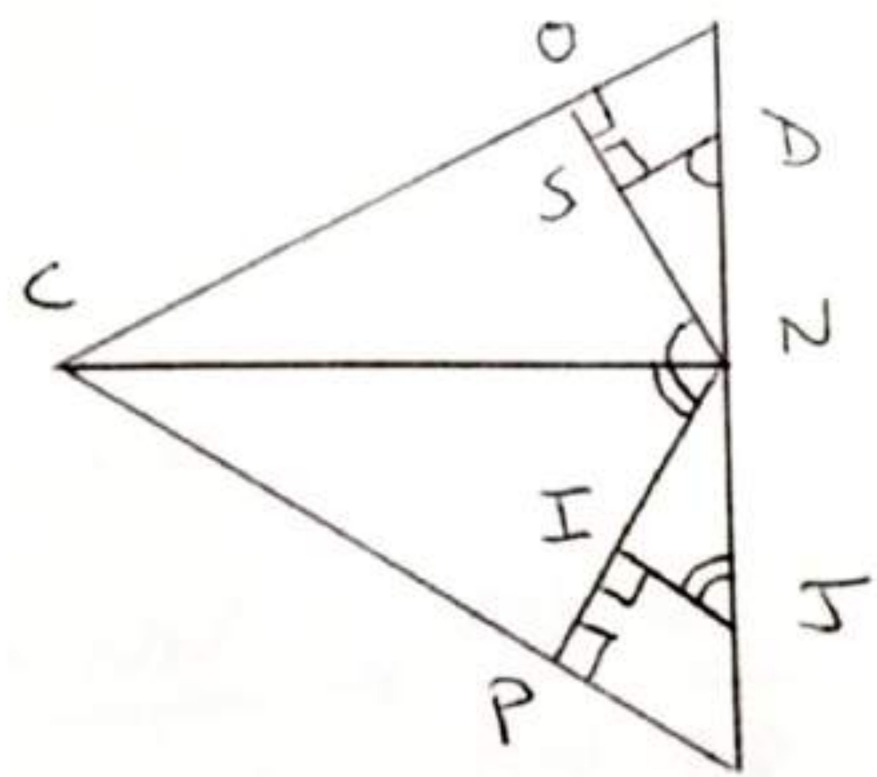
Also, when HD = QN’
and RJ = FN’



(~QG + ~NN’)/(~EF + ~NN’)
= 2(~ND)/2(~NJ) = ~ND/~NJ

As N’ ⇒ N, X ⇒ Z, and:

~DJ ⇒ line segment DJ, so:



(~QG + ~NN’)/(~EF + ~NN’)
⇒ ND/NJ

DS/JI = CO/CP
JI/JN = NP/NC
DN/DS = NC/NO
ND/NJ = (NP/NO)(CO/CP)

As N’ ⇒ N, X ⇒ Z, and:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ (NP/NO)(CO/CP)

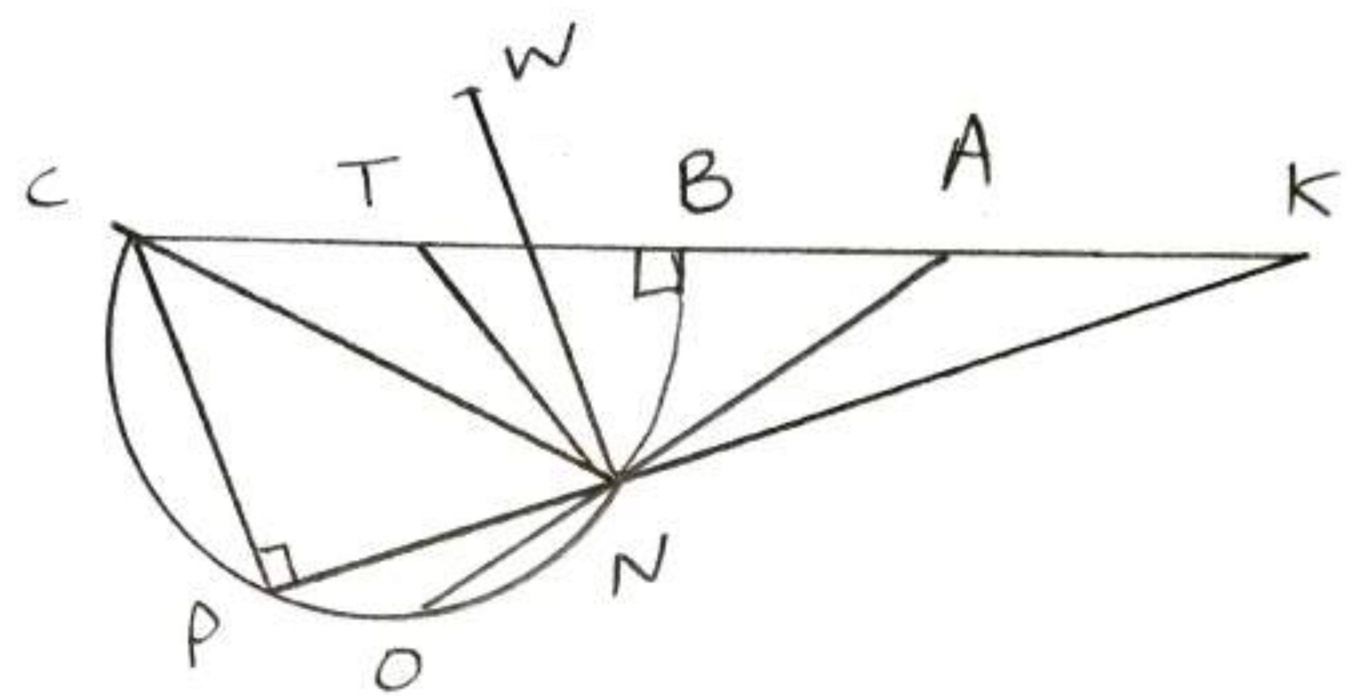
and therefore:
(AO/AN)(ZN/ZP) ⇒ (NP/NO)(CO/CP)

Thus $\mathbb{R}$ = CO/CP, and Z, (along both NP and CW), is the clear image of A refracted along ~BN, when:

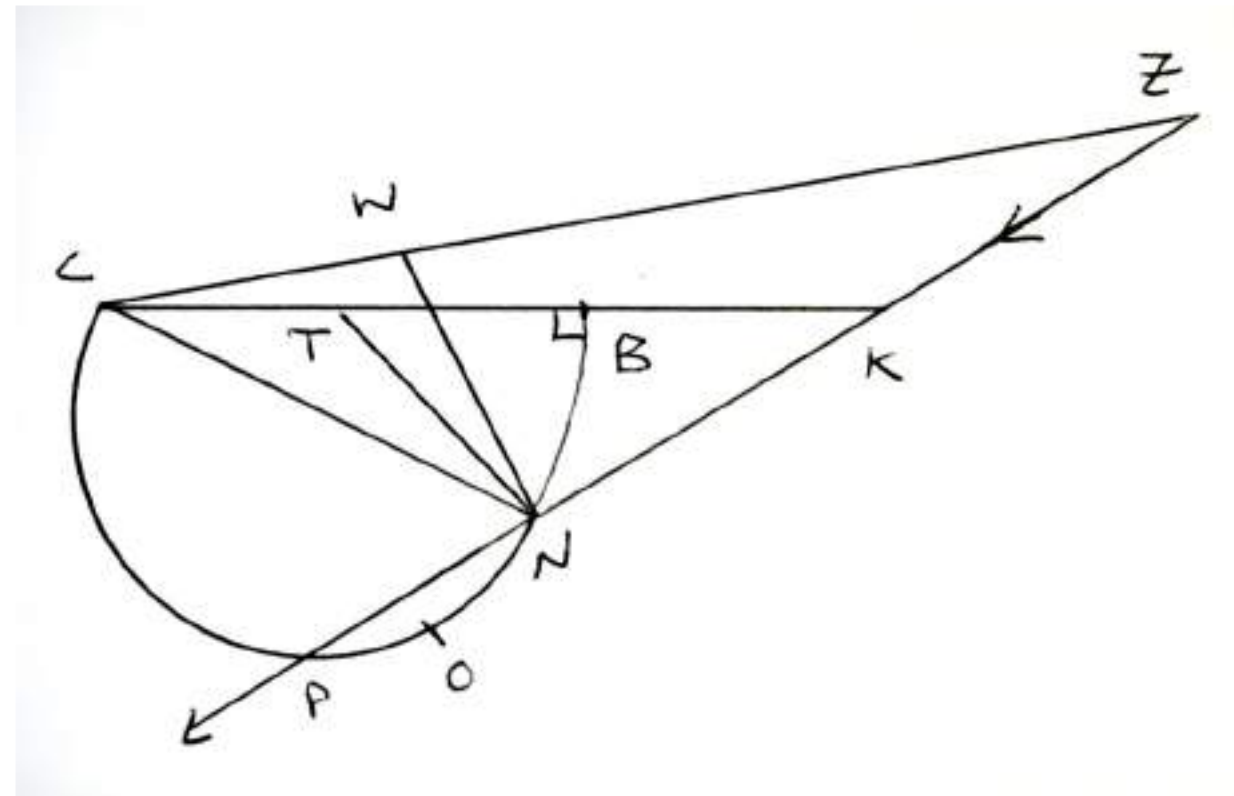
NT||CO, so:
AO/AN = CO/NT and:

NW||CP, so:
ZN/ZP = NW/CP
and:

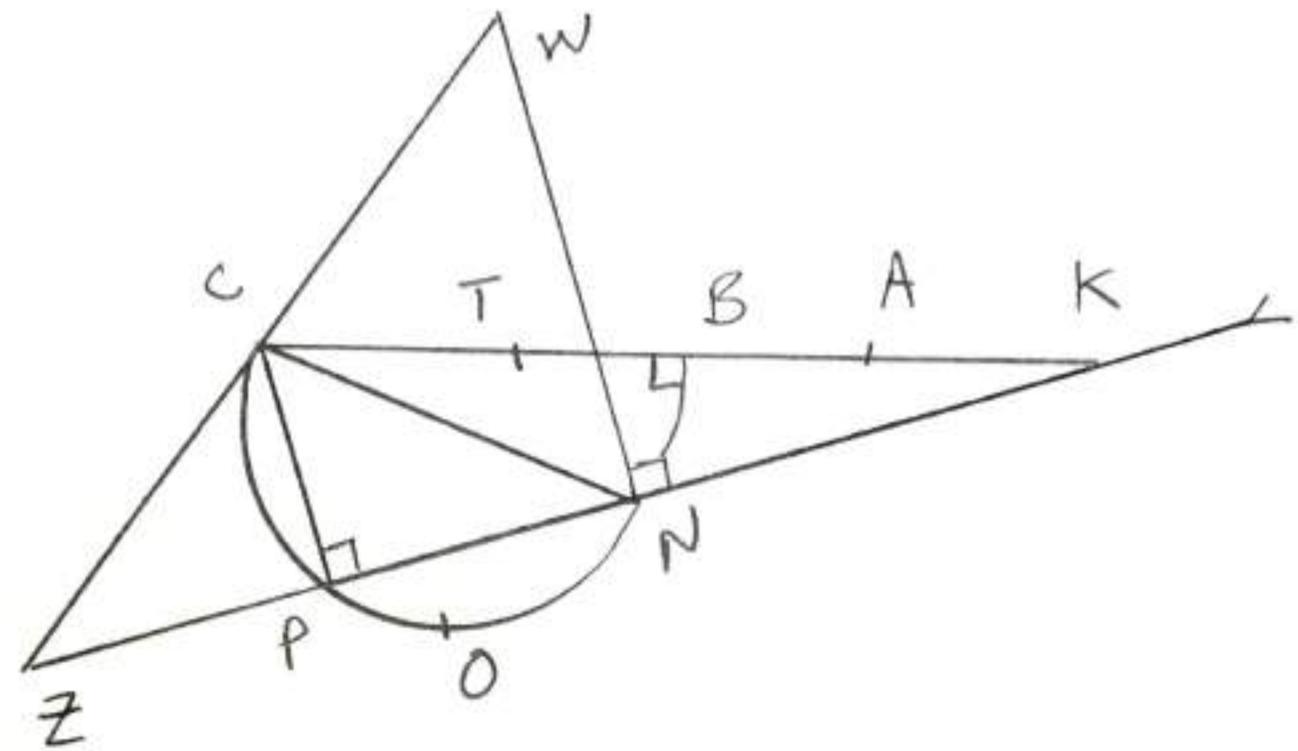
NW/NT = NP/NO
(ΔWNT ≅ ΔPNO)

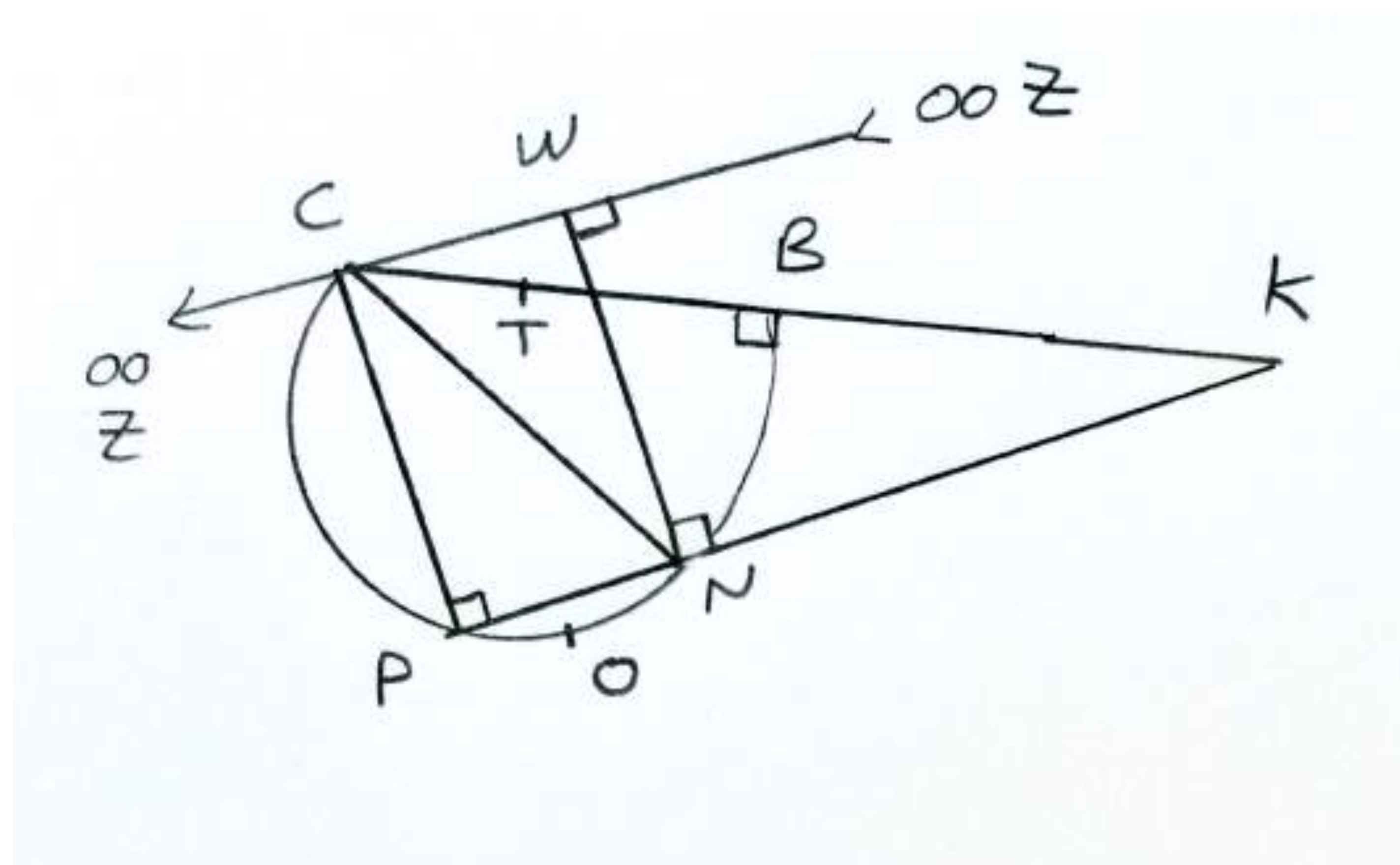


The off-axis rays from any on-axis object A, (real or virtual), can not form a virtual on-axis image at Z because NW must be less than CP for Z to be virtual; but NW must also be greater than NT.

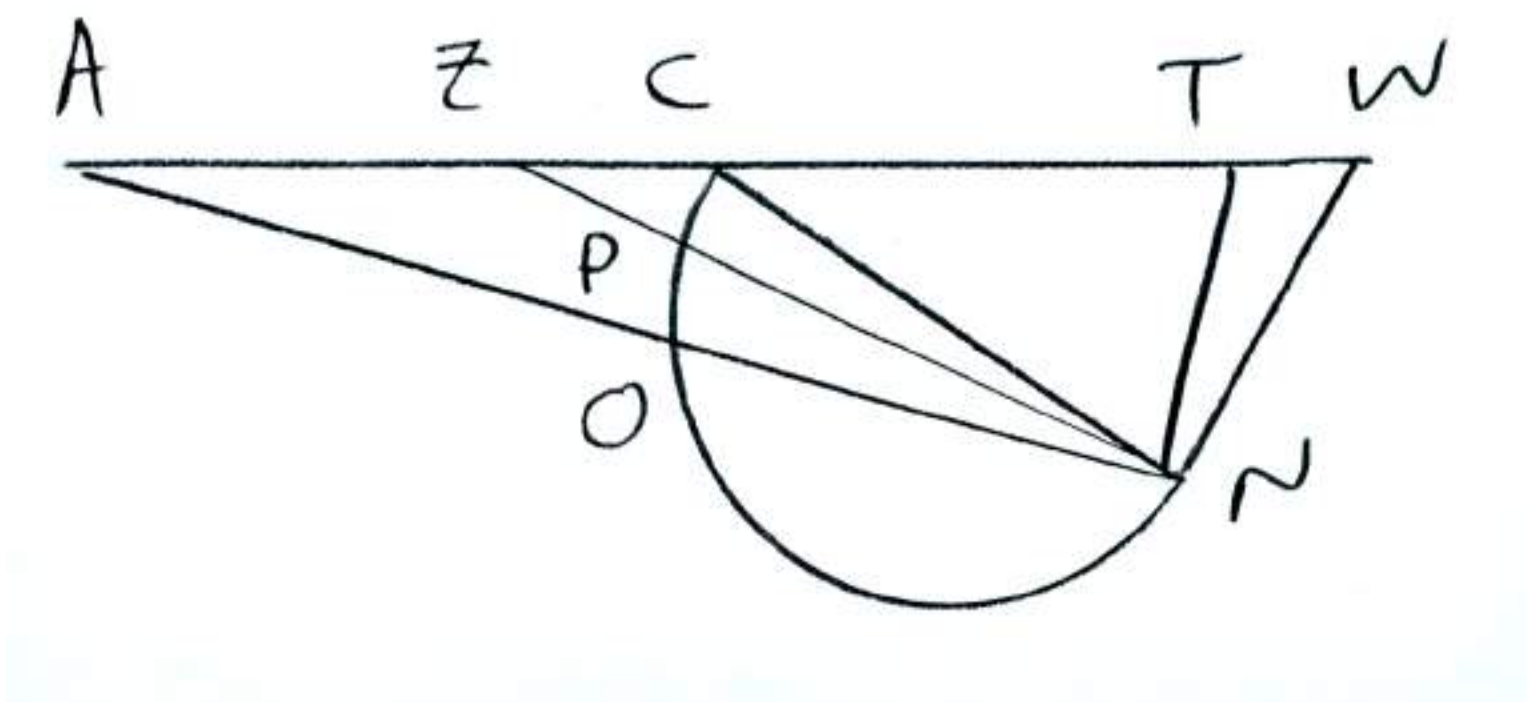


The off-axis rays from any real on-axis object A can not form a real on-axis image at Z because NW must be greater than (or equal to) CP for Z to be real; but NW must also be greater than NT.





The off-axis rays from a virtual on-axis object A *can* form a real on-axis image at Z, if NW is greater than CP, and WT lies along the axis.



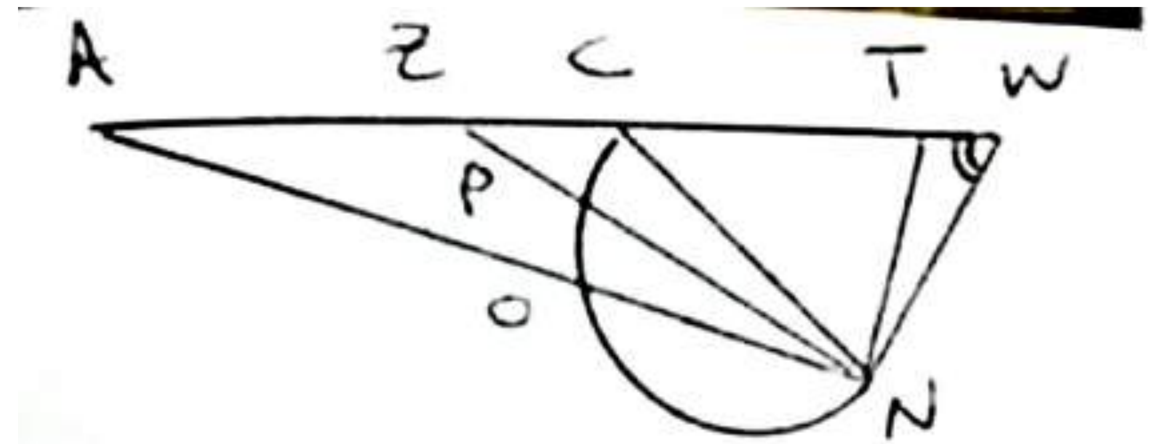
Since:

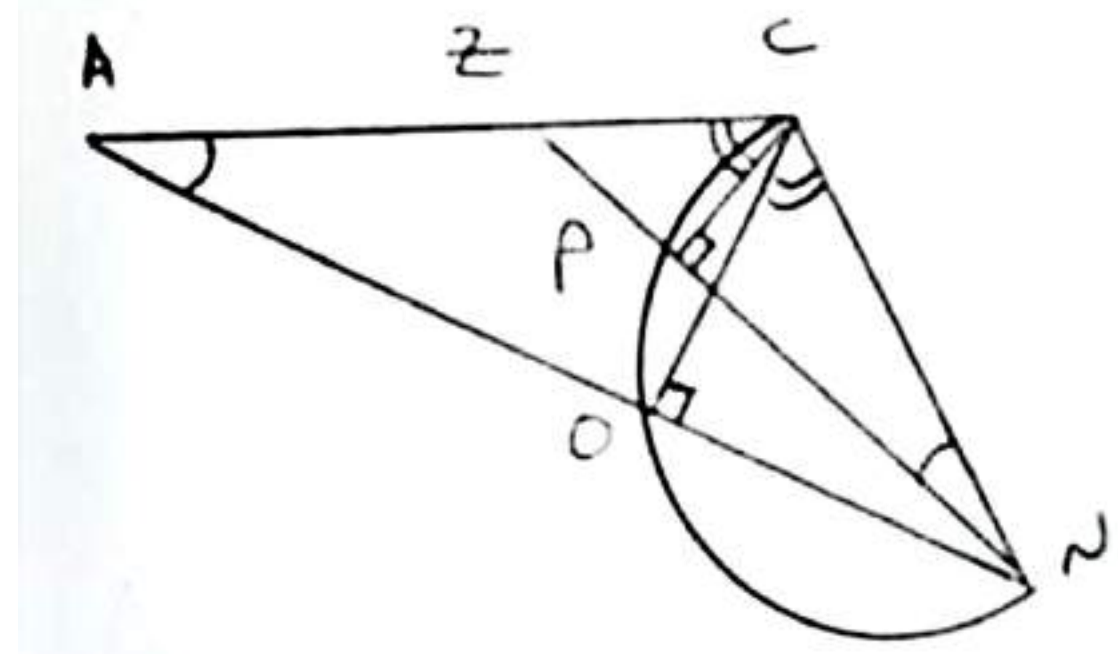
∠NWT = ∠NPO = ∠NCO

and NW || CP

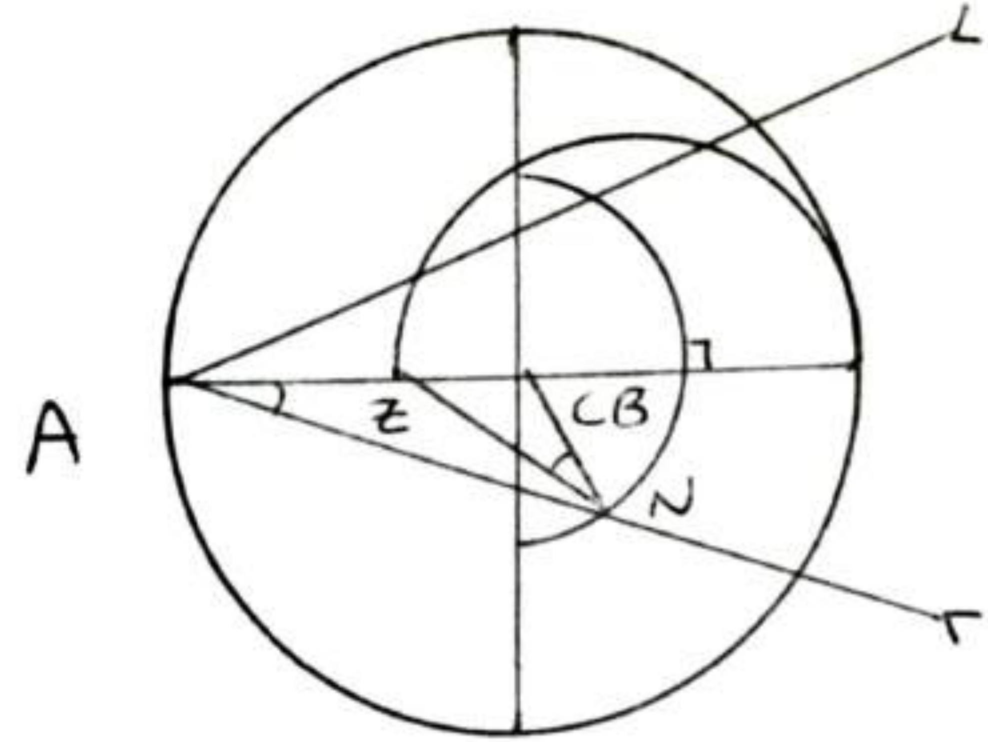
WT lies along the axis when:

ΔNCO ≅ ΔZCP





When off-axis rays from a virtual on-axis object A form a real on-axis image Z, this occurs at all points N because:



ΔACN ≅ ΔNCZ for all N,

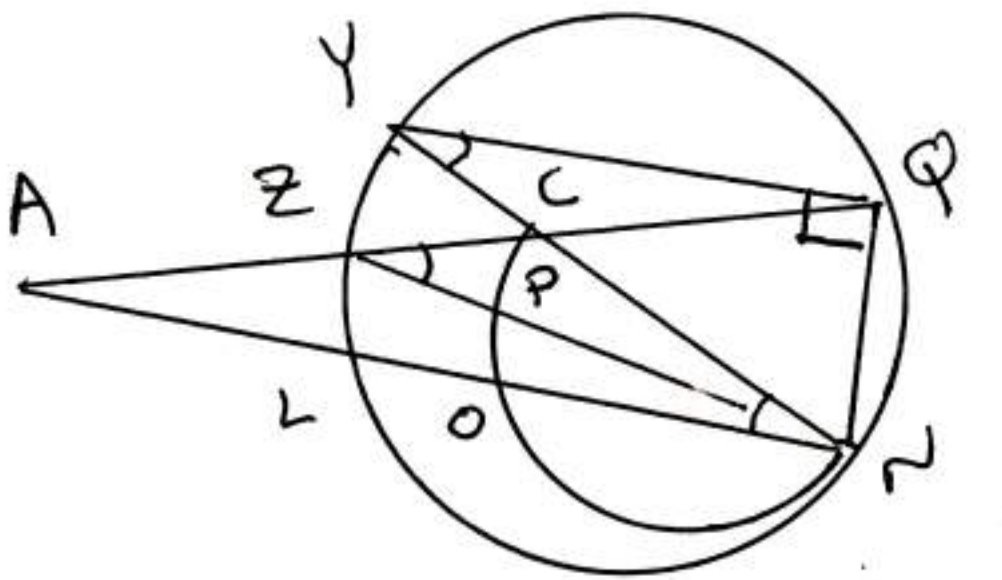
(since they share proportional sides around a common angle).

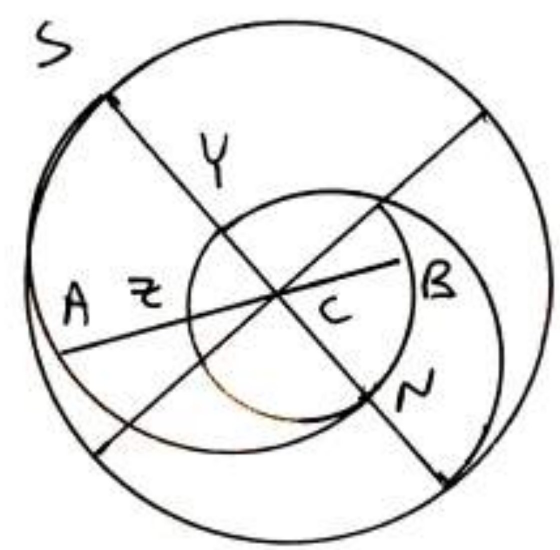
This can also be demonstrated using similar right triangles:
ΔSAN ≅ CON, and ΔYZN ≅ ΔCPN,

so that: (AO/AN)(ZN/ZP) = (SC/SN)(YN/YC).

Since: CY/CN = CN/CS = (CY + CN)/(CN + CS) = NY/NS
(SC/SN) = (NC/NY), and:

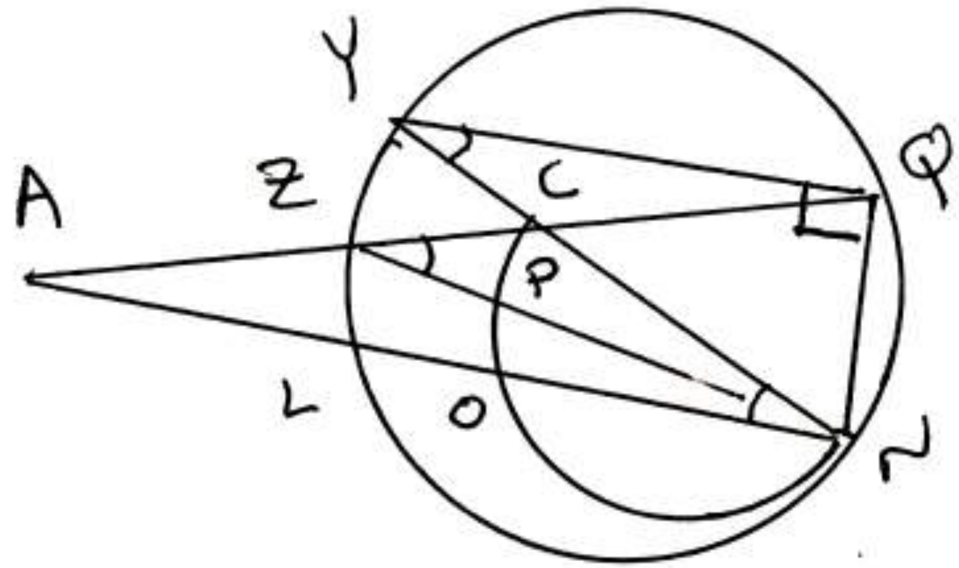
(AO/AN)(ZN/ZP) = CN/CY

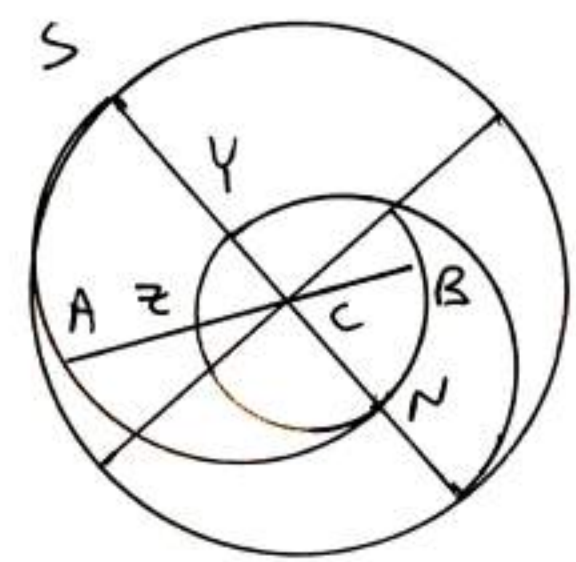




But it is also true that:
(CO/CP)(NP/NO) = CN/CY, because:

(CO/CP)(NP/NO) = (LY/LN)(PN/PC) =
= (QN/QY)(PN/PC) = (QN/QY)(ZN/ZY) =
QN (ZN)/QY(ZY) which, by the property of cyclic quadrilaterals shown in slide #7, equals CN/CY





Keeping:

$\mathbb{R}$ = (CO/CP) = (NO/NP)(AO/AN)(ZN/ZP)

constant, as N $\Rightarrow$ B:

(BC/BC)(AC/AB)(ZB/ZC) $\Rightarrow$ $\mathbb{R}$

Refraction Through a Circle's Center

(Axial Refraction)

Refraction through a circle’s center occurs when N lies at B, so that an object’s ray from A to N lies along ABC, and an image ray lies along BCZ. The locations of the object A and image Z along the optic axis BC are described by the equation:

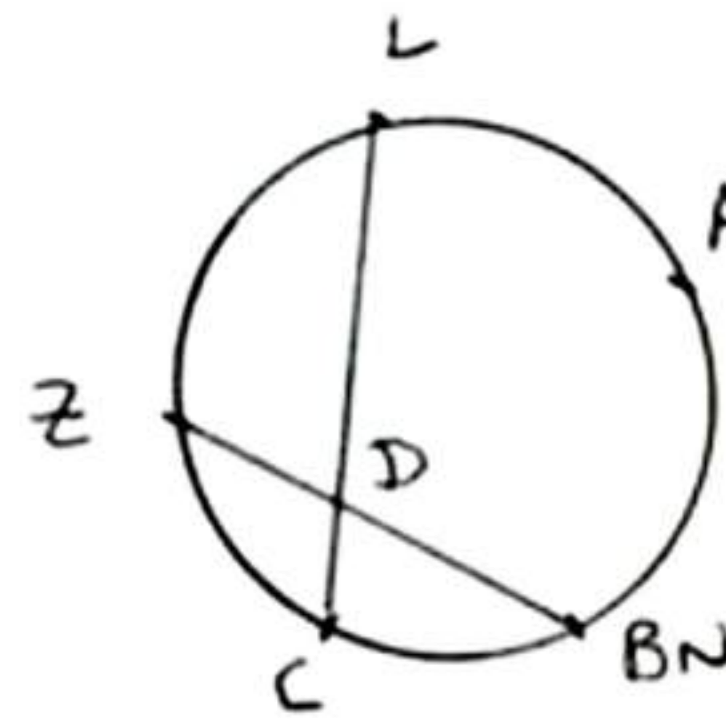
$$\mathbb{R} = CO/CP = (AC/AB)(ZB/ZC)$$

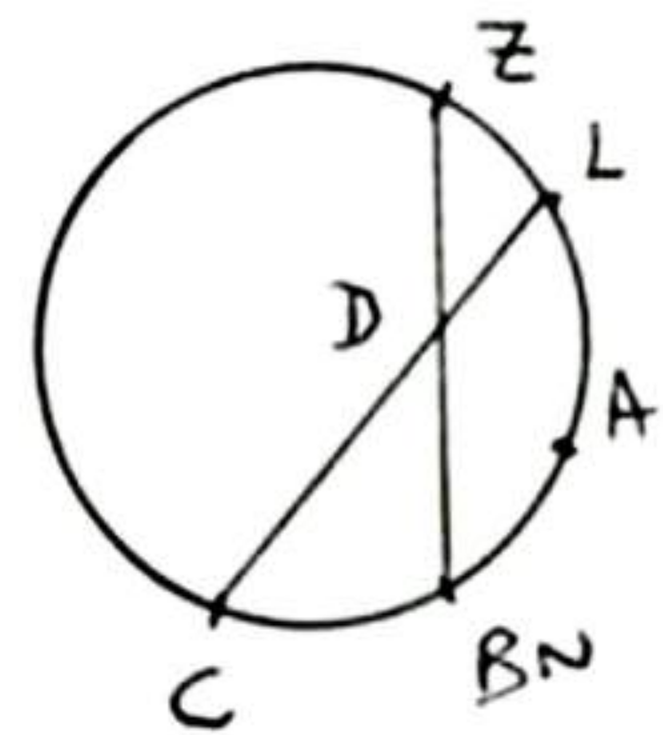
If we draw A and Z along the optic axis BC *as if* it were a circle, and draw CDL so that AL || ZB:

$\Delta ACB \cong \Delta ZCD$, and:

(AC/AB)(ZB/ZC) =
(ZC/ZD)(ZB/ZC) =
(ZB/ZD)
so as the reference circle's radius $\Rightarrow \infty$,

(ZB/ZD) $\Rightarrow \mathbb{R}$



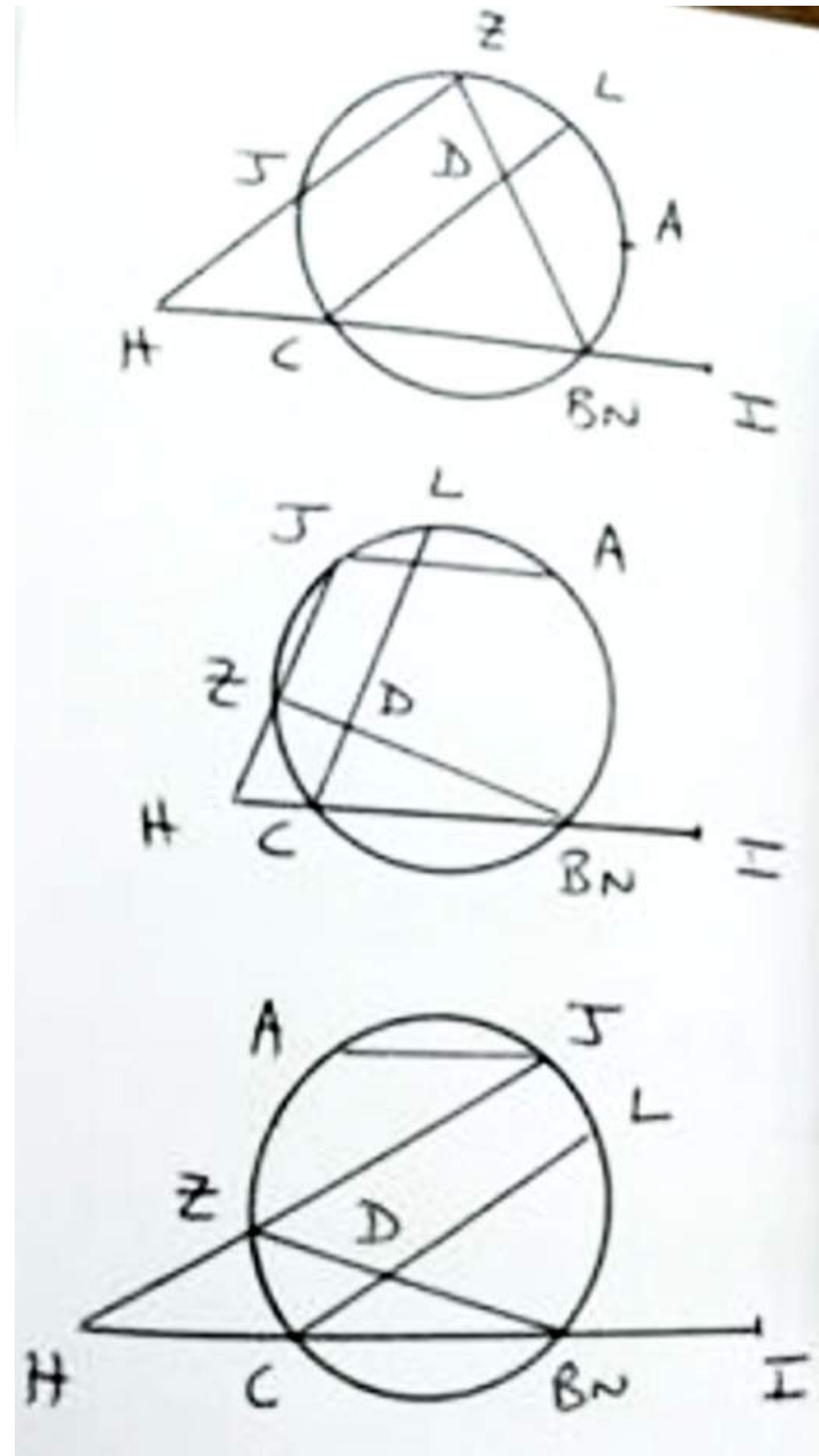


AL ll ZB
AZ = BL
~AZ = ~BL

HZ ll CL
ZC = LJ
~ZC = ~LJ

~AZ + ~ZC = ~AZC
~BL + ~LJ = ~BLJ

~AZC = ~BLJ
AJ ll CB



HZ ll CL

ZB/ZD = HB/HC

ΔHBZ ≅ ΔHJC

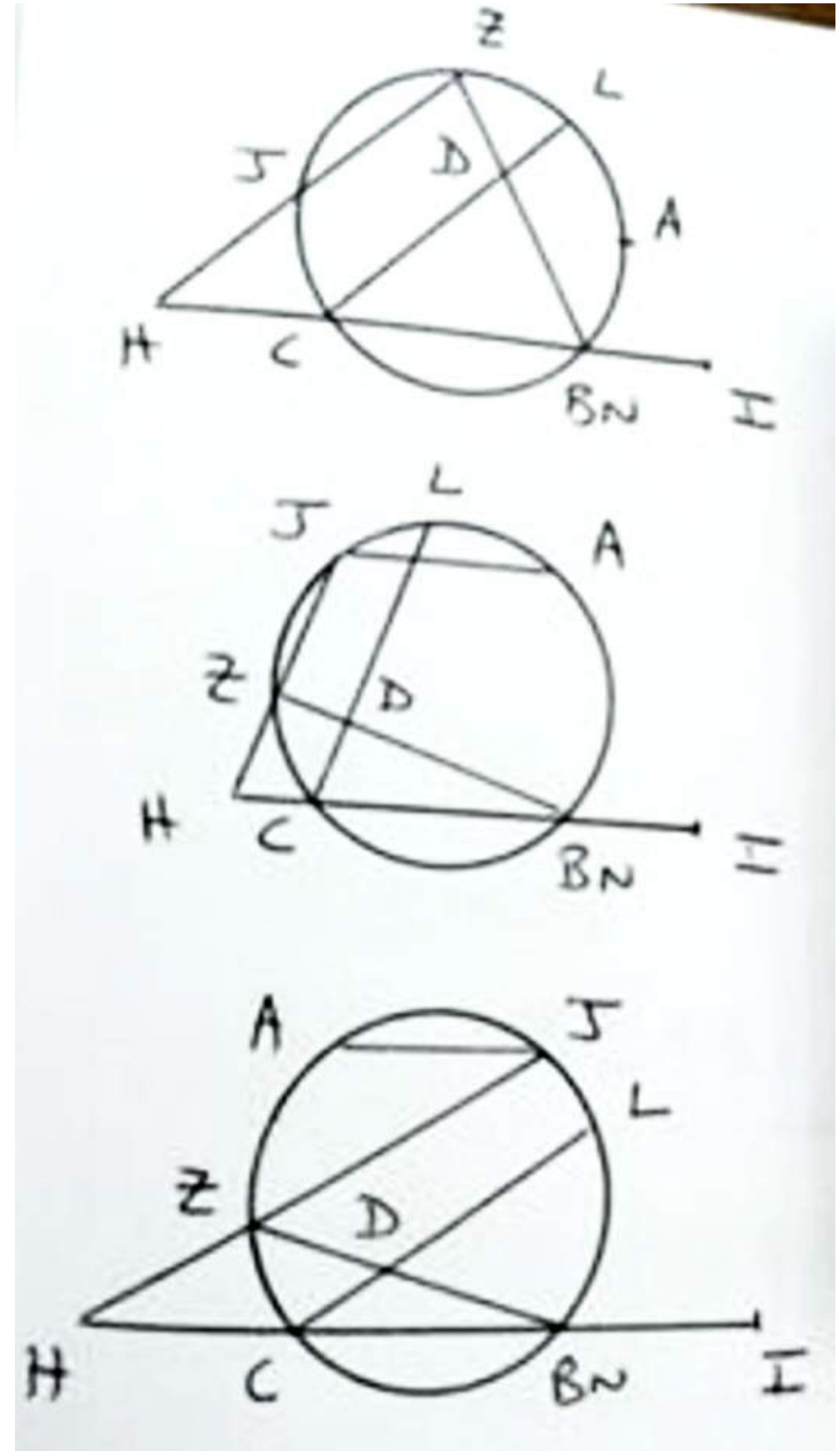
when ΔHJC = ΔIAB:

HC = IB, and:

IB/IA = HZ/HB

This results in
Newton's Equation:
as the reference circle
radius ⇒ ∞,

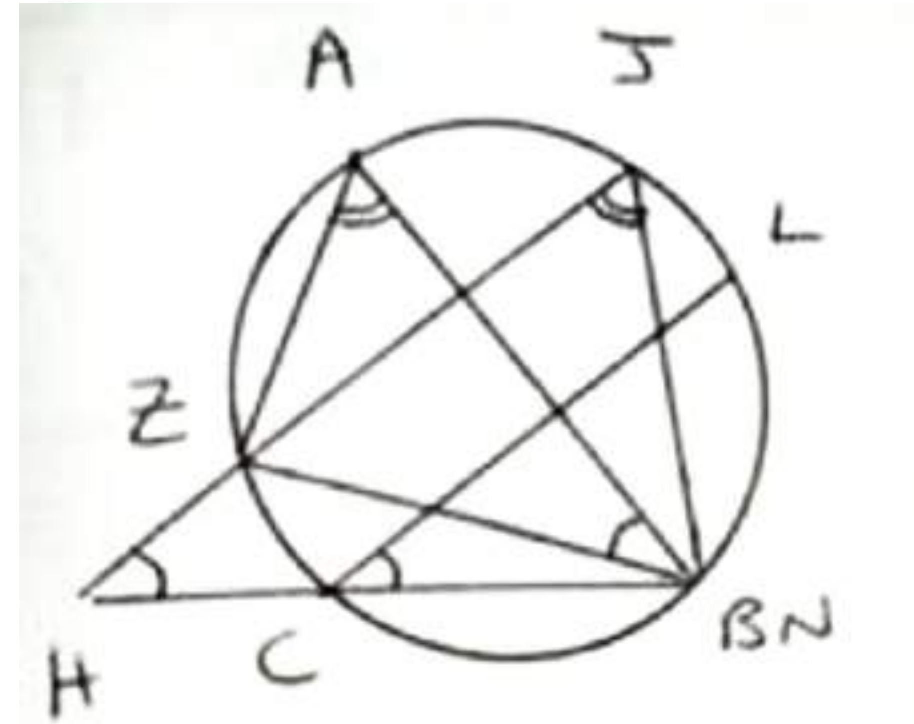
(AI)(ZH) = (BI)(BH)

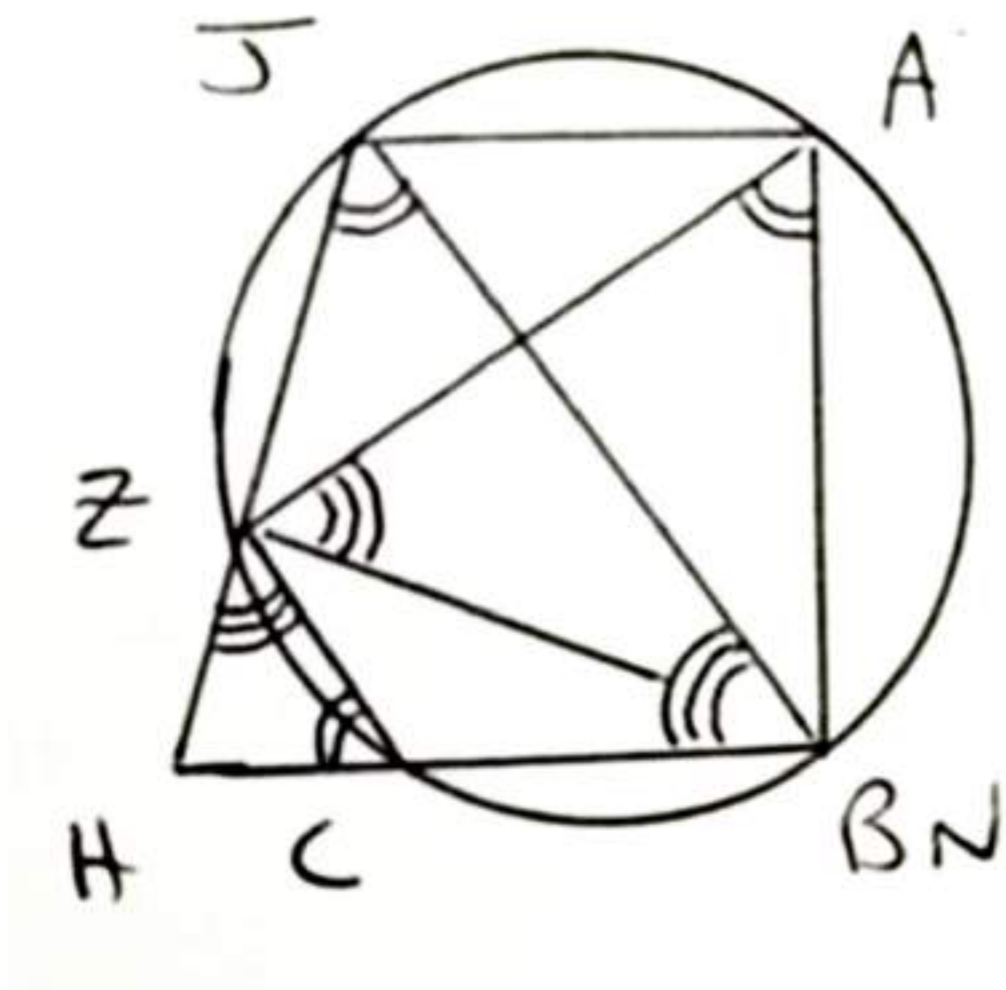


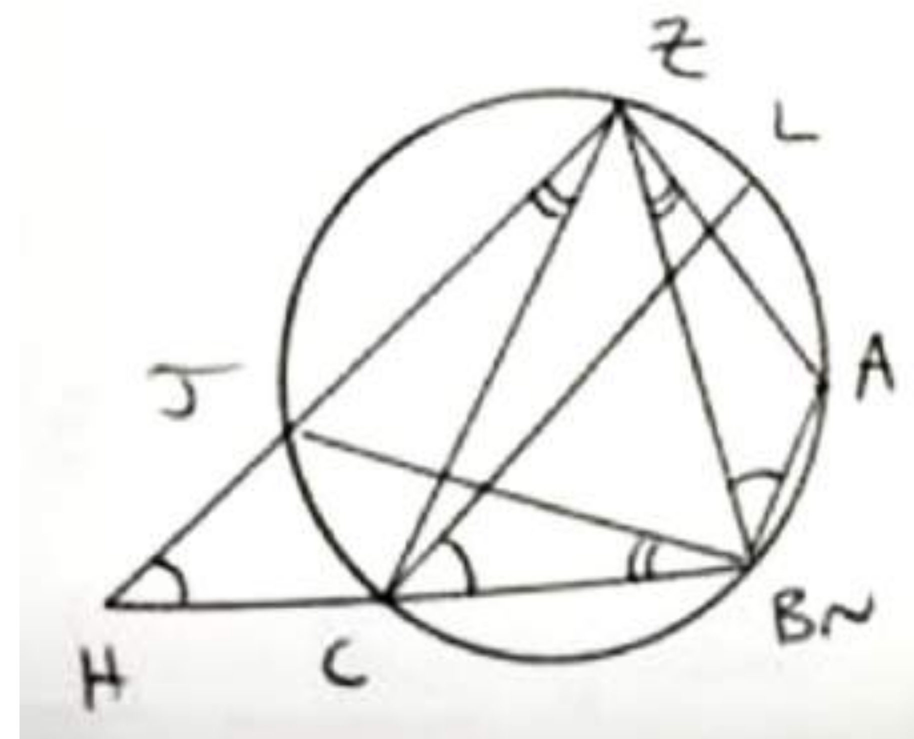
$\Delta HCZ \cong \Delta HJB \cong \Delta BAZ$

$(HC/HZ) = (BA/BZ)$

$[1/(HZ)(BA)] = [1/(HC)(BZ)]$







as the reference circle's radius ⇒ ∞,

[1/(HZ)(BA)] = [1/(HC)(BZ)] ⇒ ℝ/(HB)(BZ)

and the resulting possible sums occur:

HZ = HB + BZ
HB = HZ + BZ
BZ = HZ + HB

which, when multiplied by the above three factors, form the conjugate foci equations.

The conjugate foci equations allow for the effect of axial refraction at a circle to be expressed as the term:

$$(1/HC) = (\mathbb{R}/HB)$$

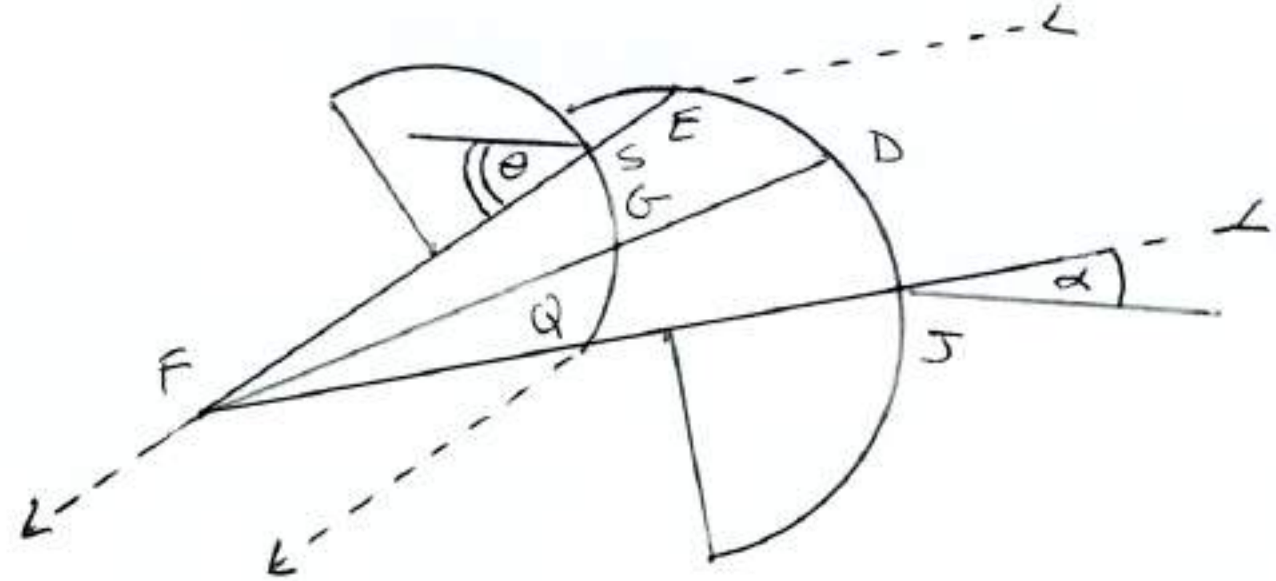
which is then additive with object vergence, defined as $(1/BA)$; or image vergence, defined as $(\mathbb{R}/BZ)$.

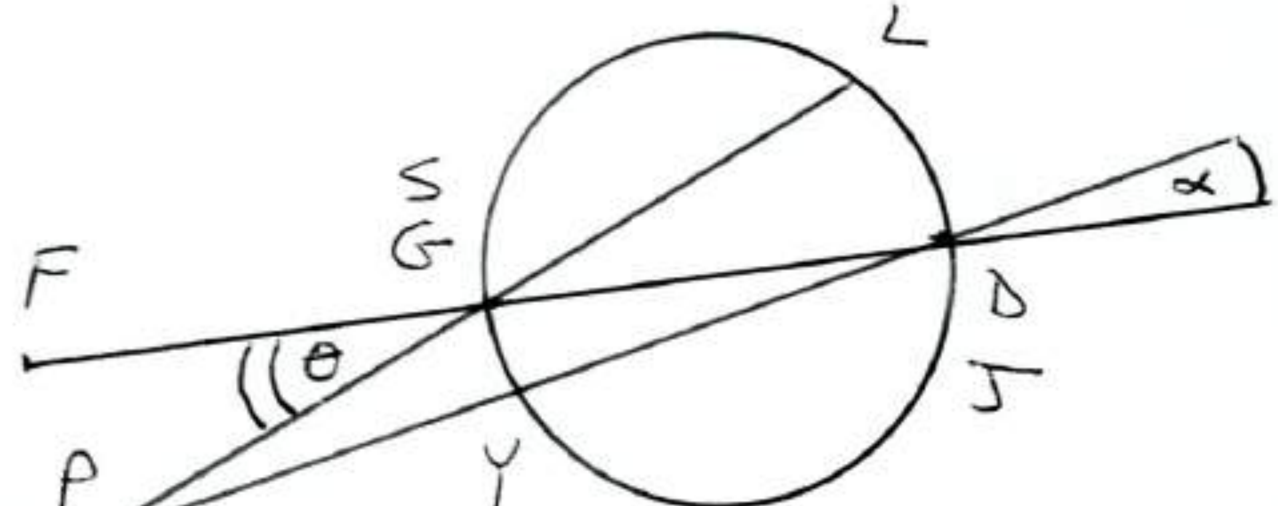
Afocal Angular Magnification/Minification

When off-axis distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis DGF as shown;

as ∠JFD = ∠SFG,

and both approach zero:

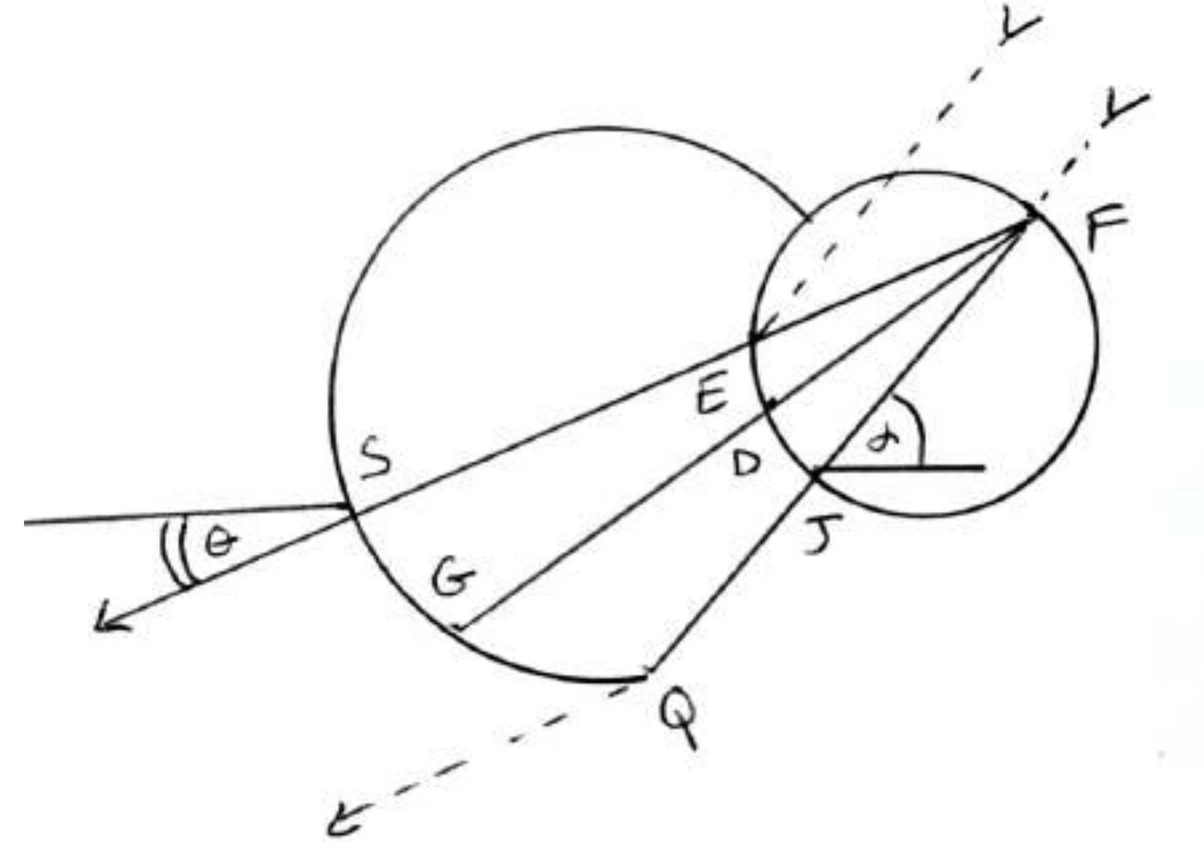


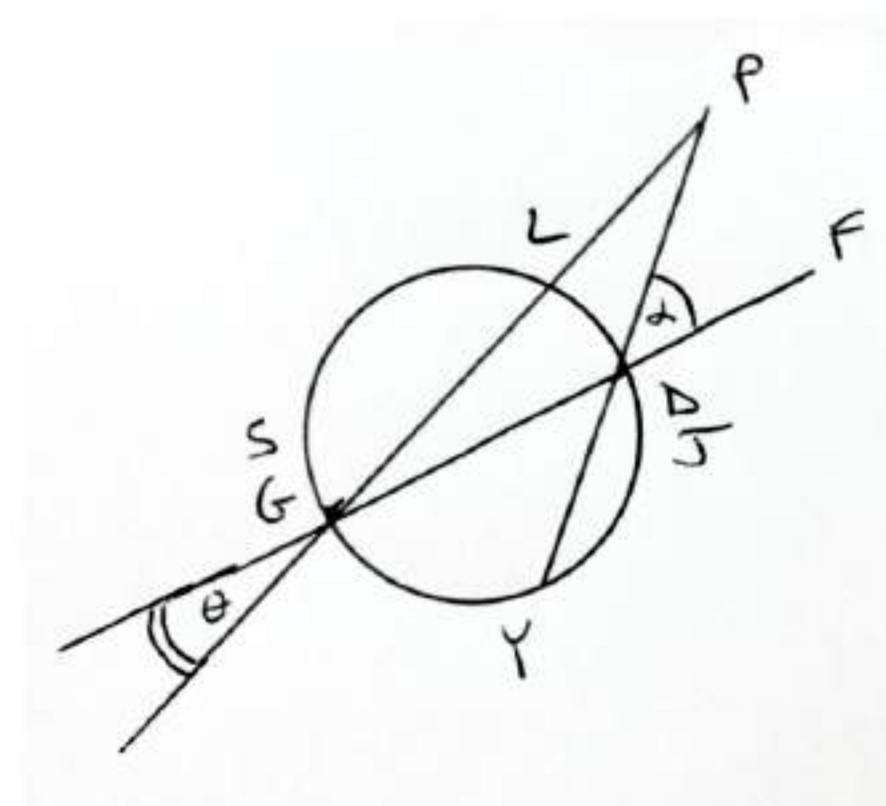


Or when off-axis distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis FDG, as shown;

as ∠JFD = ∠SFG,

and both approach zero:





$\theta/\alpha \Rightarrow$ (~LD/GD)/(~YG/GD) as P $\Rightarrow$ F

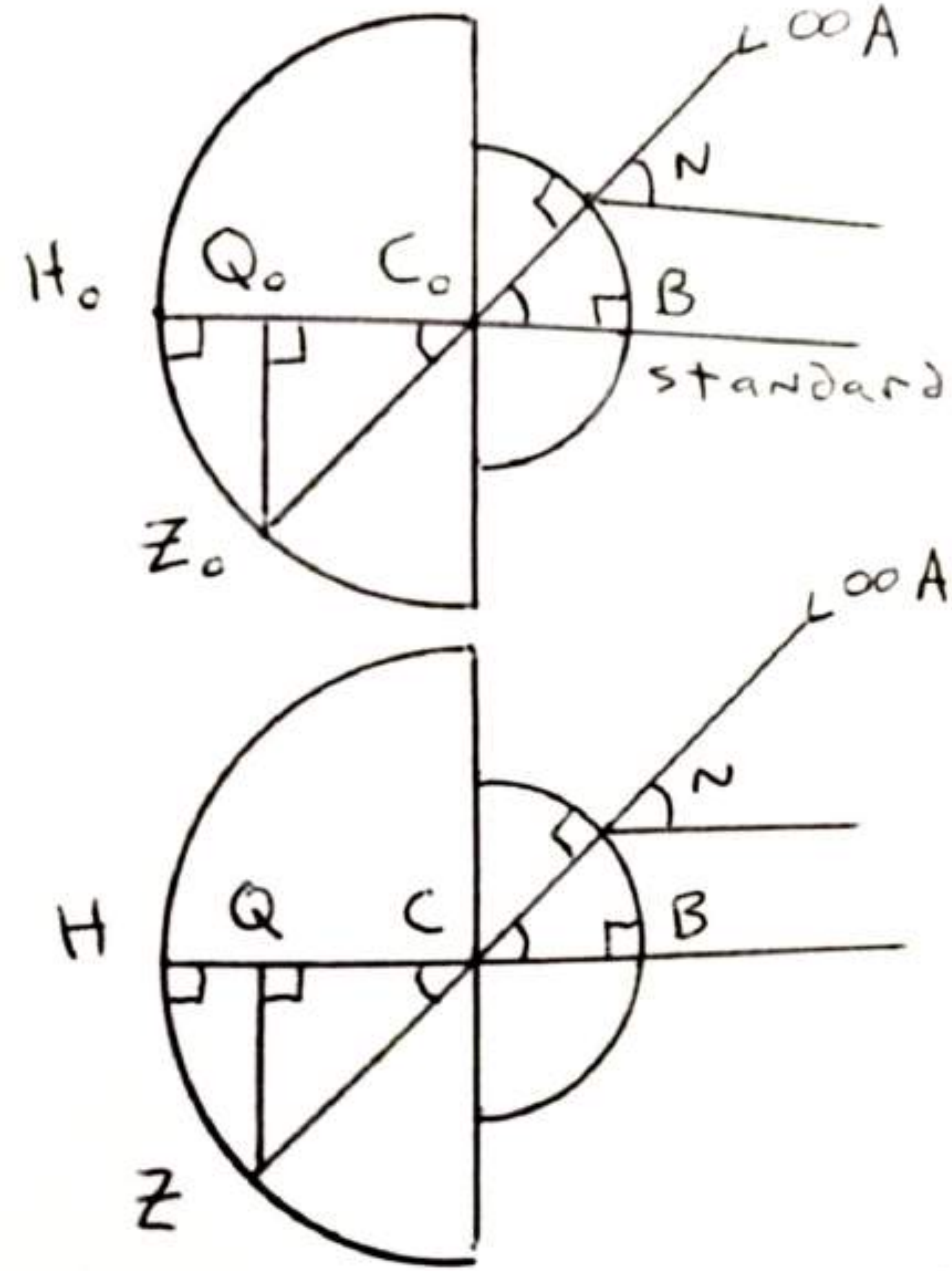
$\theta/\alpha \Rightarrow$ (FD/FG) as P $\Rightarrow$ F

so that afocal axial angular magnification/minification equals:

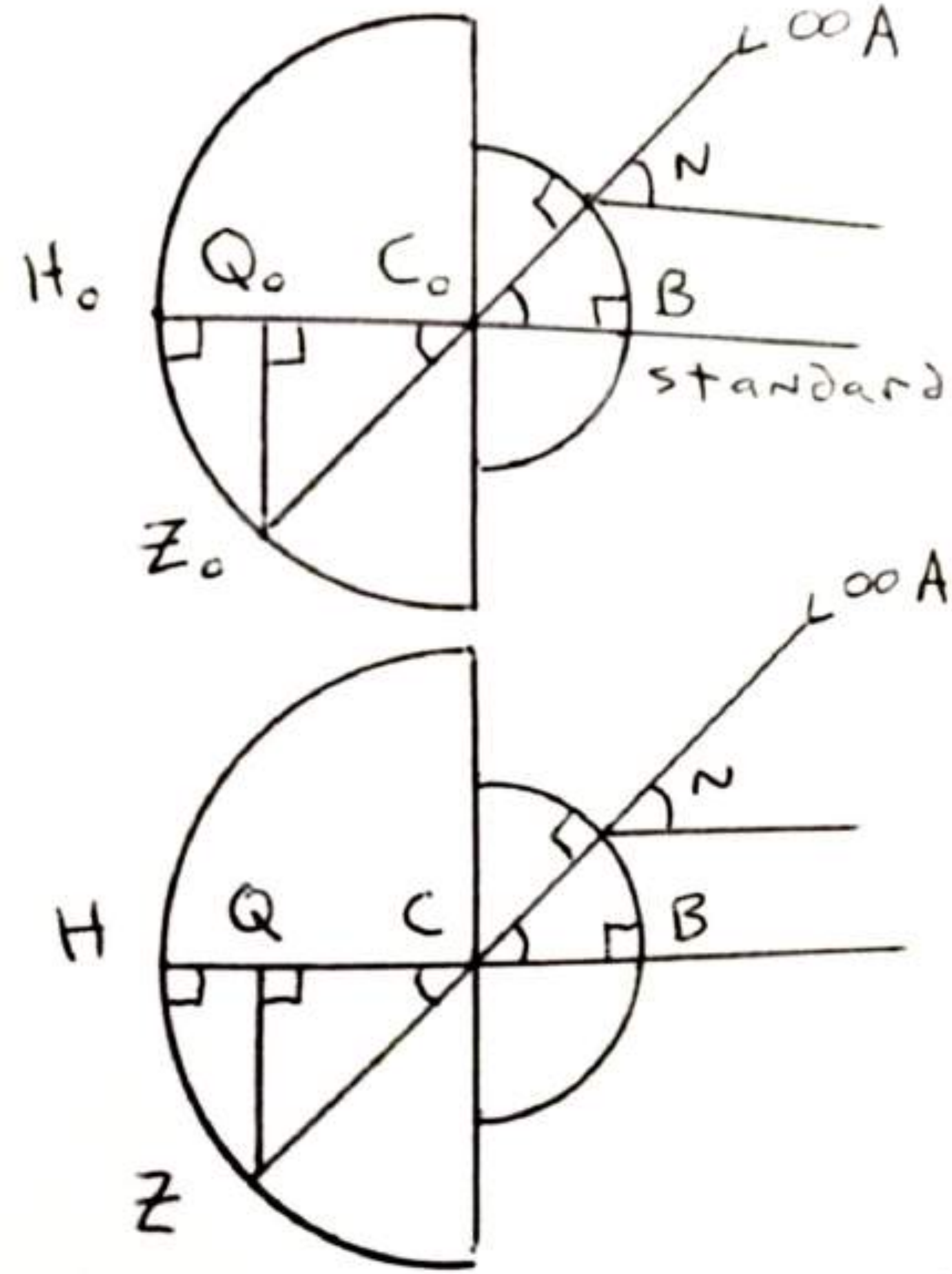
FD/FG

Retinal Image Size
Magnification/Minification

The top diagram illustrates a standard single-surfaced eye with a distant object A, and resulting retinal image size H_oZ_o.



The bottom diagram illustrates any single-surfaced eye with a distant object A, and resulting retinal image size HZ.



As N $\Rightarrow$ B, the retinal image size magnification, $\sim ZH/Z_oH_o$, approaches its axial value:

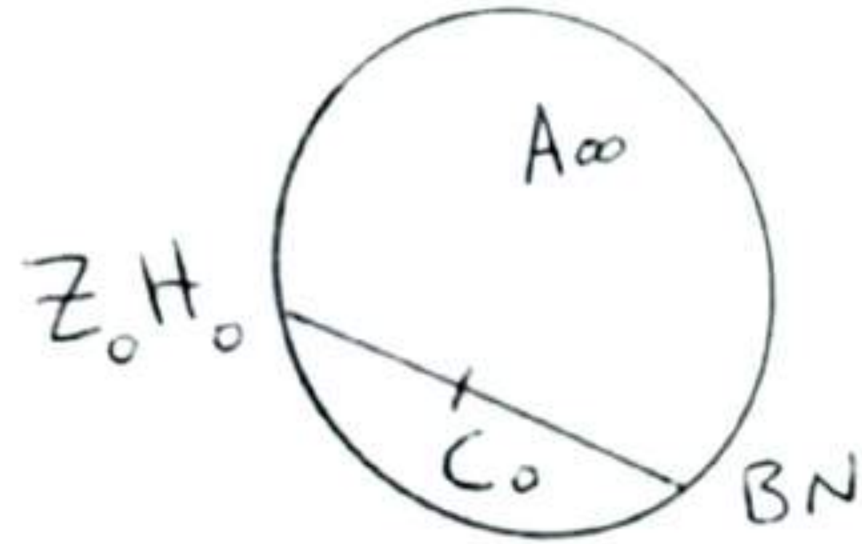
$$ZQ/Z_oQ_o = ZC/Z_oC_o = HC/H_oC_o$$

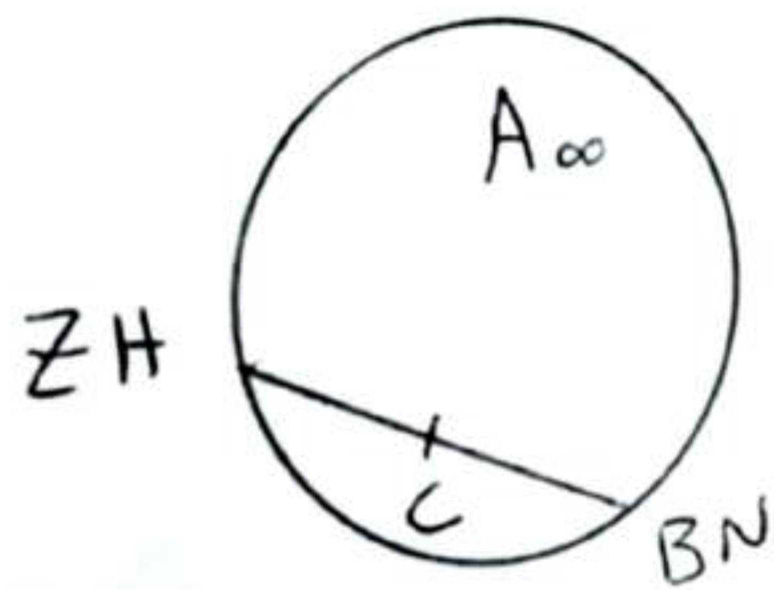
$$= (BH/\mathbb{R})/(BH_o/\mathbb{R}) = BH/BH_o$$

The retinal image size magnification for eyes with single refracting surfaces will factor out standard values when comparisons are made between non-standard eyes, and this is clinically valid to the degree that optical components within the eyes introduce no magnification differences.

Distance Correction
Magnification/Minification

Once again representing the optic axis BCZ as a circle of infinite radius, the distant object A at ∞ is focused by the radius BC of the presumed single refracting surface towards the axial image Z, which lies at the retina H when there is no distance refractive error. (BH_o represents the standard axial length, and BC_o represents the standard single refracting curvature radius).



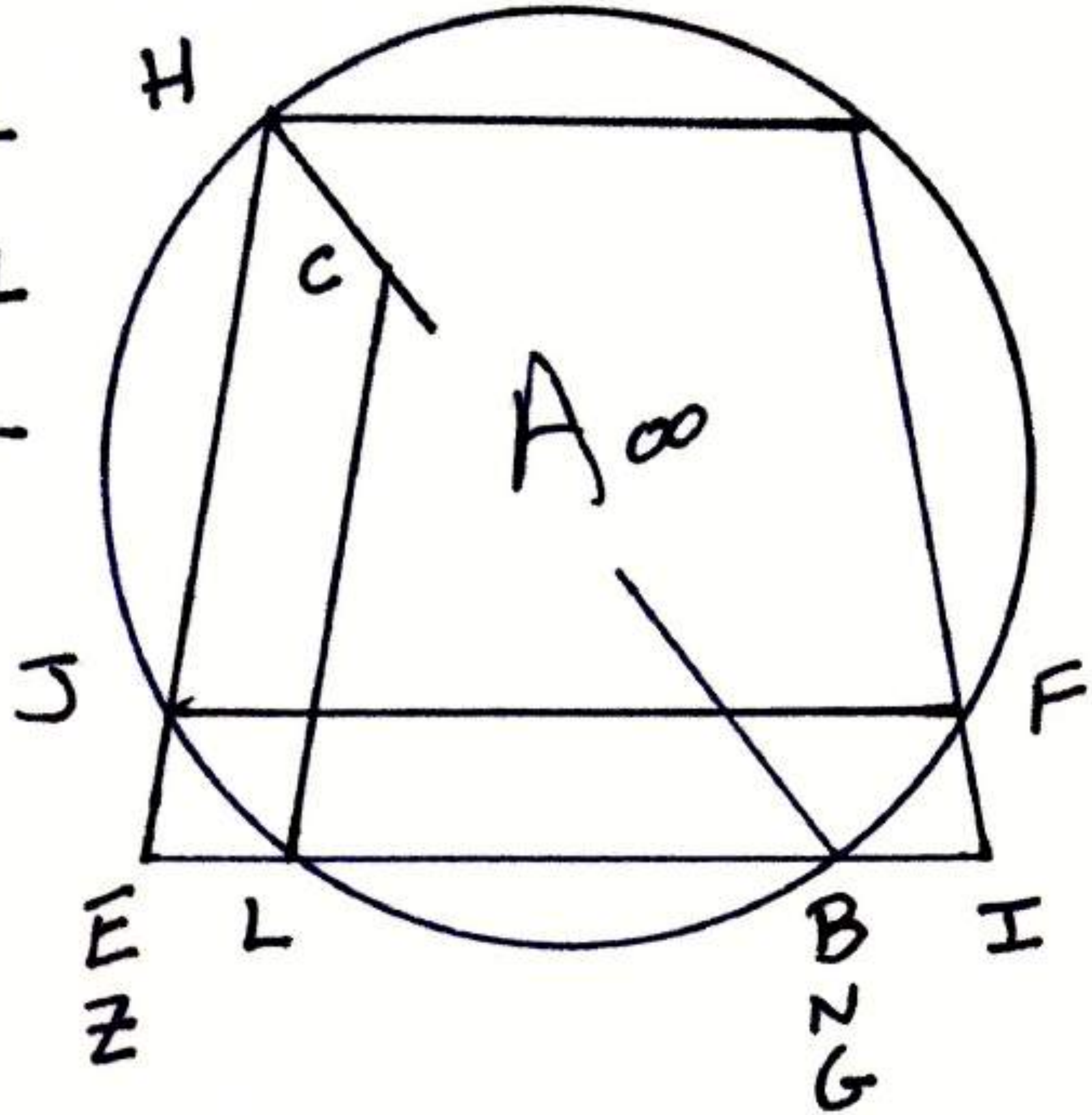


As pictured in the next three slides, additional refraction G (at B) will create an "ametropic" eye, with "distance refractive error," and a combination curvature effect with total radius BL instead of BC, moving image Z from the retina at H to its erroneous location at E. The "front focal point" of the "ametropic" eye is defined as point I.
A "distance correction" must focus the distant object towards F, so that JF || BL, in order to move image Z back to the retina H.

$JF \parallel BL$

$HZ \parallel CL$

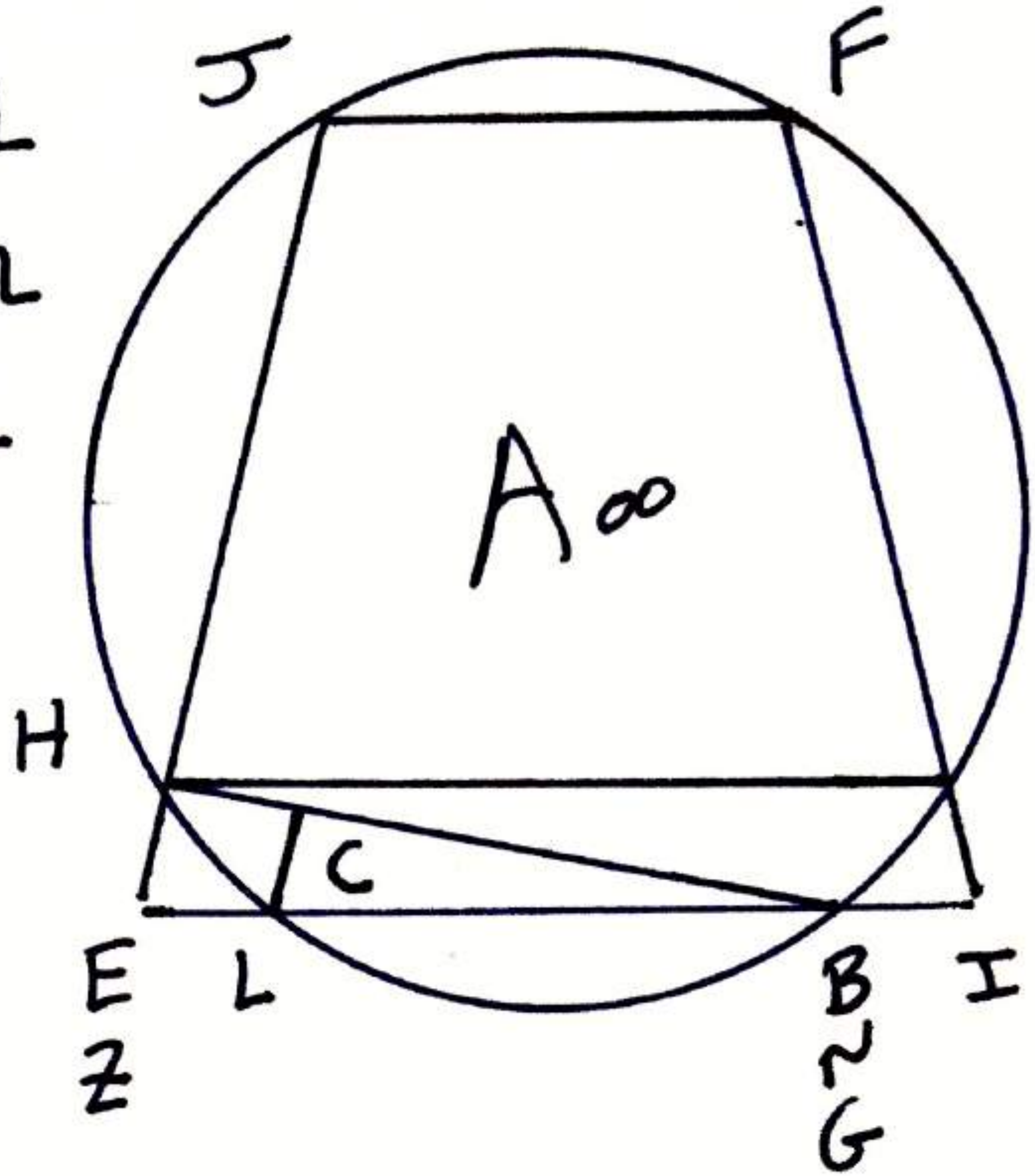
$BI = EL$

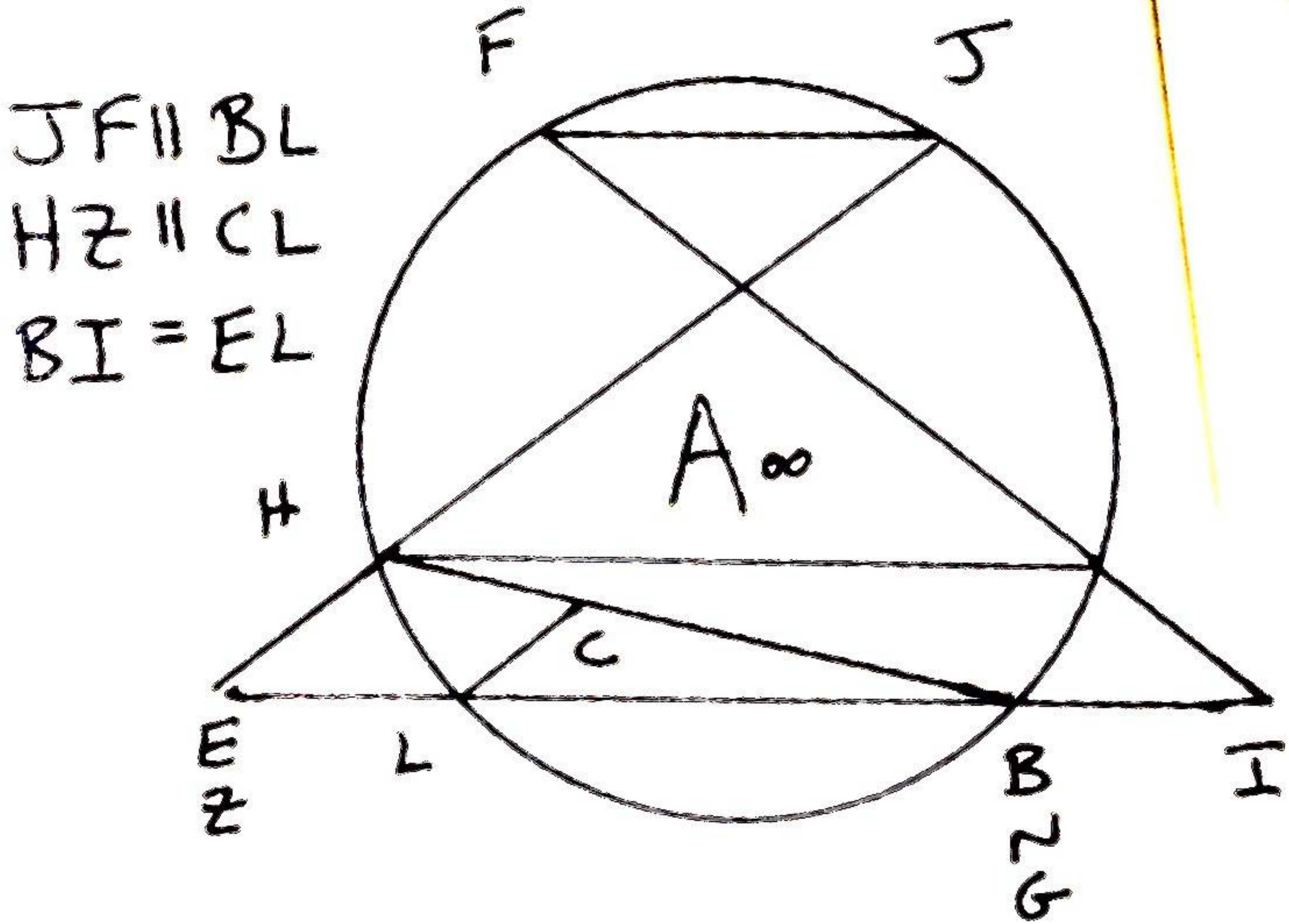


JF || BL

HZ || CL

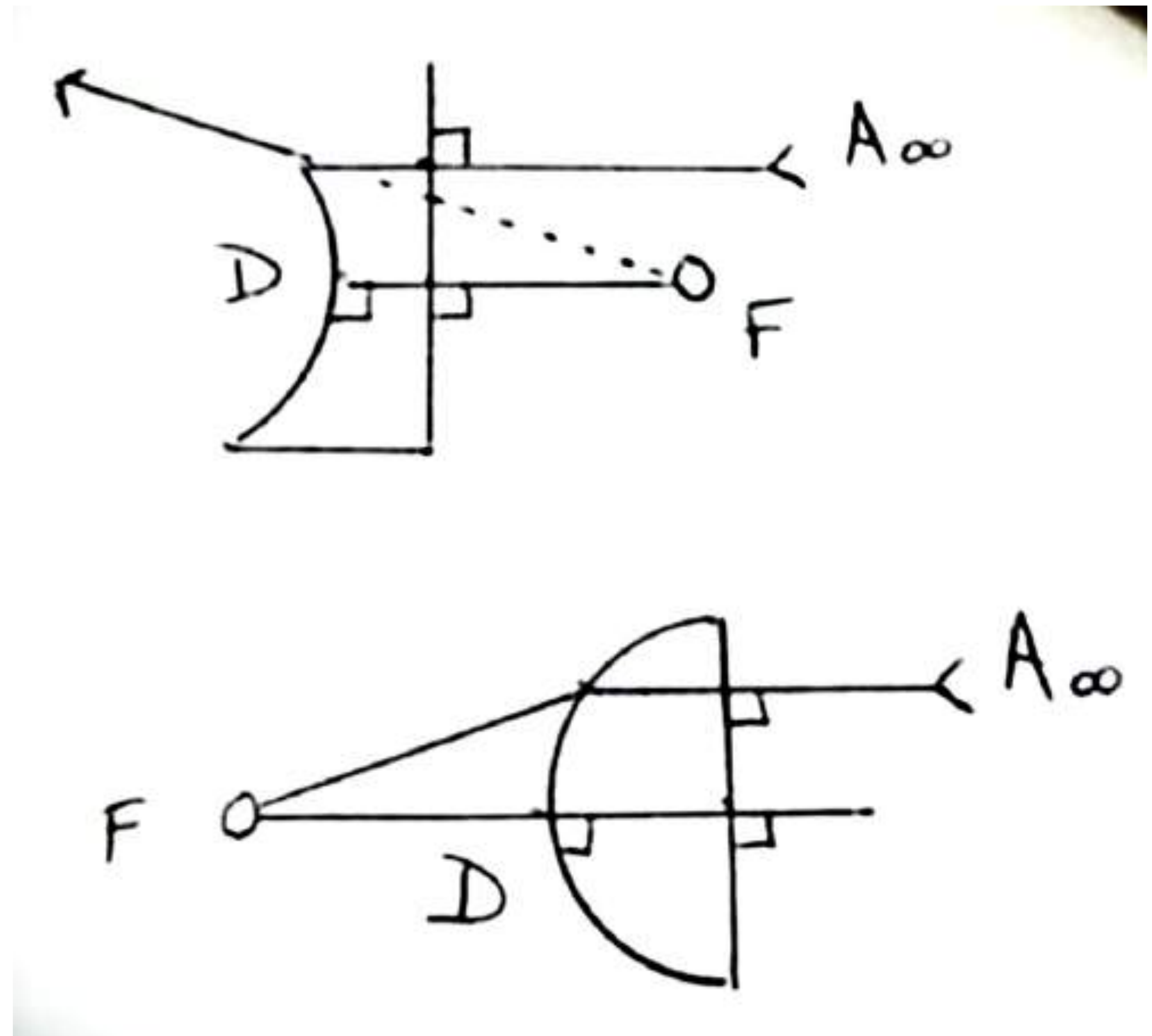
BI = EL





JF ll BL
HZ ll CL
BI = EL
F
J
A∞
H
C
E
Z
L
B
~
G
I

The distance correction at D:



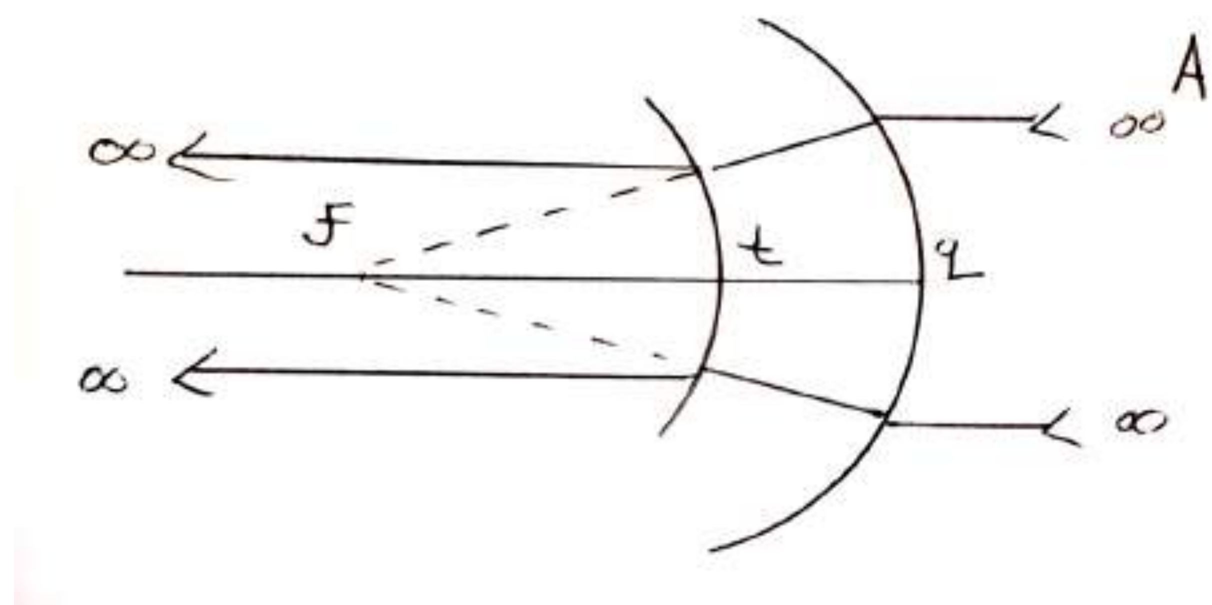
Since the distance correction D moves image Z from E to retina H, rays leaving the refractive error G (at B) after this correction is in place must be afocal. This results in afocal axial angular magnification equaling:

FD/FG (= FD/FB)

Therefore, the total axial magnification of distance correction equals:

$M = (BH/BH_o)(FD/FB)$

When the front surface of a spectacle lens that corrects distance refractive error is not flat, it is convex; and adds an additional “shape” factor, (fq/ft), to the afocal axial magnification of distance correction. (Point “t” lies at D, and FD/FB remains the “power” factor of the afocal axial magnification of distance correction).



“Axial Ametropia” occurs when E is at H_o, (and point I is therefore at I_o, the front focal point of the standard eye). The distance refractive error is then completely due to an axial length BZ, (or BH), that is not standard.

$$\Delta H_oBH \;=\; \Delta EBH \cong \Delta EJL \;=\; \Delta I_oFB$$

$$(BH/BH_o) \;=\; (FB/FI_o)$$

$$M \;=\; (FB/FI_o)(FD/FB) \;=\; FD/FI_o$$

Therefore, in the case of axial ametropia, there is no total axial magnification of distance correction if the correction D lies at I_o.

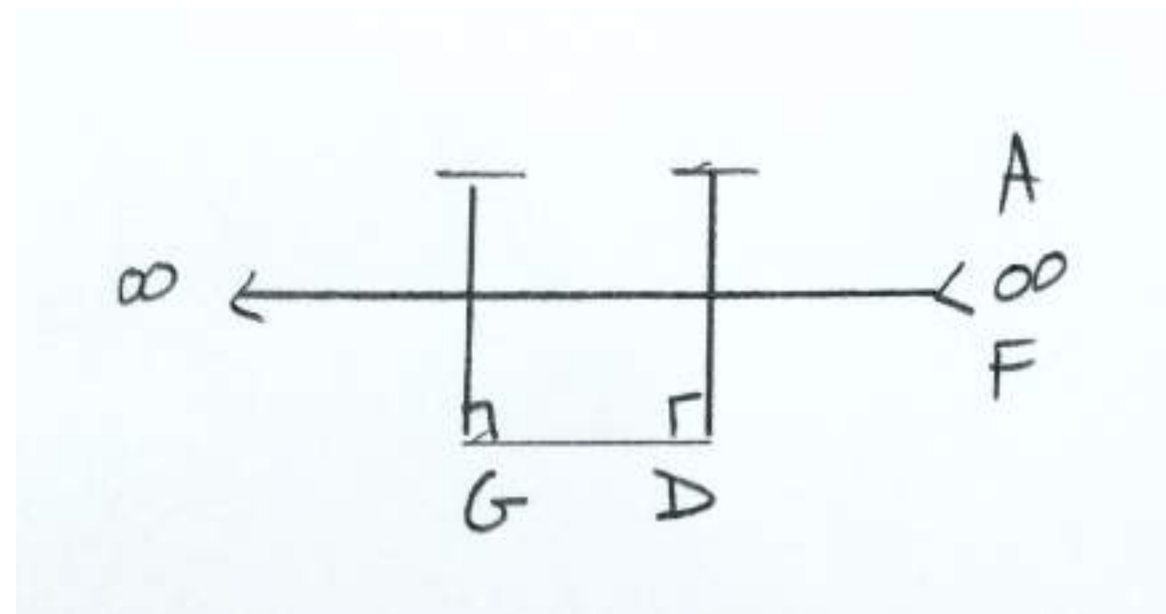
“Refractive Ametropia” occurs when the retina H is at at H_o. The distance refractive error at G moving image Z to E is then completely due to a refracting radius BL that is not the standard BC_o.

When the distance correction D lies at B:

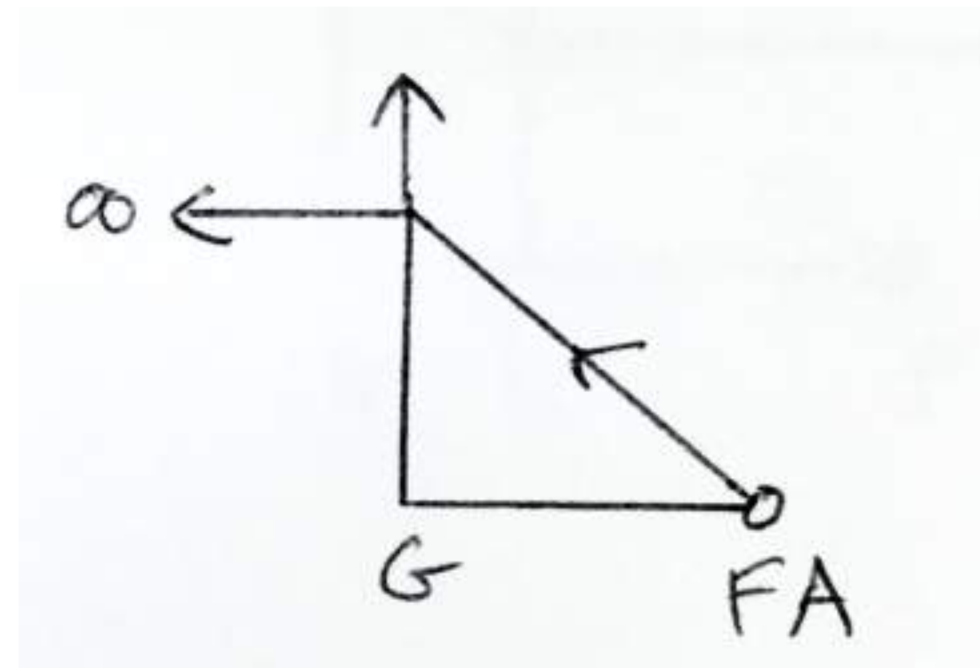
$$M = (BH/BH_o)(FD/FB) = 1$$

Near Correction Magnification

There is no afocal axial angular magnification of distance correction with a distant object "A," and an emetropic eye whose refractive error at G (at B) is by definition zero, (with its focal point F at infinity).

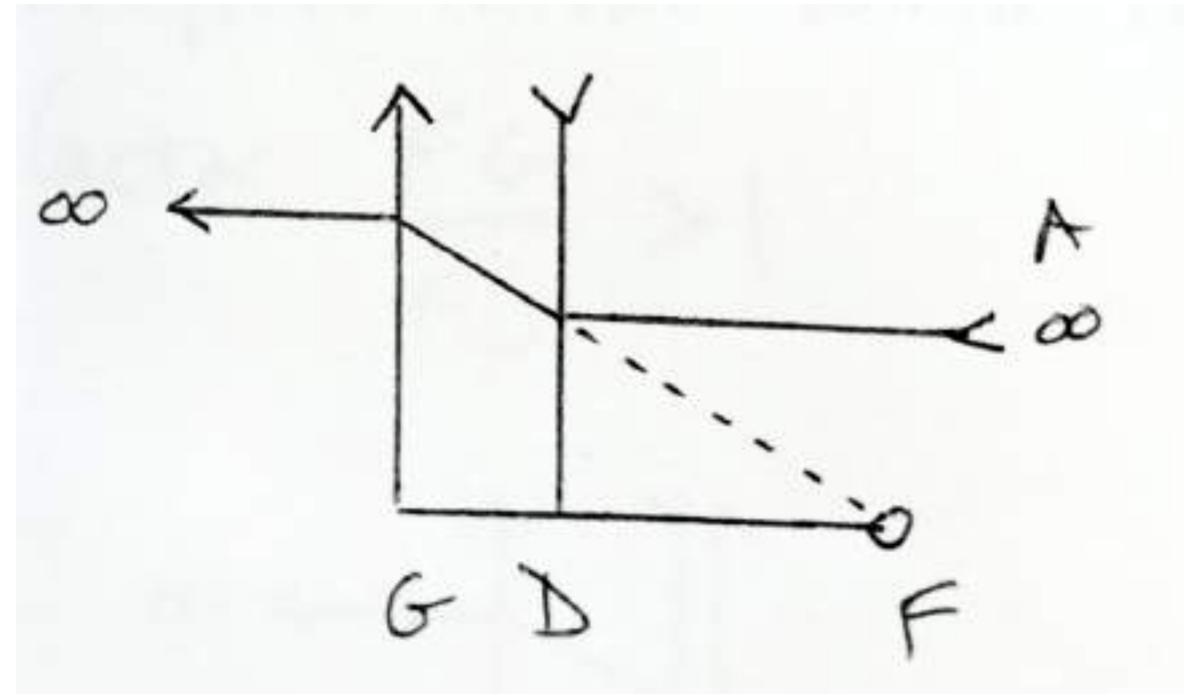


There is also no afocal axial angular magnification when object A is at the front focal point F of an uncorrected ametropic eye as shown, since this “myopic” system is not afocal, and involves only one refracting element G.



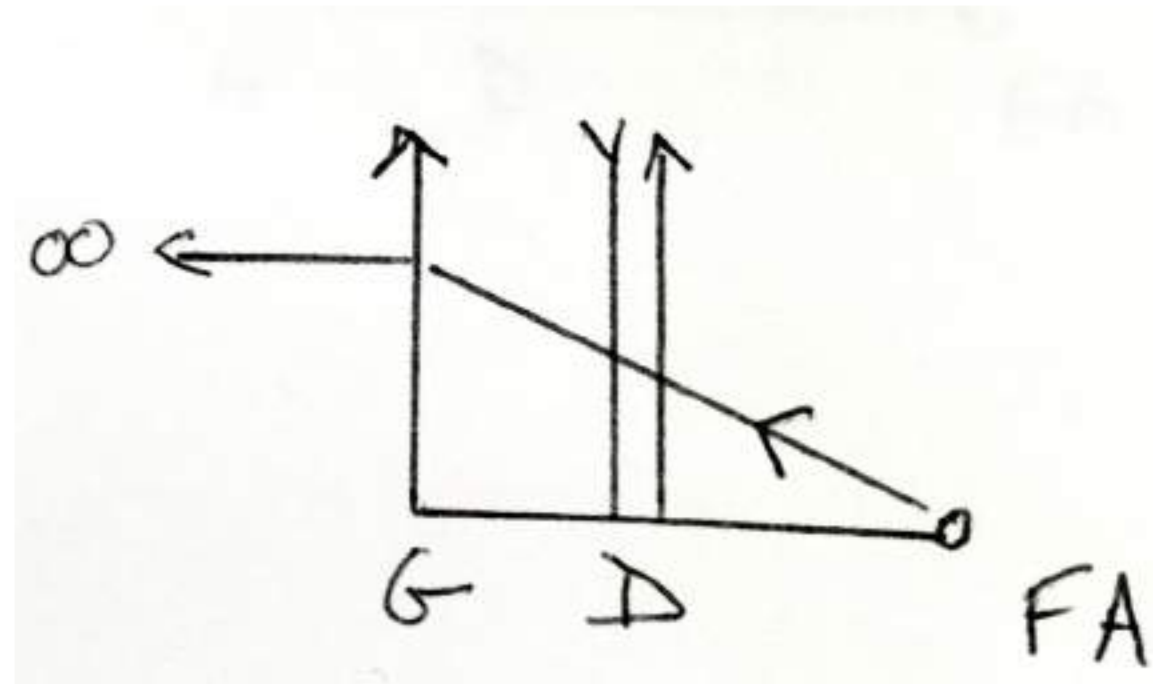
A distance myopic correction at D creates afocal axial angular minification:

FD/FG $<$ 1



and this is relative to either the myopic eye with object A at its front focal point F, or the emetropic eye with object A at distance.

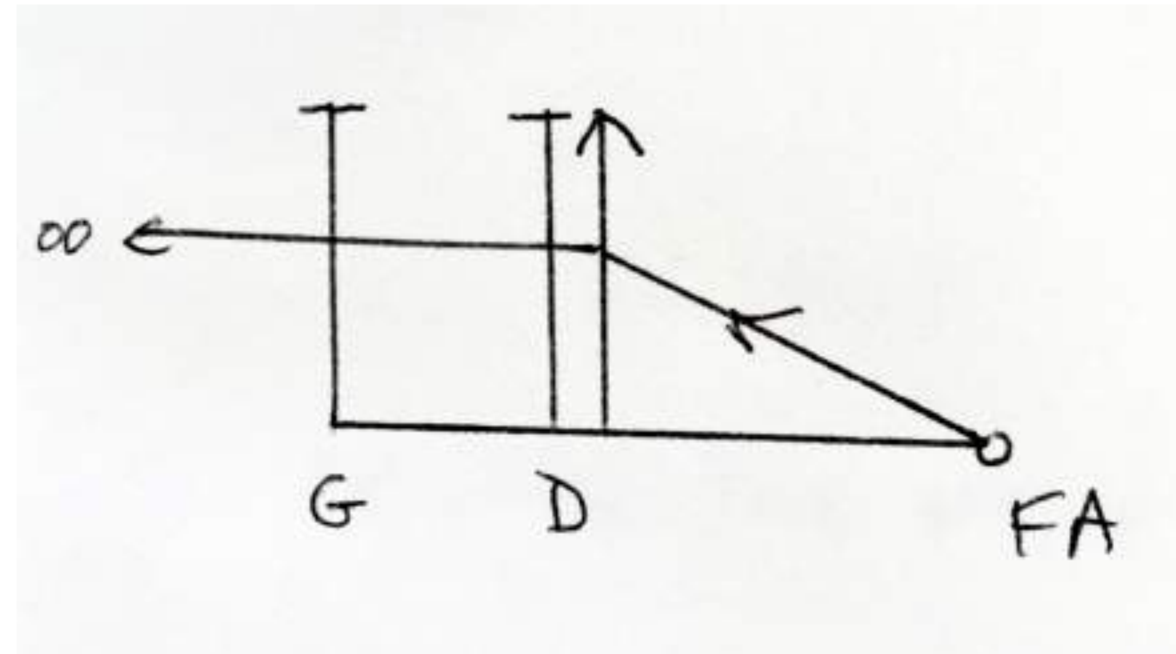
Removing the myopic distance correction at D with a converging lens at D removes this afocal axial angular magnification with the factor:



FG/FD > 1

and this magnification of near correction is relative to the distance corrected myope.

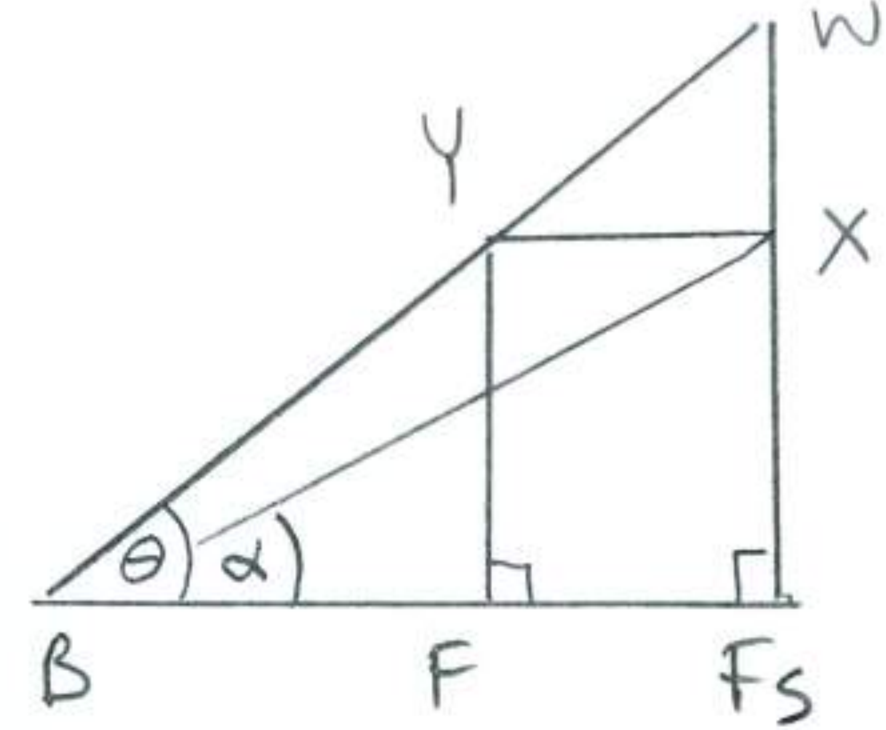
If additional converging power is added to the converging lens so that the near focal point is in focus for an emetropic eye, which we then consider to be the reference eye, the magnification of near correction is still that which is removed with the factor:



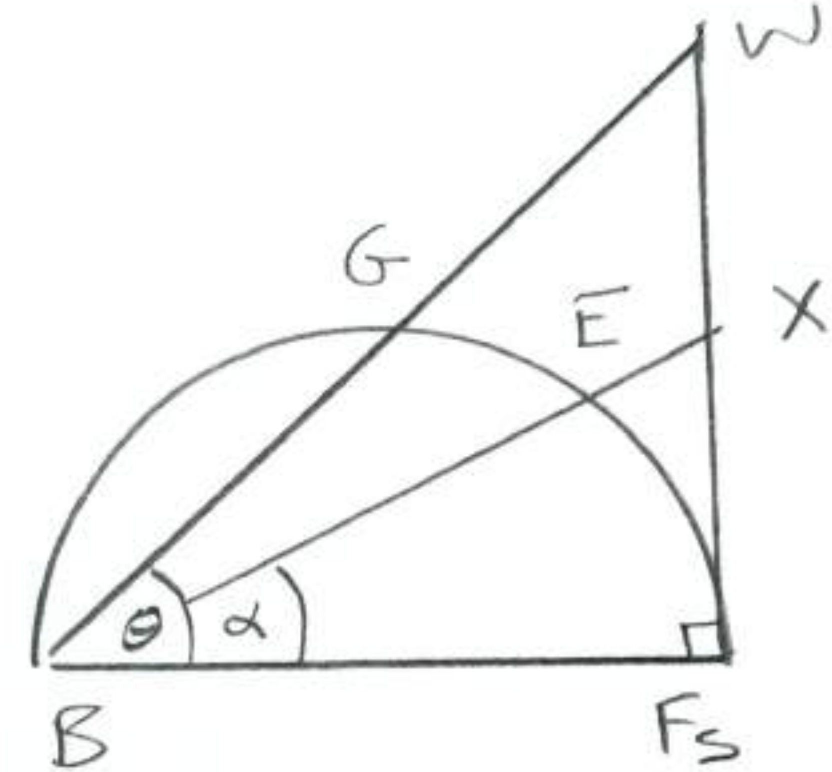
FG/FD > 1

Near Object Positional Magnification

When an object at a standard distance Fs is moved to F:



The object angular subtense magnification equals:



$\theta/\alpha = (\sim GFs/BFs)/(\sim EFs/BFs)$

as XFs $\Rightarrow$ 0

the object angular subtense magnification approaches its axial value:

$\theta/\alpha \Rightarrow$ WFs/XFs = WFs/YF = BFs/BF

which equals the axial
object angular subtense magnification.

Total Near Magnification

The ratio describing axial object angular subtense magnification:

BFs/BF

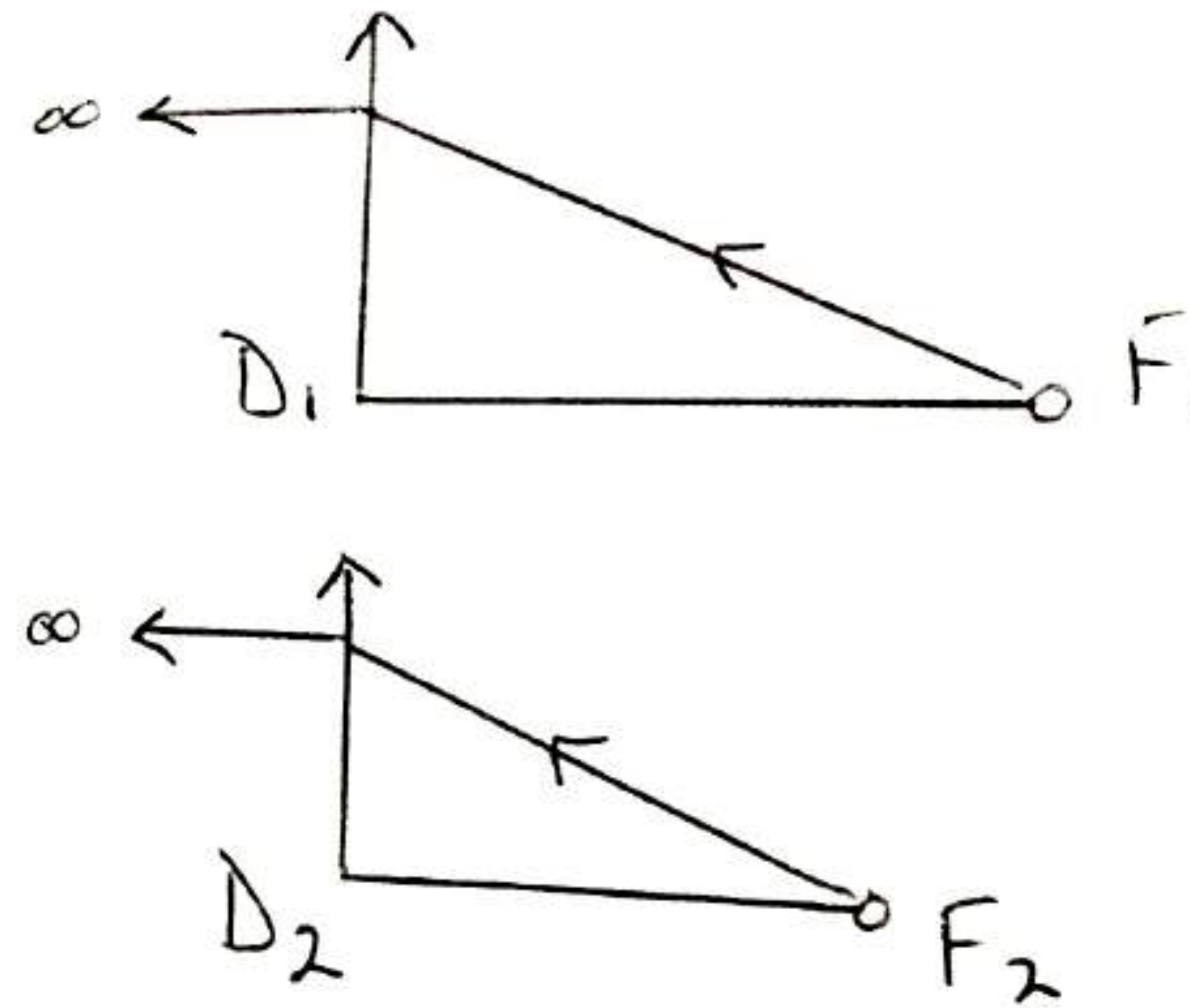
when multiplied by the ratio describing near magnification due to a single converging lens producing parallel light for an emmetropic eye:

FB/FD

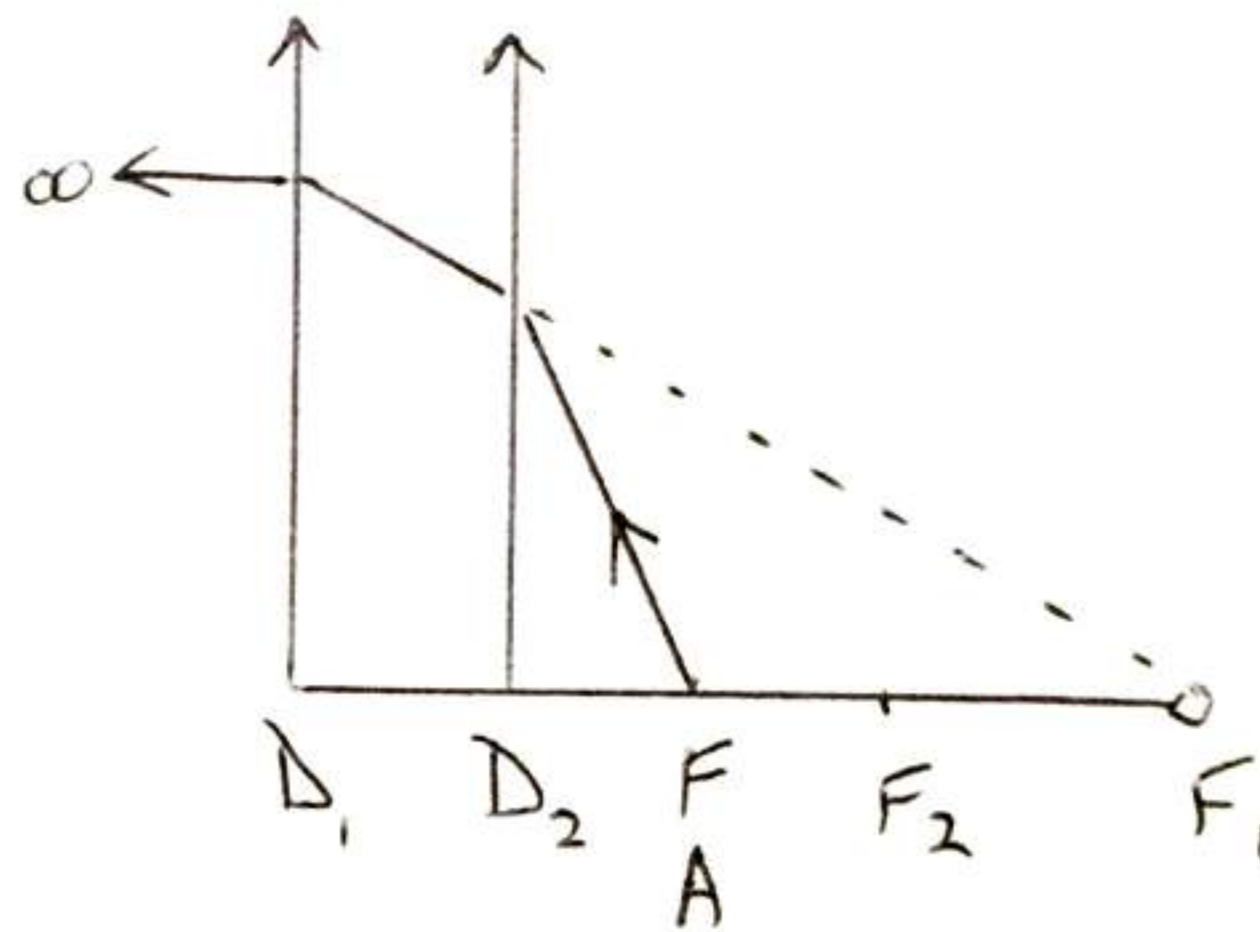
produces a ratio which factors out the object's actual distance to the eye, confirming that when a converging lens is used with its front focal point at the object, so that parallel light leaves the converging lens from the object, the image size is the same regardless of the object-to-eye distance.

Double Refraction Systems

When the converging lens at D is split into two converging lenses:



with the same combined focus F:



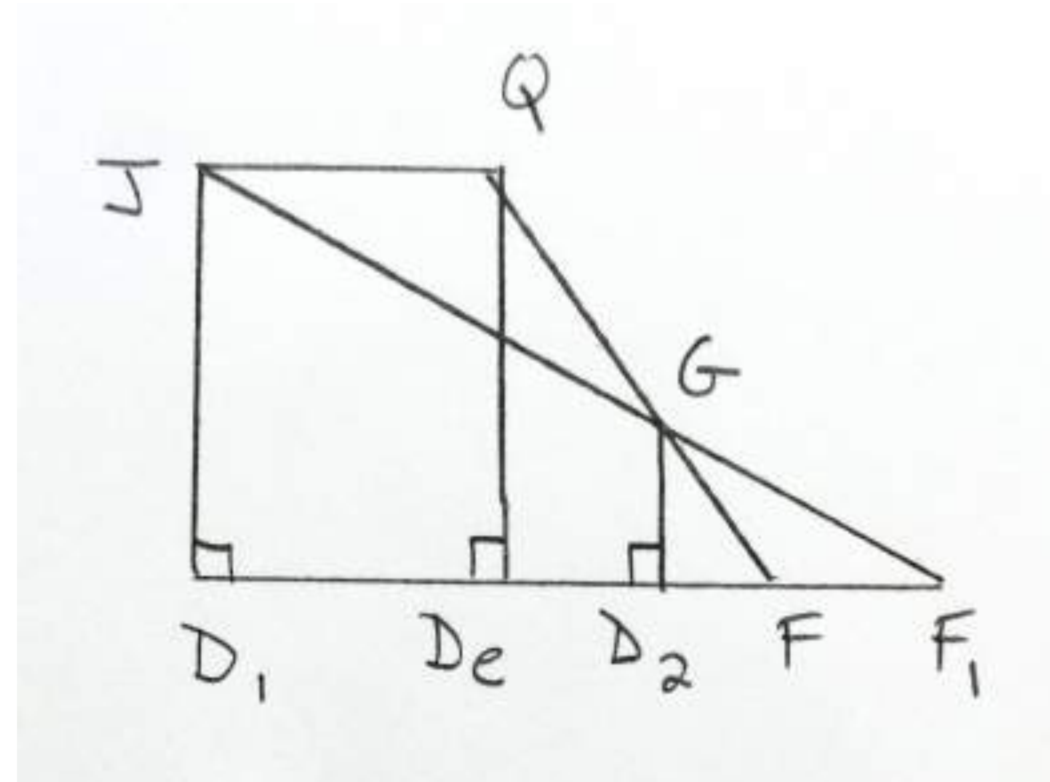
the ratio describing axial near magnification due to a single converging lens producing parallel light for an emmetropic eye:

FB/FD

must be expressed ***as if*** all convergence occurred at a single unknown axial point De:

FB/FDe

De can be located using triangles.



$D_2G/D_2F = DeQ/DeF$

$D_2G/D_2F_1 = D_1J/D_1F_1$

$D_2F(DeQ/DeF) = D_2F_1(D_1J/D_1F_1)$

$DeQ/DeF = (D_2F_1/D_2F)(D_1J/D_1F_1)$

$1/DeF = (D_2F_1/D_2F)(1/D_1F_1)$

$FB/FDe = (D_2F_1/D_2F)(FB/D_1F_1)$

Multiplying the axial object subtense magnification by the axial magnification of near correction (relative to the same eye without refractive error) produces:

$$BFs/FDe = (D_2F_1/D_2F)(BFs/D_1F_1)$$

The converging lens D_2 creates a virtual image F_1 of an object at F. When considering a stand magnifier with lens D_2, constant stand height D_2F, and reading spectacle add or ocular accommodation D_1, the stand magnifier's (constant) enlargement of the object at F equals:

$$E = D_2F_1/D_2F$$

The stand magnifier's axial magnification is its (constant) enlargement factor E, multiplied by what would be produced by D_1 alone, if the object A were at F_1.

Crossed Ophthalmic Cylinders

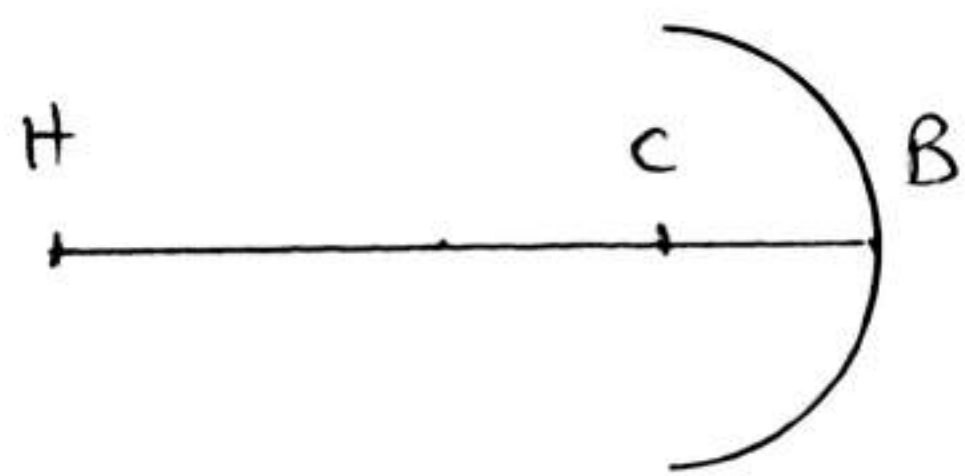
It is useful to know the meridian of maximum axial refraction when combining the effects of two ophthalmic cylinders crossed obliquely. To do this, we need to first describe how the axial radius of curvature of an ophthalmic cylinder changes from infinity along its axis to its minimum value perpendicular to that axis. Ophthalmic cylinder meridional sections are ellipses of variable shape that transform from initial front and back parallel lines along the cylinder axis to a circular section perpendicular to that axis.

Assume that the meridian of minimum ophthalmic cylinder radius occurs in a parabolic section, rather than a circular one.
Now assume that meridional sections maintain a parabolic shape as they vary towards a single tangential point represented as the cylinder axis with an infinite radius of curvature relative to that point.

This will allow for the following *relatively* easy approximation of the axial radii of curvature of meridional sections. If these approximate axial radii of curvature are expressed in forms that are additive in terms of refraction, we can then approximate the sum of those expressions for any meridional section of obliquely crossed ophthalmic cylinders, and we can approximate the maximum sum of those expressions with the required meridional axis.

We know that with any axial radius of curvature CB, and index of refraction $\mathbb{R}$, the axial image of a distant object lies at H when:

$\mathbb{R}$ = HB/HC



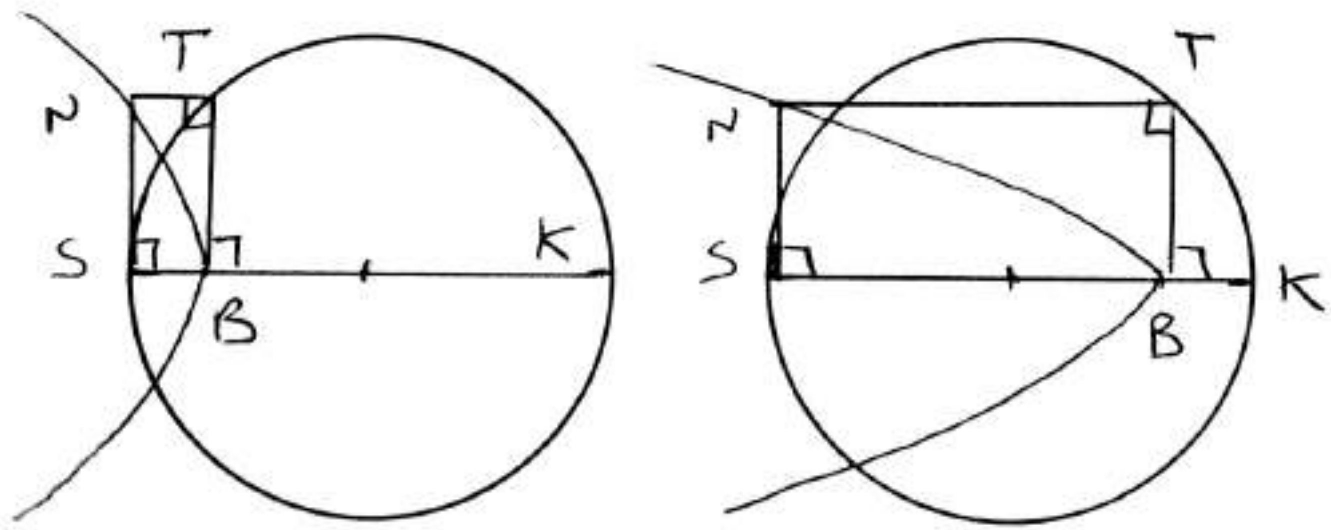
We also know that the axial refractive effects of compound refractive surfaces at B are additive only as their refractive "powers," which equal:

$$\mathbb{R}/HB \;=\; 1/HC \;=\; [(HB - HC)/HC]/CB \;=$$

$$(\mathbb{R} - 1)/CB$$

All parabolas have the same shape, in the same way that all circles have the same shape. However, while circles have a single (internal) determining constant, the radius of curvature, parabolas have both a determining constant internal and external to the curve, and can be defined by either.

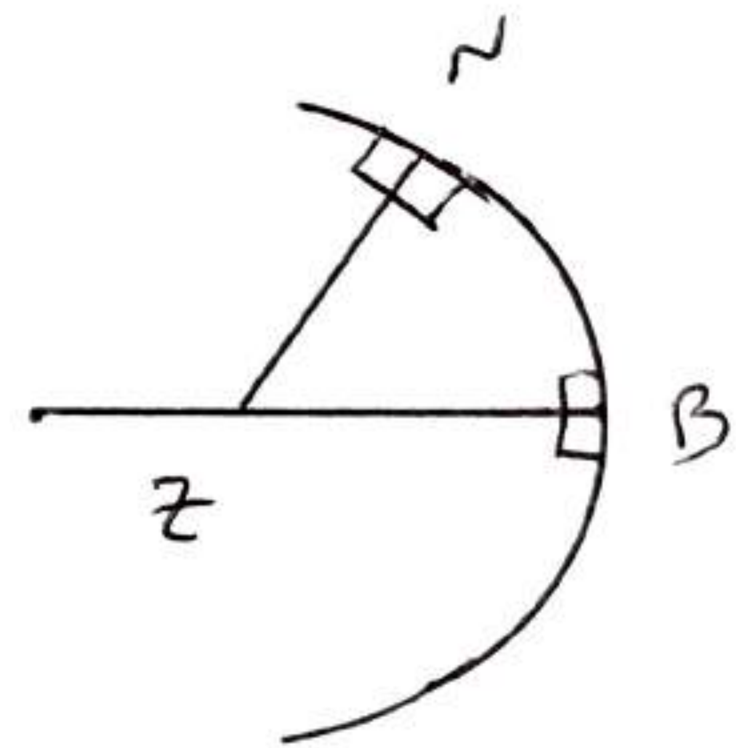
For example, a parabola's external determining constant equals BK when:



$$\frac{SB}{BT} = \frac{BT}{BK}$$

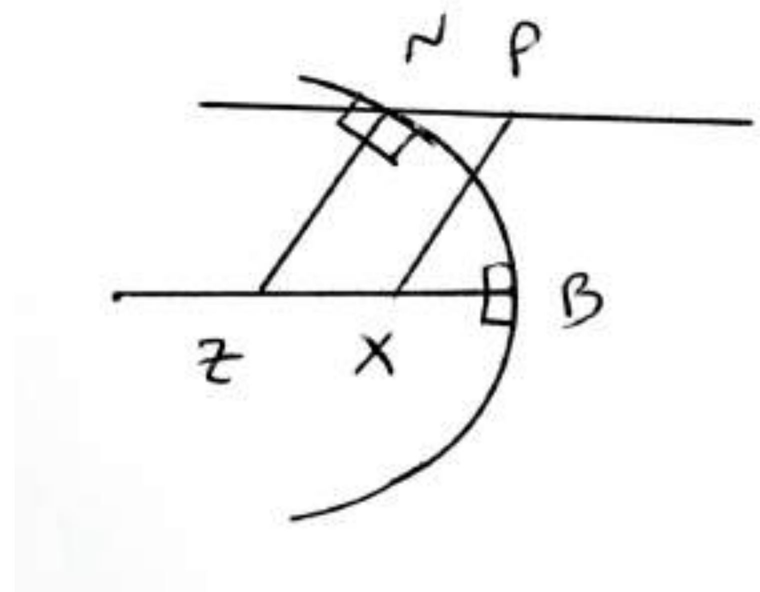
Both these curves have the same shape. The one on the left simply represents a “zoomed in” look at the vertex of the one on the right.

We can set up the necessary off-axis conditions to determine a parabola's axial center of curvature in terms of its internal determining constant XB, by involving ZN in the geometric solution for XB.



In order to keep the determining geometrical relationships axial as N ⇒ B, they should also

depend on line NP being parallel to the axis, and XP being parallel to ZN.



We know X lies between Z and B, since parabolas flatten in their periphery.

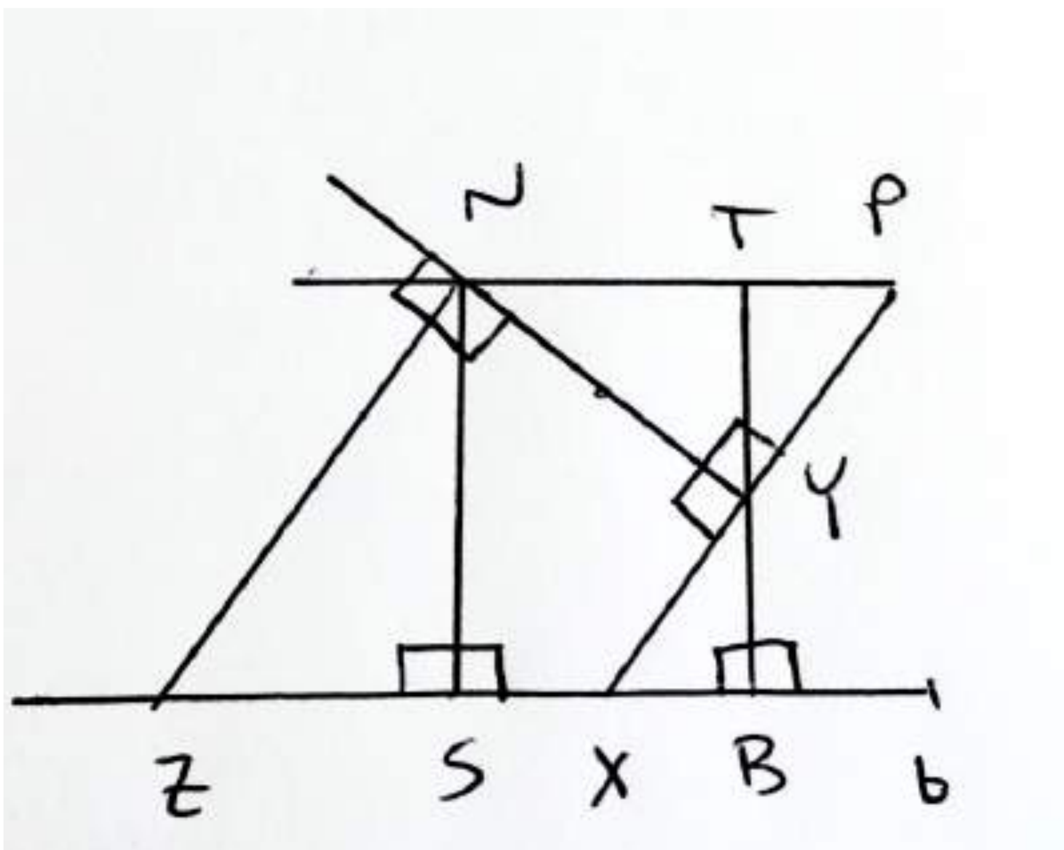
Since as N $\Rightarrow$ B,

Z $\Rightarrow$ C by definition,

and since XP = ZN,

P will remain external to the curve, and X can therefore not be its axial center of curvature, but must instead lie somewhere along CB.

In order to maintain ZN perpendicular to the parabola at N as N ⇒ B, the same geometrical relationships must exist that allow for that when N lies at B.



In other words:

YP = YX and
Bb = BX so
CB = 2(XB)

$$\frac{TN}{TB} = \frac{TN}{2(TY)} = \frac{YB}{2(XB)} = \frac{YB}{CB} = \frac{TB}{2(CB)}$$

Since TN = SB, the external determining constant BK equals 2(CB).
Since TB = 2(YB), the internal determining constant XB equals (CB)/2.

Refracting power equals: $(\mathbb{R} - 1)/CB$

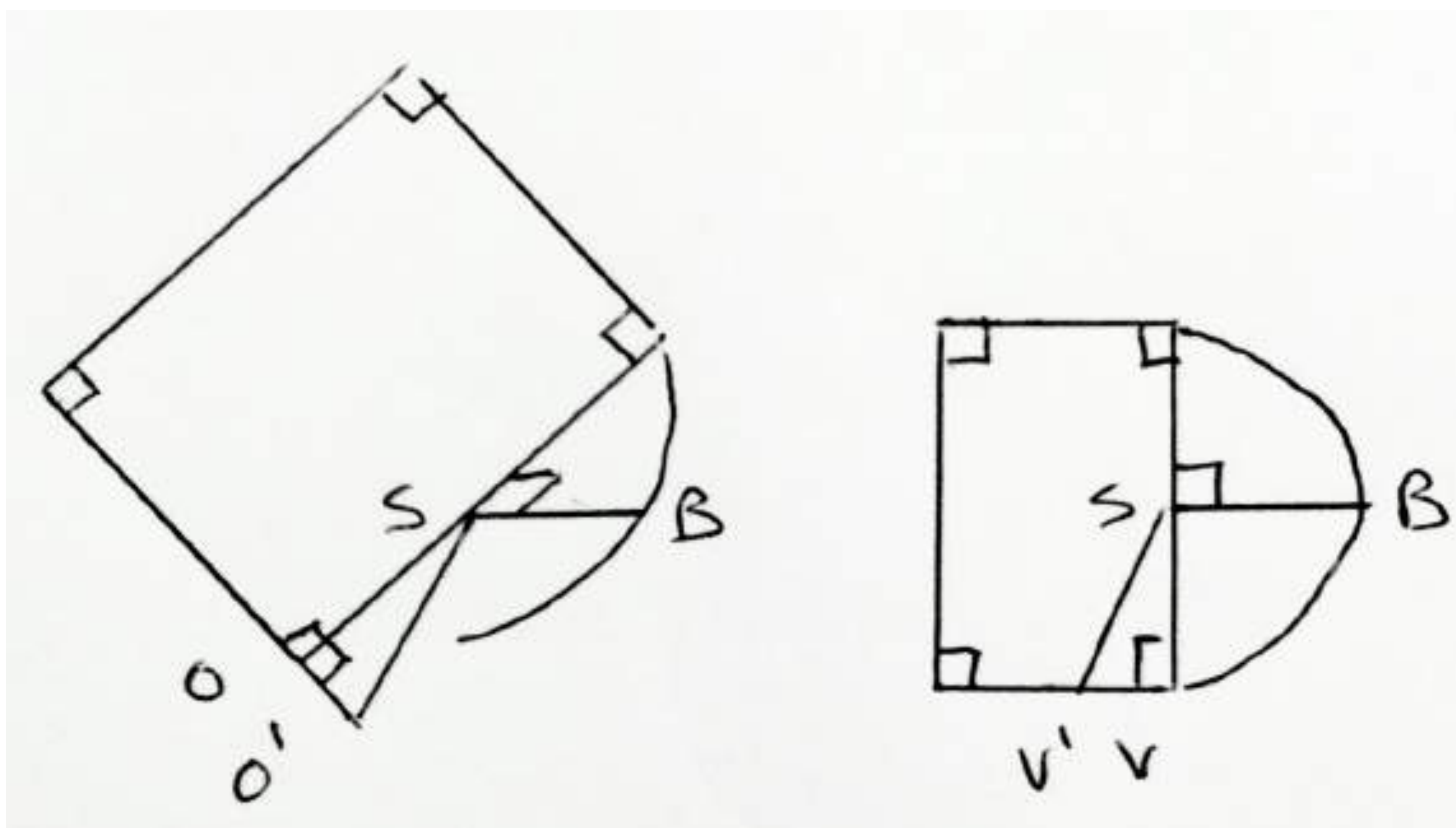
If $\mathbb{R} = 1.5$, this equals: $1/[2(CB)]$

For a parabola: $SB/BT = BT/BK = BT/[2(CB)]$

so its axial refracting power then equals:

$SB/TB^2 = SB/SN^2 = 1/BK$

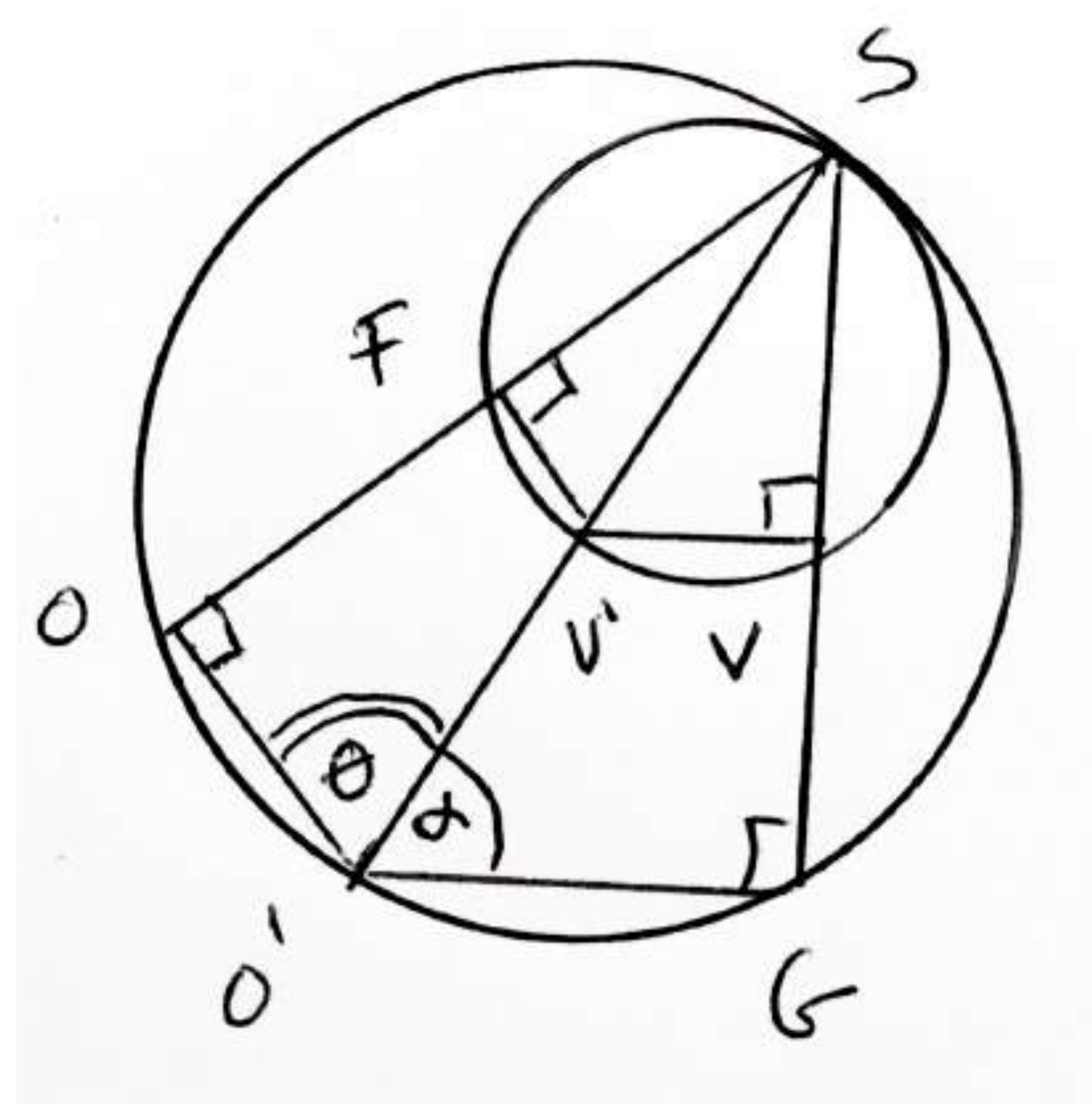
When 2(SO) equals the minimum sagitta of an oblique parabolic cylinder, and when with equal sagittal depth SB, 2(SV) equals the minimum sagitta of a more highly curved parabolic cylinder with a horizontal axis:



Keeping ΔOSV constant, as we rotate circle SOG with variable diameter SV'O' around point S:

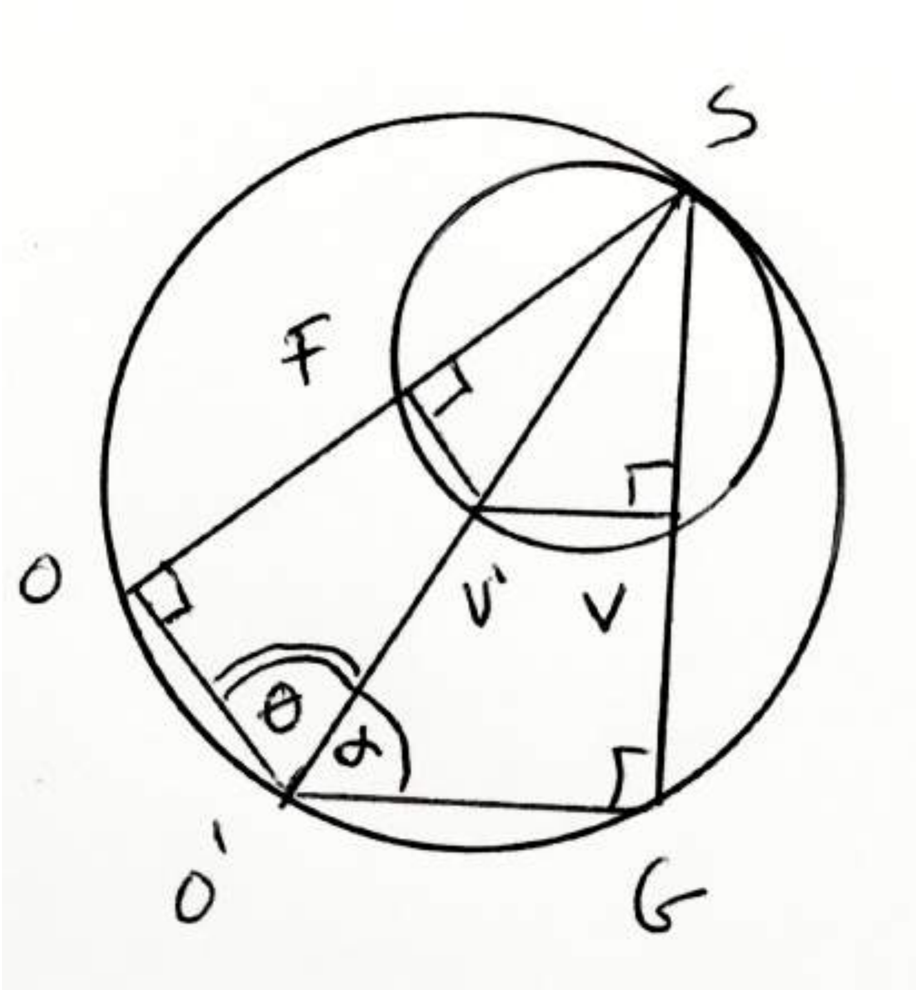
∠OO'G is constant because ∠OSG is constant,

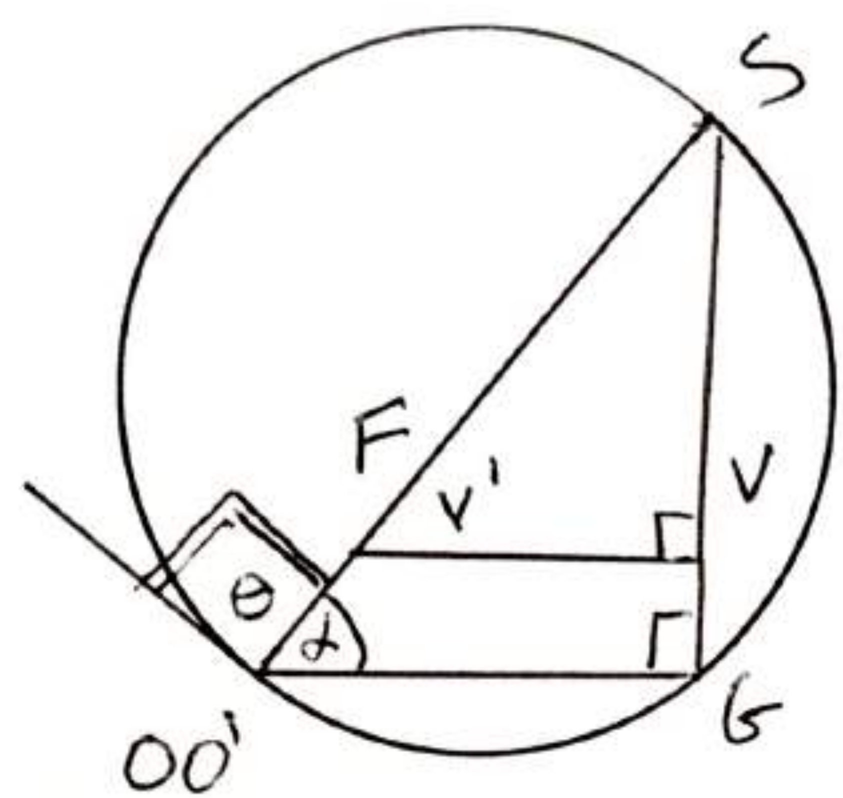
so $\Delta\theta = -\Delta\alpha$



Keeping ΔOSV constant,   as O' ⇒ O:

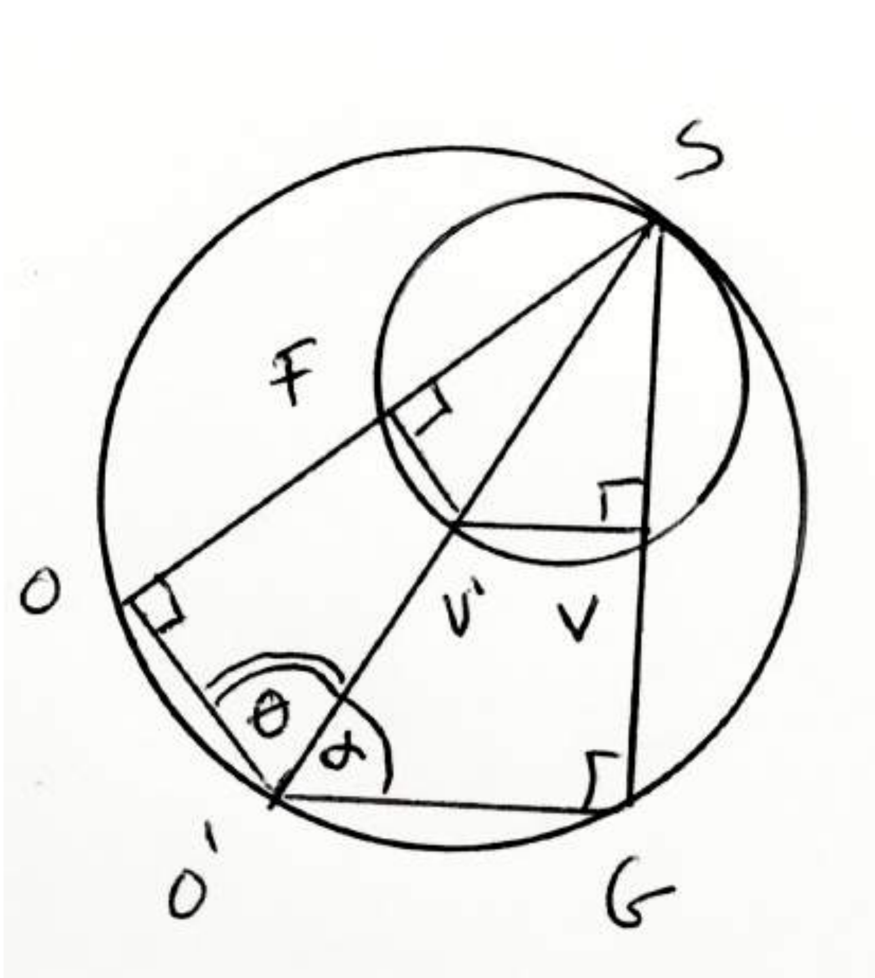
SG and diameter SO' decrease.
SV' increases more than SO' decreases.

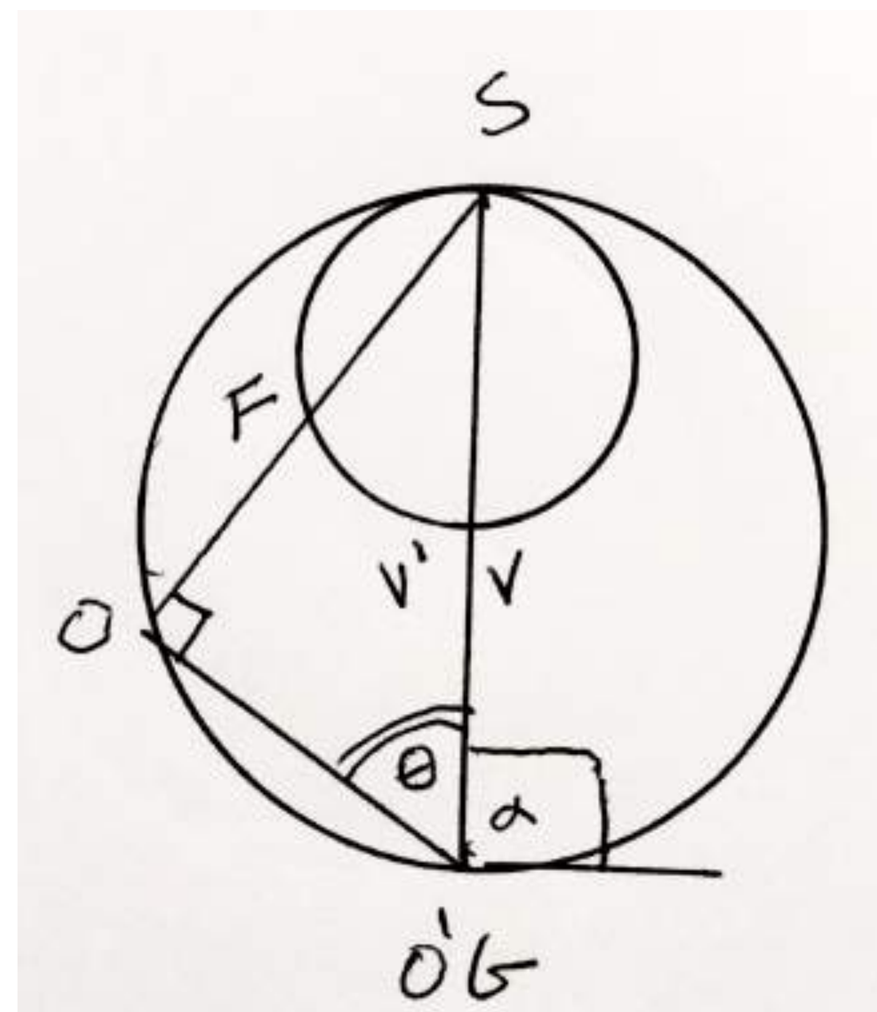




Keeping ΔOSV constant, as V' ⇒ V:

SG and diameter SO' increase.
SO' increases more than SV' decreases.





Since the sum (SO' + SV') increases when either:

O' ⇒ O,   or   V' ⇒ V

there must be a specific SV'O' within ΔOSV producing a minimum sum (SO' + SV'), which must be near where small rotations of SV'O' about S produce only minimal changes in the sum (SO' + SV').

Since as when one term of the sum (SO' + SV') increases, the other always decreases, the minimum (SO' + SV') must occur near where small rotations of SV'O' within ΔOSV produce equal but opposite changes in SO' and SV'. Therefore, the minimum (SO' + SV') can be found by finding the position of SV'O' where:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta(SO') = \lim_{\Delta\alpha \Rightarrow 0} \Delta(SV')$$

However, the combined effects of refraction are additive only as refractive powers, which, when $\mathbb{R} = 1.5$, equal:

$SB/(SO')^2$ and $SB/(SV')^2$

Therefore, the meridian with the maximum combined effects of this refraction can be found by finding the position of SV'O' where:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta \left[SB/(SO')^2\right] = \lim_{\Delta\alpha \Rightarrow 0} \Delta \left[SB/(SV')^2\right]$$

To solve this equation, each expressed limit must be transformed into the variable that approaches zero, so the equation must be transformed into:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta\{[SB(SO/SO')^2]/SO^2\} = \lim_{\Delta\alpha \Rightarrow 0} \Delta\{[SB(SV/SV')^2]/SV^2\}$$

$$\lim_{\Delta\theta \Rightarrow 0} \Delta\{[(SB)\sin^2\theta]/SO^2\} = \lim_{\Delta\alpha \Rightarrow 0} \Delta\{[(SB)\sin^2\alpha]/SV^2\}$$

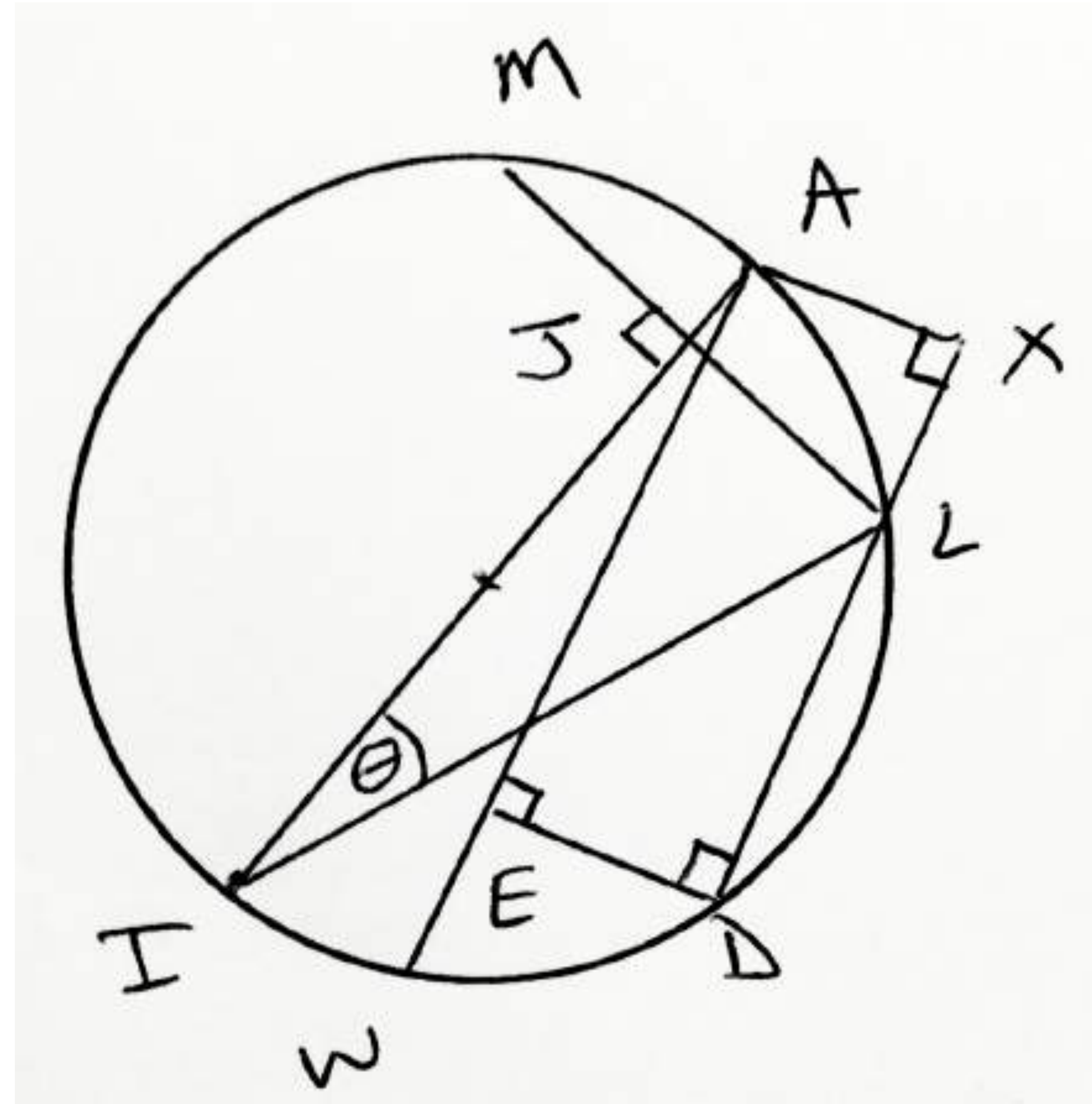
$$(SB/SO^2) \lim_{\Delta\theta \Rightarrow 0} \{\Delta\sin^2\theta\} = (SB/SV^2) \lim_{\Delta\alpha \Rightarrow 0} \{\Delta\sin^2\alpha\}$$

$$\frac{\text{Limit as } \Delta\theta \Rightarrow 0 \text{ of } (\Delta\sin^2\theta)}{\text{Limit as } \Delta\alpha \Rightarrow 0 \text{ of } (\Delta\sin^2\alpha)} = SO^2/SV^2$$

Solve for

Limit $\Delta \sin^2 \theta$
$\Delta\theta \Rightarrow 0$

on the reference circle:



AW ≥ LD || AW

∠ALD = ~AID/AI

≥ ~AI/AI = π

First establish the necessary functions of θ in terms of arcs and chords.

$$\theta = \sim AL/AI$$

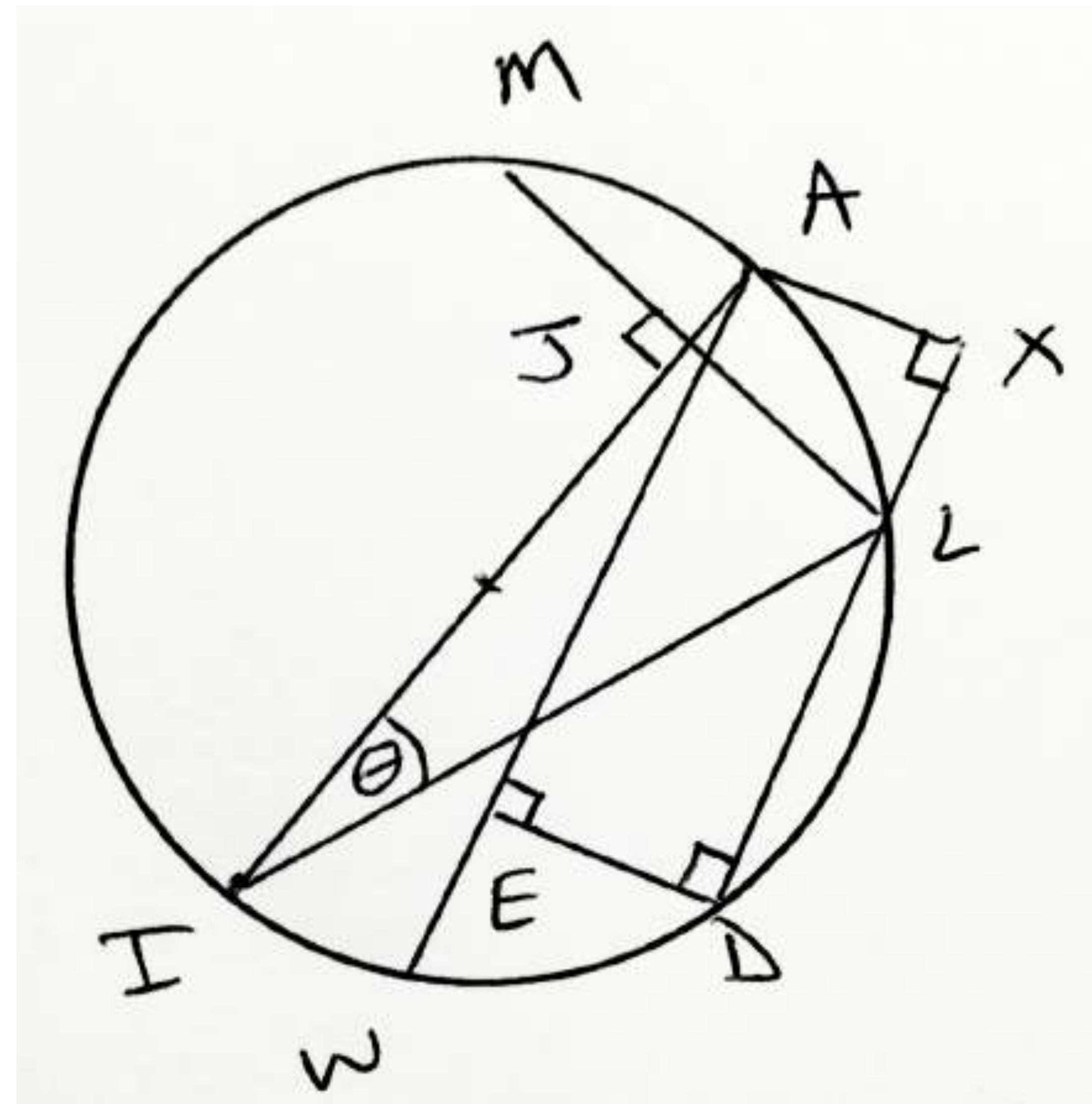
$$\sin^2\theta = AL^2/AI^2$$

$$\Delta\theta = \sim LD/AI$$

$$\sin^2\Delta\theta = LD^2/AI^2$$

$$(\theta + \Delta\theta) = \sim ALD/AI$$

$$\sin^2(\theta + \Delta\theta) = AD^2/AI^2$$



$\cos\theta = IL/AI$

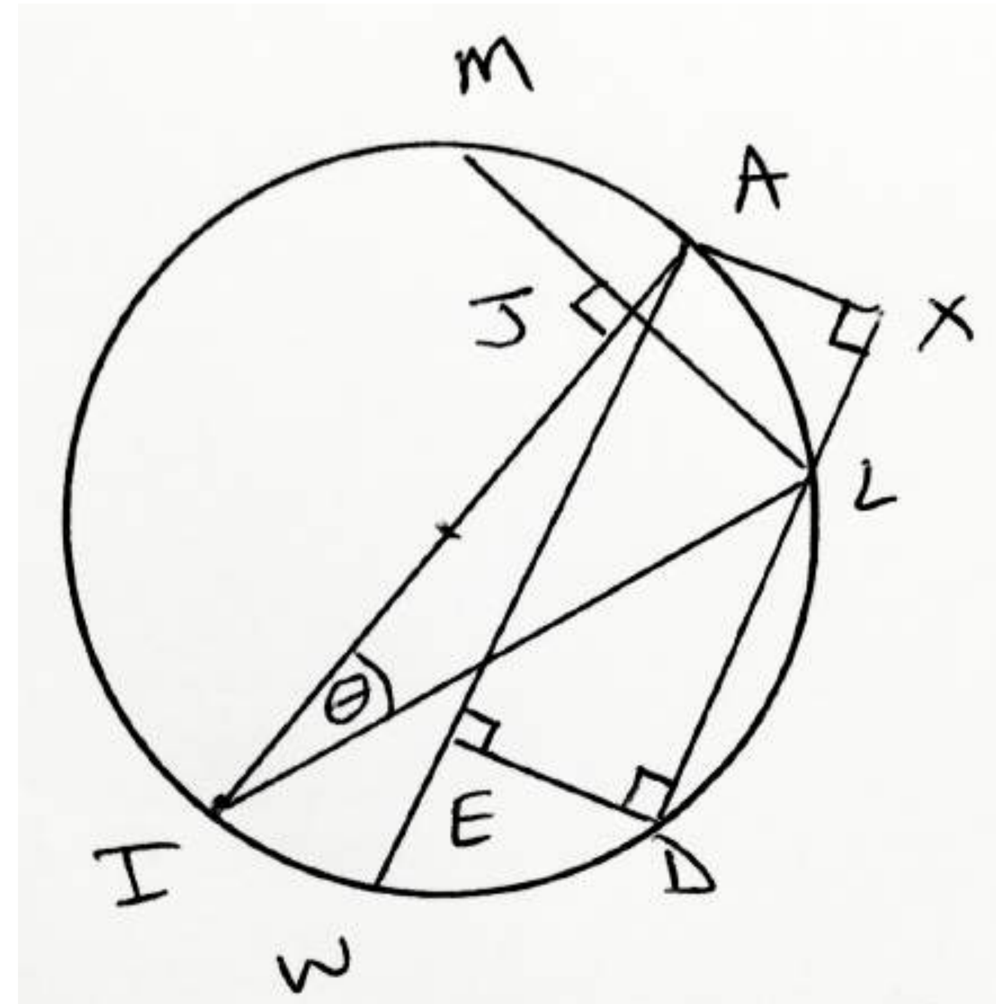
$\cos(\theta + \Delta\theta) = DI/AI$

$\sin\theta = AL/AI = JL/IL$

$\sin\theta\cos\theta = (JL/IL)\ (IL/AI)$

$2(\sin\theta\cos\theta) = ML/AI$

$2(\sin\theta\cos\theta) = \sin 2\theta$



Then consider the following property of the cyclic quadrilateral circle ALDW:

$$AD\,(LW) = AL\,(DW) + LD\,(AW)$$

$$AD^2 = AL^2 + LD\,(AW)$$

$$AW = LD + 2\,(XL) = LD + 2\,(AL)\,(XL/AL)$$

$$\Delta DIA \cong \Delta EWD = \Delta XLA$$

$$AW = LD + 2\,(AL)\,(ID/IA)$$

$$\mathbf{AD^2 - AL^2 = LD^2 + 2\,(LD)\,(AL)\,(ID/IA)}$$

$$\mathbf{AD^2 - AL^2 = LD^2 + 2\,(LD)\,(AL)\,(ID/IA)}$$

$$AD^2/AI^2 - AL^2/AI^2 =$$
$$LD^2/AI^2 + 2\,(LD/AI)\,(AL/AI)\,(ID/IA)$$

$$\sin^2(\theta + \Delta\theta) - \sin^2\theta =$$
$$\sin^2\Delta\theta + 2(\sin\Delta\theta)\,(\sin\theta)\cos(\theta + \Delta\theta)$$

$$\Delta(\sin^2\theta) = \sin^2(\theta + \Delta\theta) - \sin^2\theta =$$
$$\sin^2\Delta\theta + 2(\sin\Delta\theta)\,(\sin\theta)\cos(\theta + \Delta\theta)$$

$$\lim_{\Delta\theta \Rightarrow 0} \frac{\Delta(\sin^2 \theta)}{\Delta\theta} =$$

$$\lim_{\Delta\theta \Rightarrow 0} \frac{\sin^2 \Delta\theta + 2(\sin \Delta \theta)(\sin \theta) \cos (\theta + \Delta\theta)}{\Delta\theta} =$$

$$= 2 \sin \theta (\cos \theta) = \sin 2\theta$$

because:

$$\lim_{\Delta\theta \Rightarrow 0} \frac{\sin^2 \Delta\theta}{\Delta\theta} = 1 \; ; \quad \lim_{\Delta\theta \Rightarrow 0} \frac{\sin \Delta\theta}{\Delta\theta} = 1$$

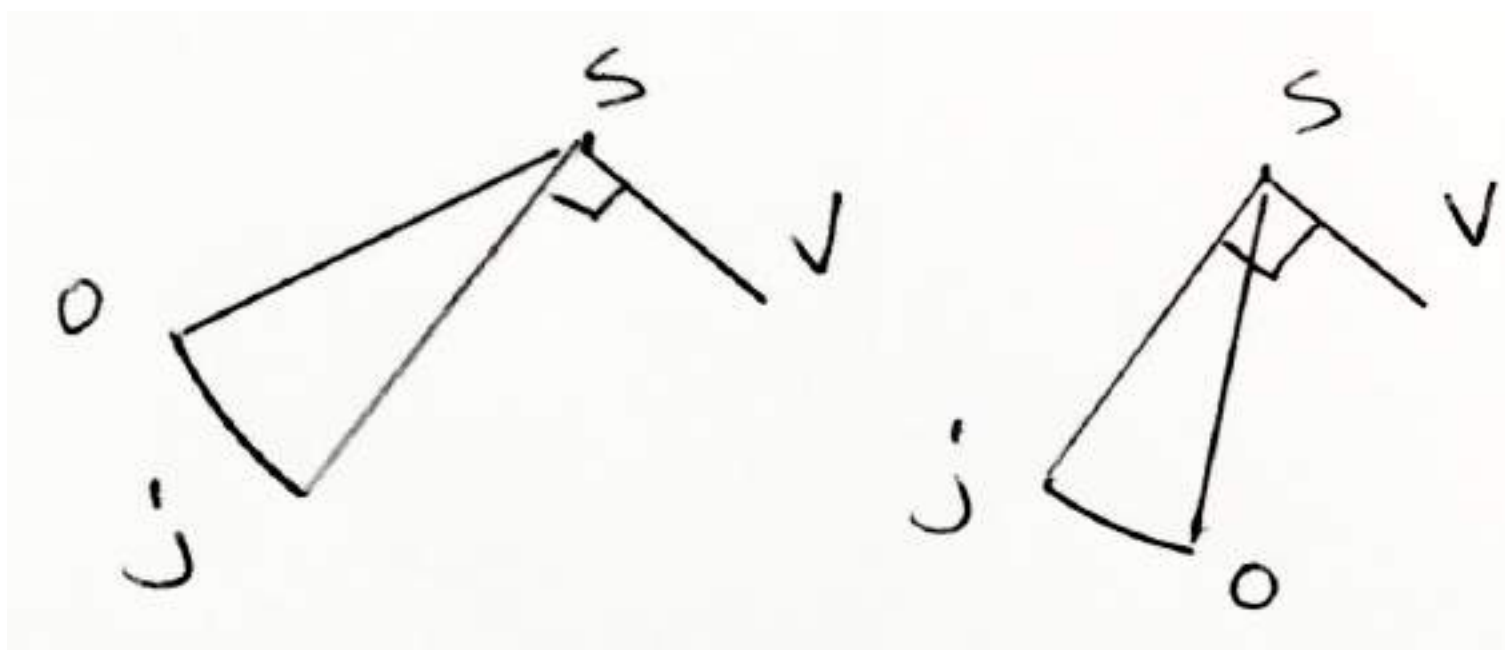
Therefore, the meridian with the maximum combined effects of refraction can be found using:

$$\frac{\sin 2\theta}{\sin 2\alpha} = \frac{SO^2}{SV^2}$$

The first step to solve this problem is to divide SV into SaV so that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

Make SO = Sj ⊥ SV
to construct:



Draw Sb so that:

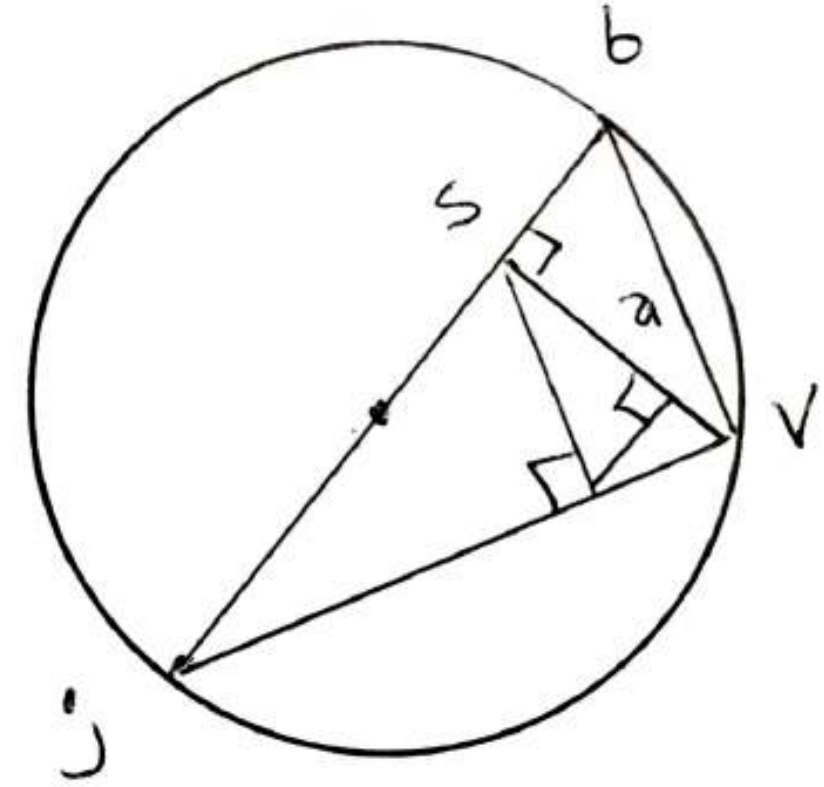
$SO^2/SV^2 = Sj^2/SV^2 = Sj/Sb$

by making:

$Sj/SV = SV/Sb$

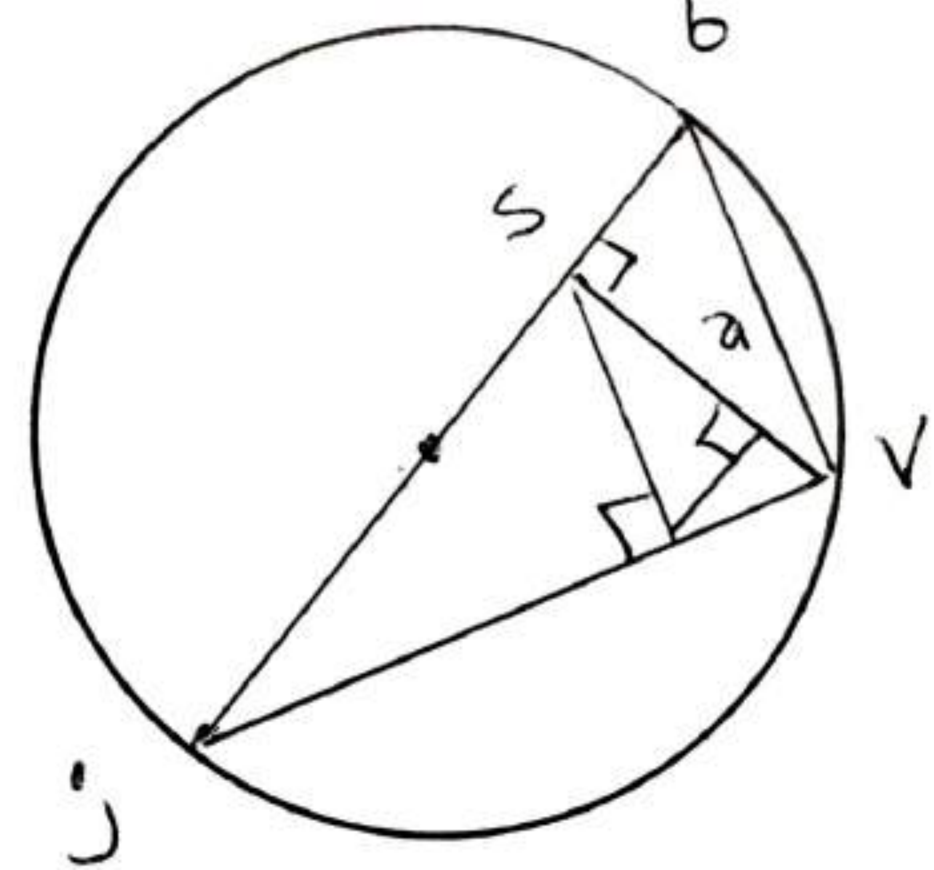
so that:

$Sj^2/SV^2 = Sj/Sb = SO^2/SV^2$



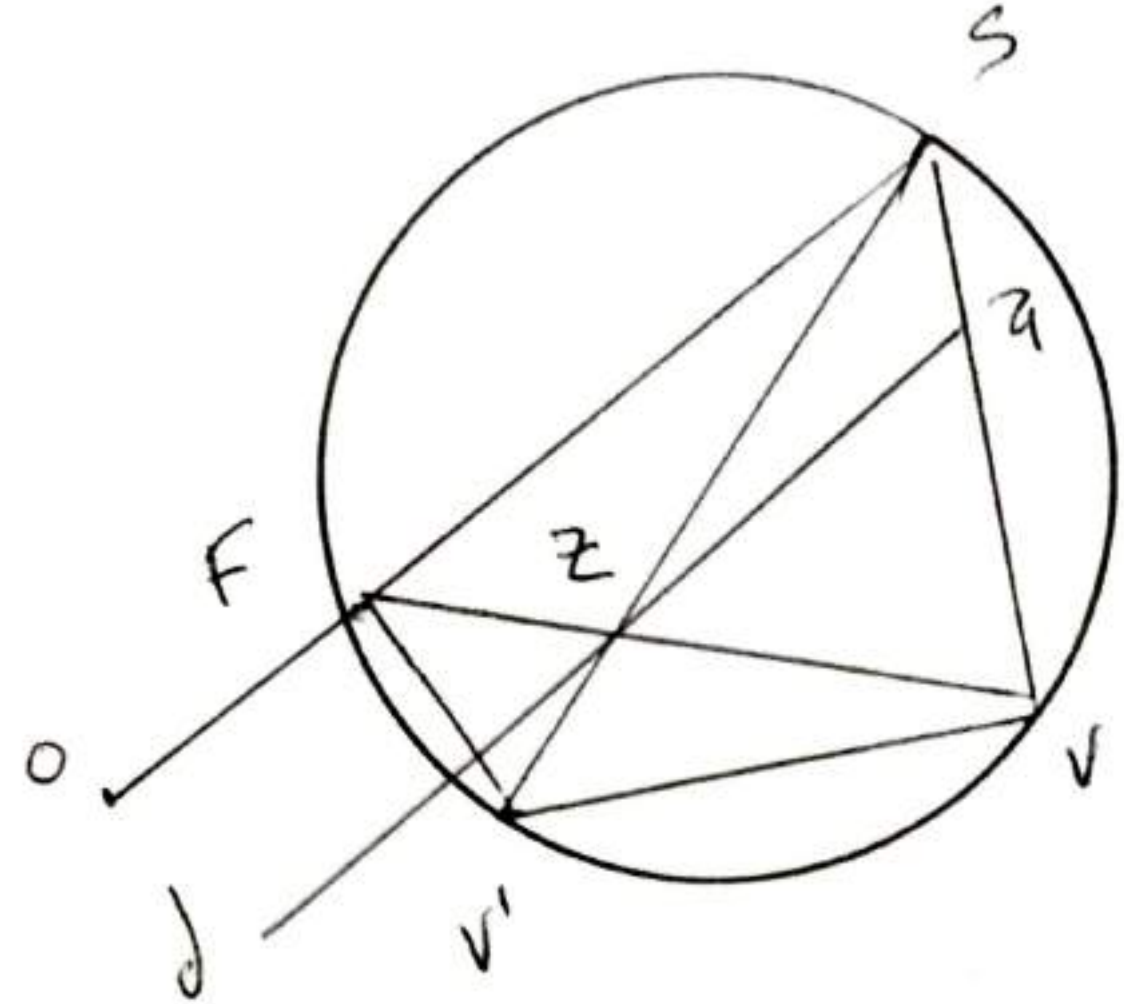
Similar triangles
then show that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

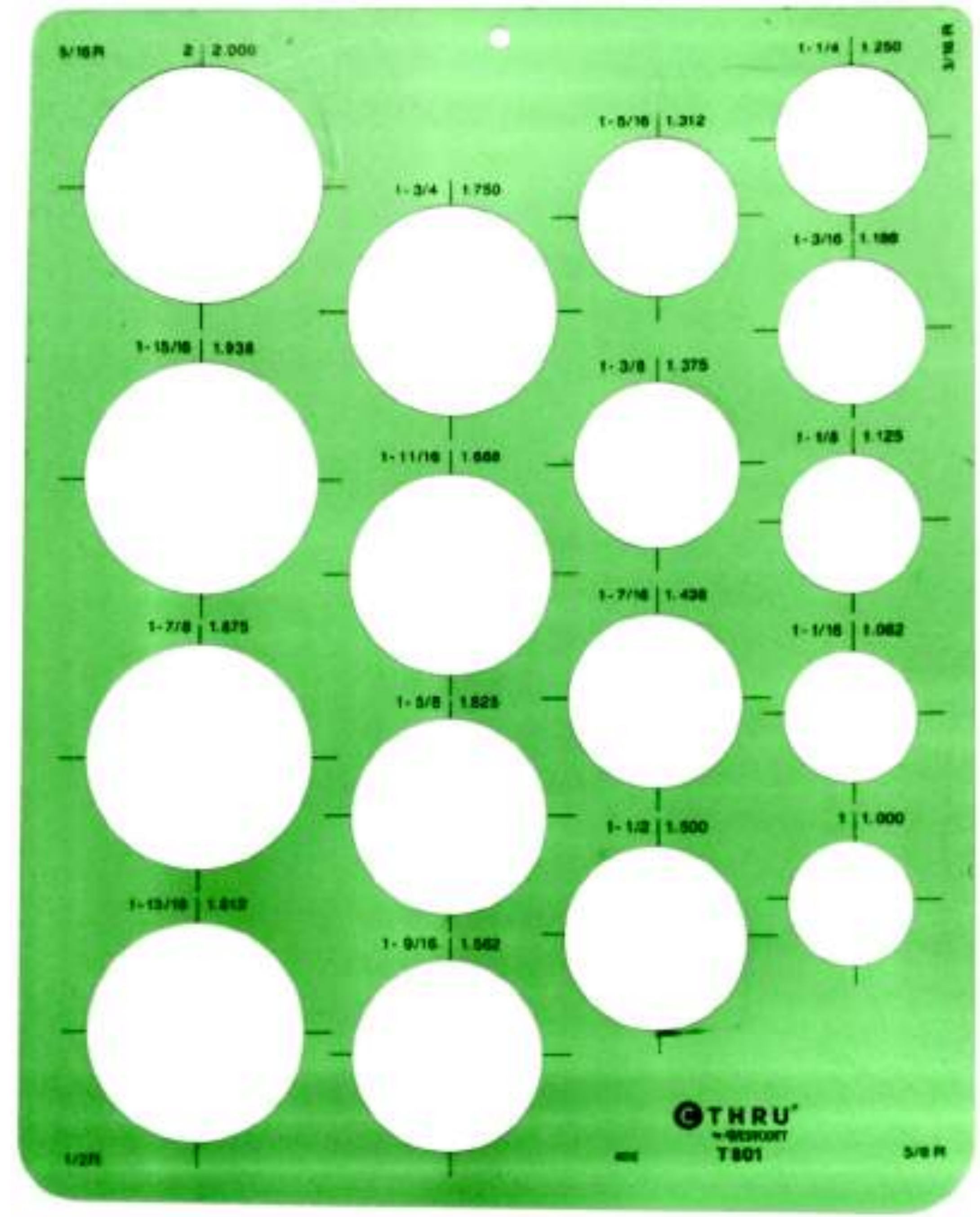


Draw ad || SO

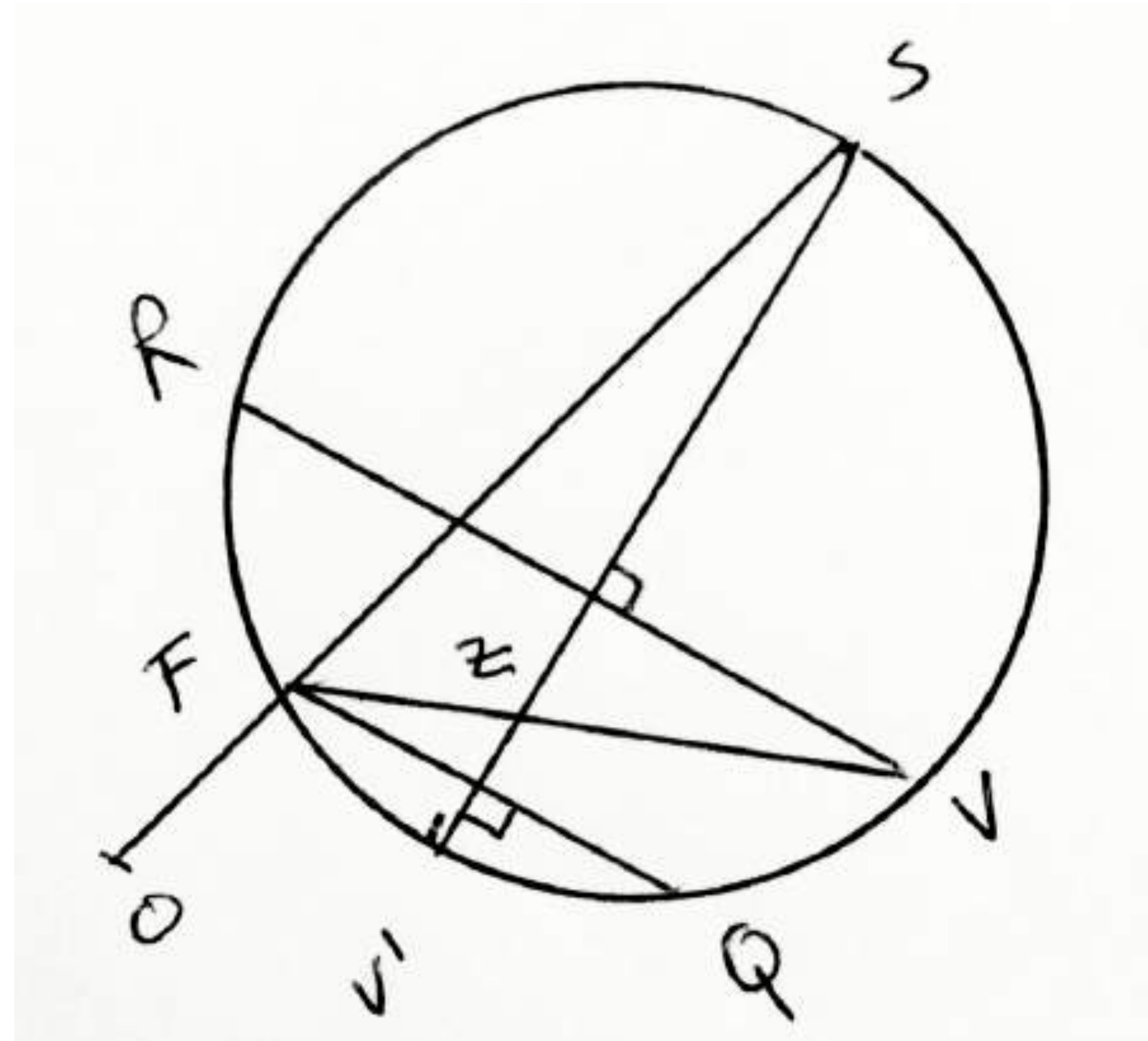
Choose a circle through S and V with a variable diameter SV' so that FZV lies on a common chord.



The easiest way to do this involves a template of various circles, each with the location of their diameters already marked.



SV' is the meridian with the maximum combined effects of refraction because:



$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{FZ}{ZV} = \frac{FQ/2}{RV/2} = \frac{FQ}{RV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

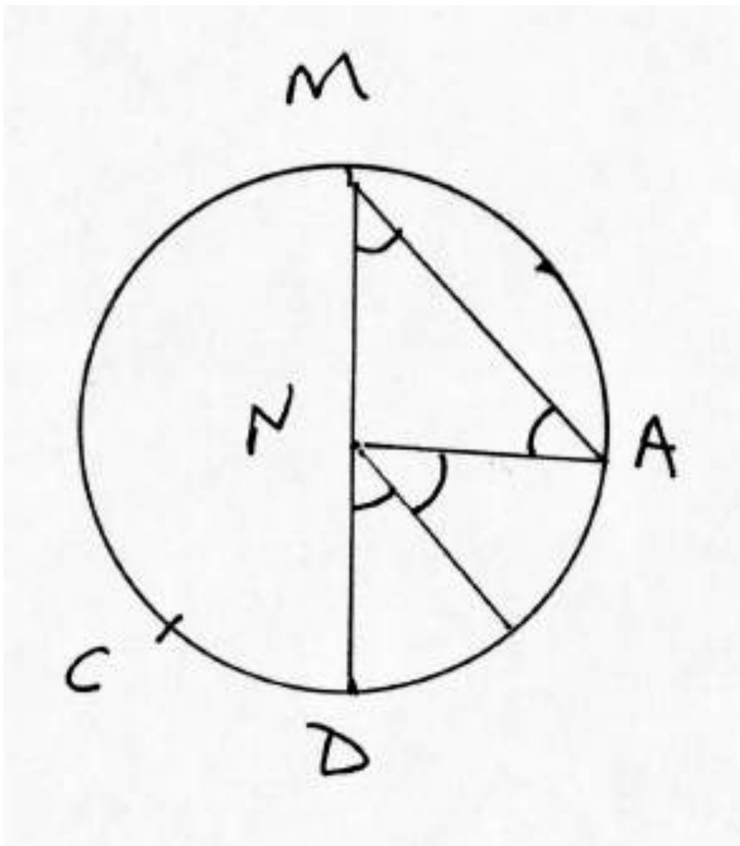
Double-angle Method

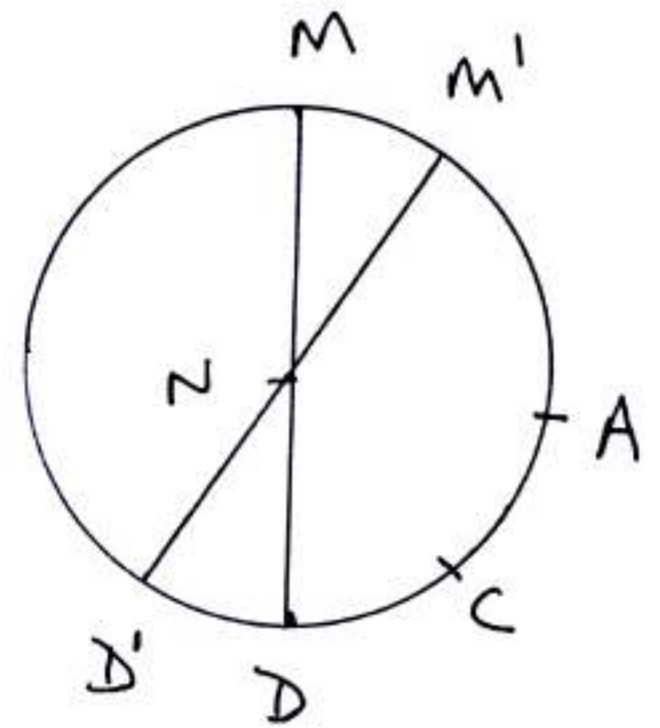
We have already shown how to find angle θ, and angle α, so that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

An additional method, the double angle method, employs the fact that an arc subtends twice the angle at a circle’s center as it does at its circumference, and that the entirety of a circle subtends π radians any a point on its circumference. To illustrate:

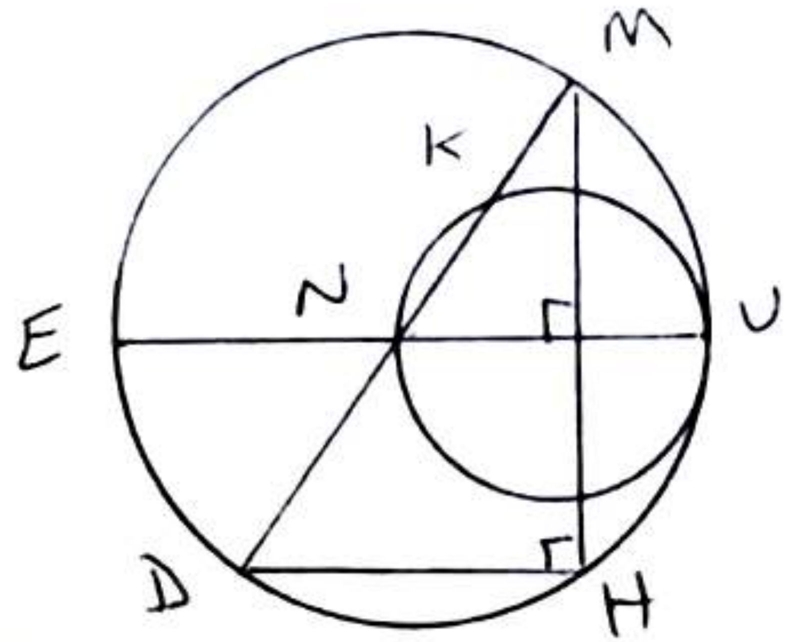
$\angle DNA = 2\angle DMA$; $\angle DNC = 2\angle DMC$

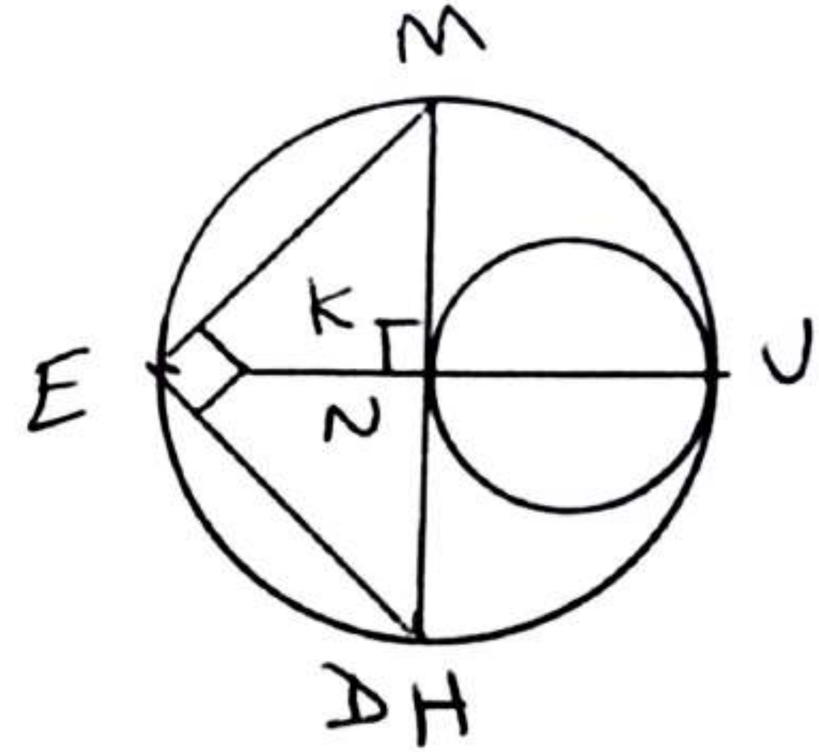




$\angle ANC = \angle DNA +/- \angle DNC = 2(\angle DMA +/- \angle DMC) =$

$2\angle AMC = 2\angle AM'C$





$\sim UK/UN = \sim MH/MD = 2\sim UM/UE = 2\sim UM/2UN$

$\sim UK = \sim UM$

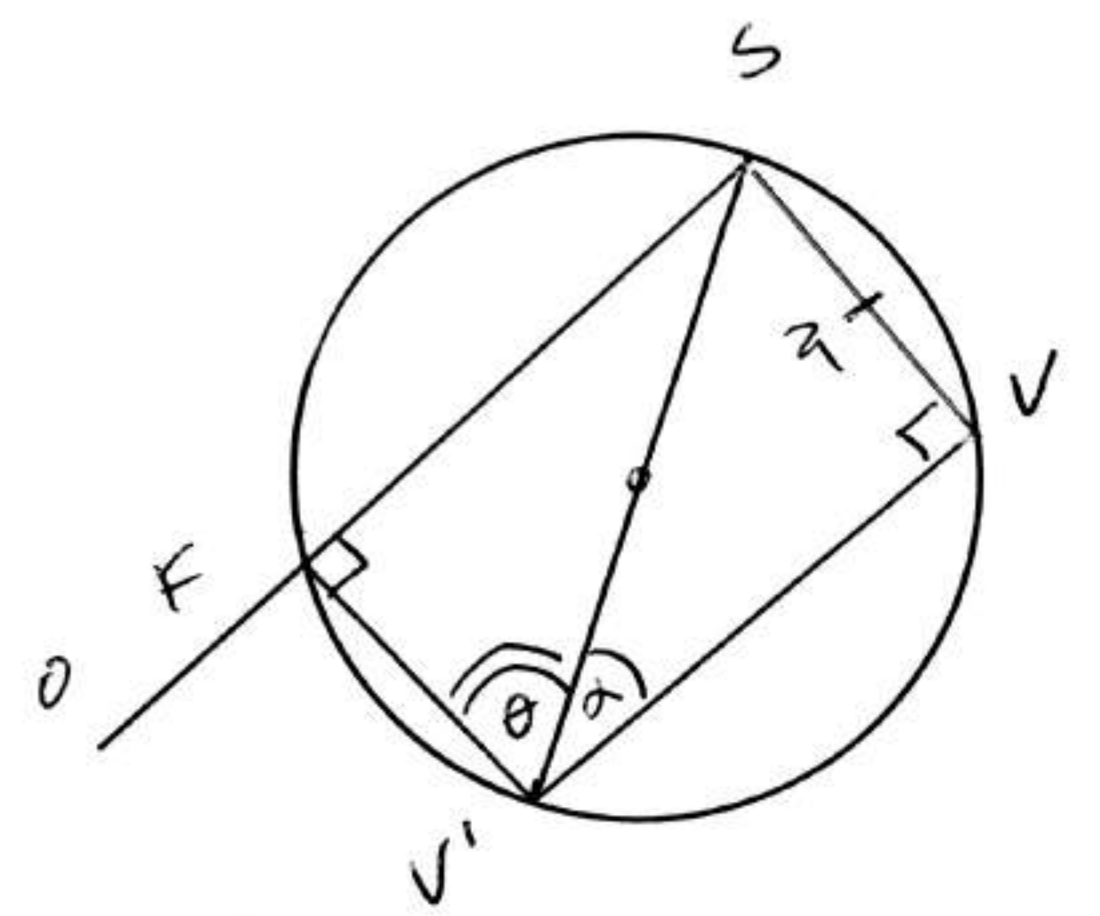
As $K \Rightarrow N$, and $D \Rightarrow H$:

$2\sim KU/UN = 2\angle MNU = \angle MNH \quad \Rightarrow \quad \pi$ radians

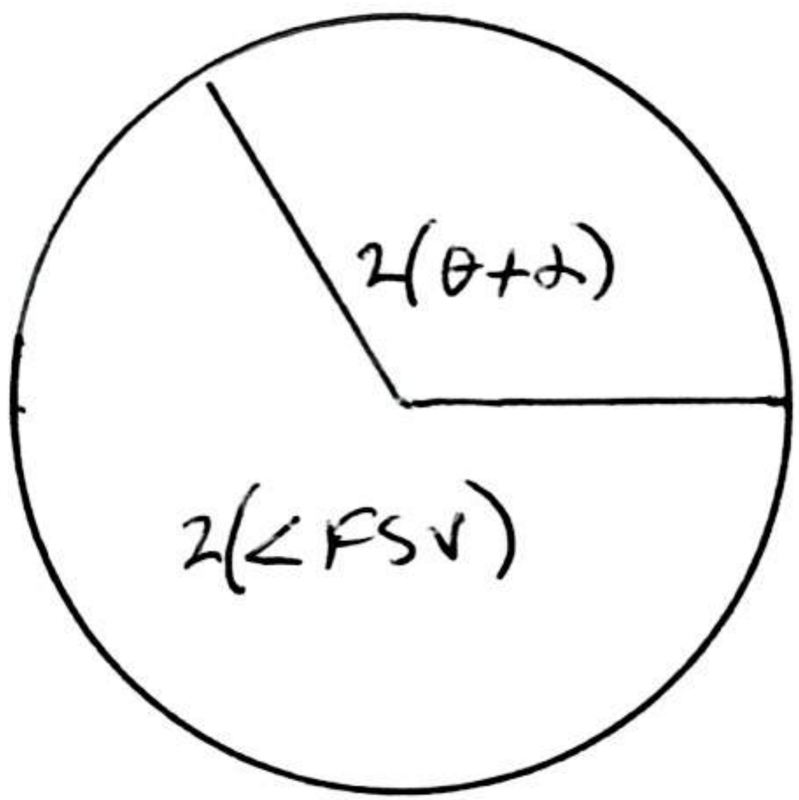
Given constant ΔOSV:
∠FSV is constant.

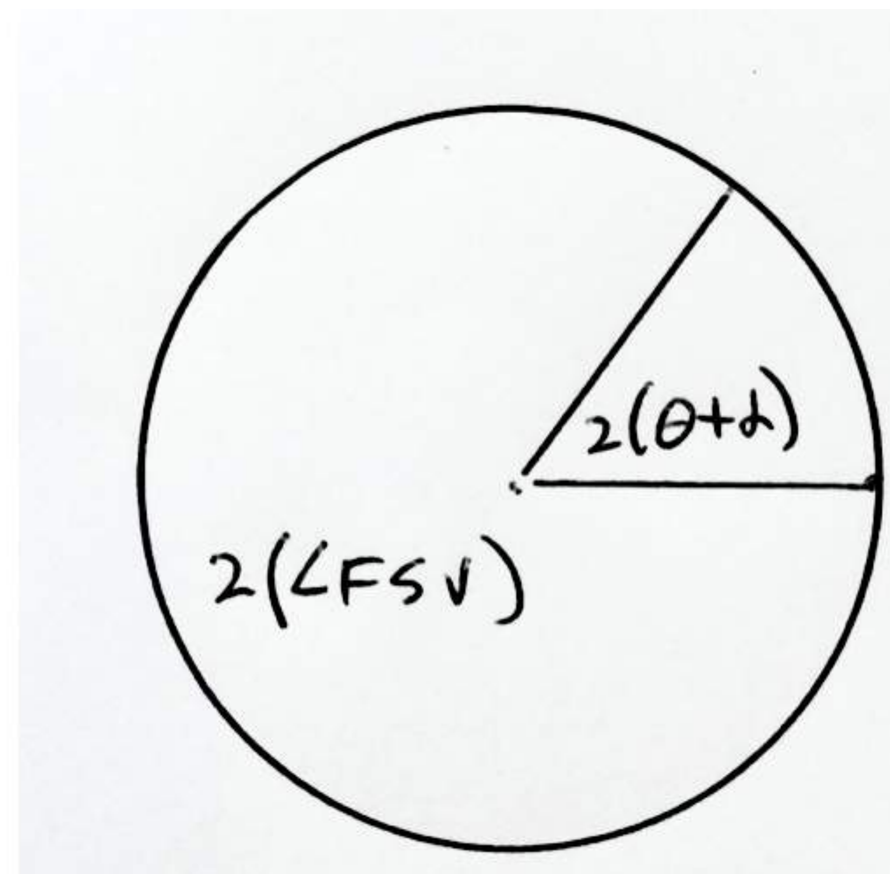
and since:
∠FSV + (θ + α) = π radians,

(θ + α) is also constant.



$$2(\angle FSV) + 2(\theta + \alpha) = 2\pi$$

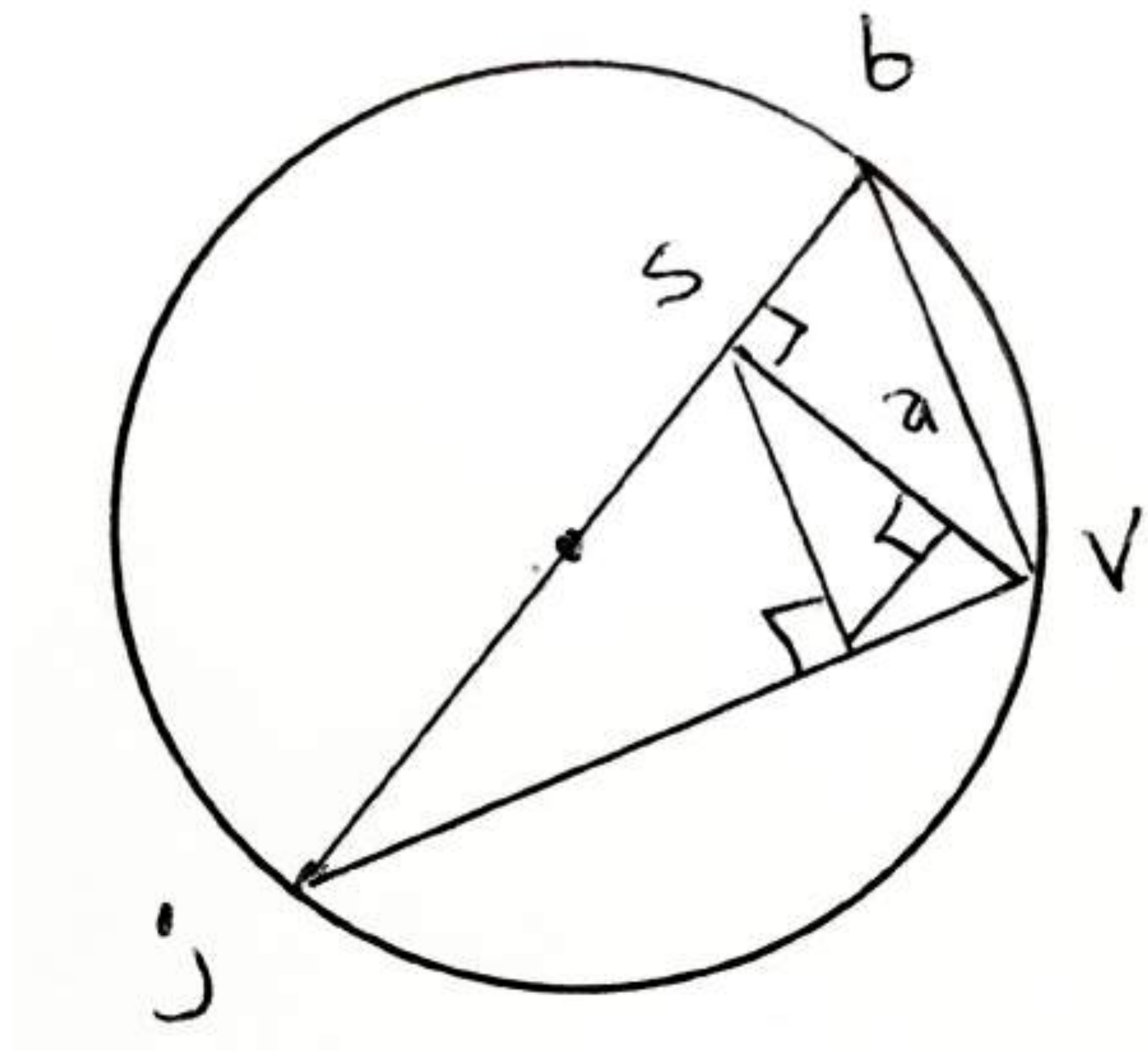




When:

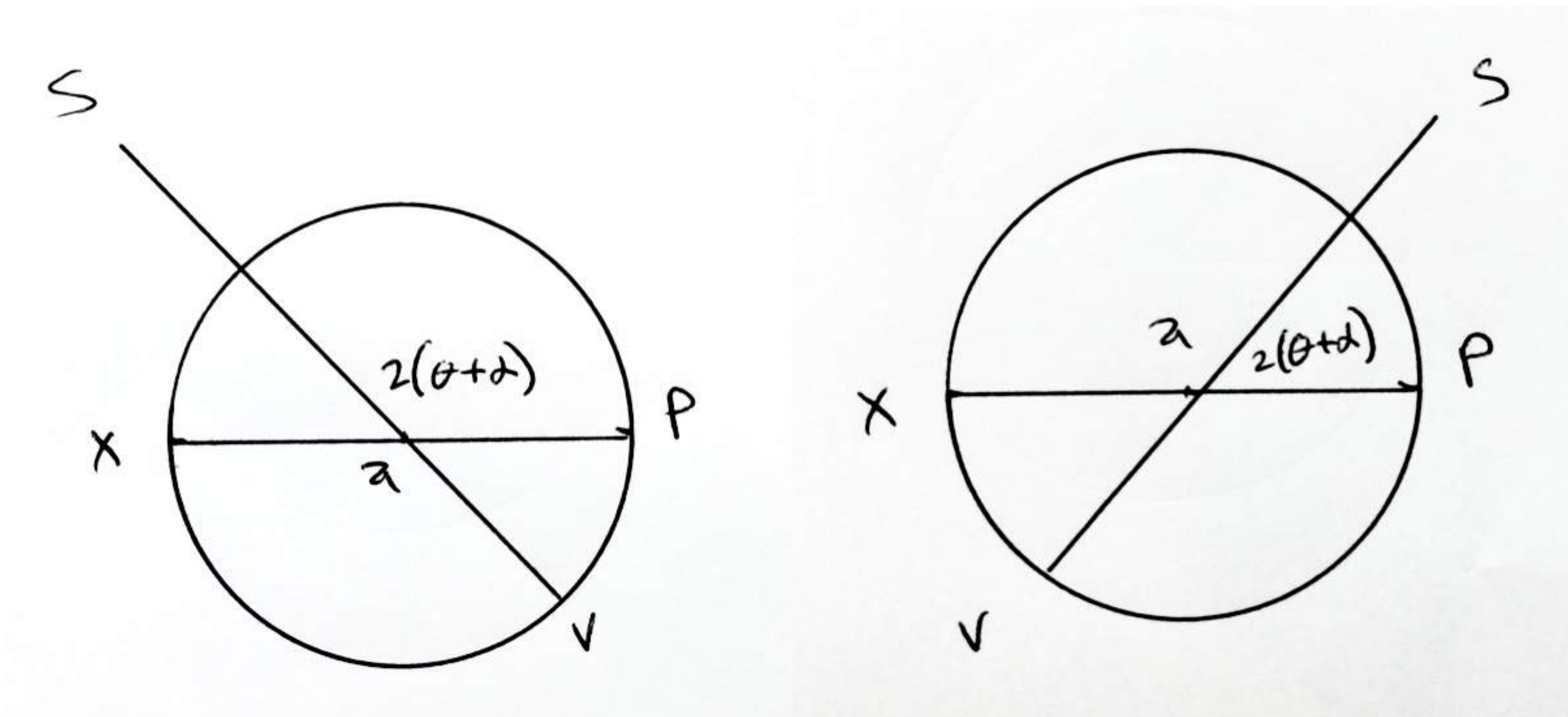
$$\frac{SO^2}{SV^2} = \frac{Sj^2}{SV^2} = \frac{aS}{aV}$$

as drawn:

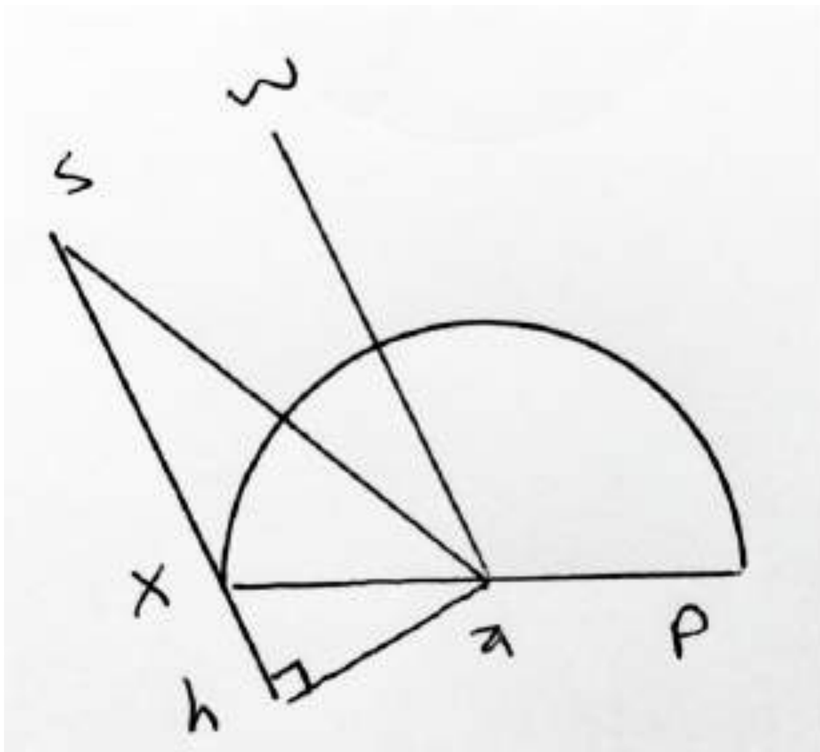


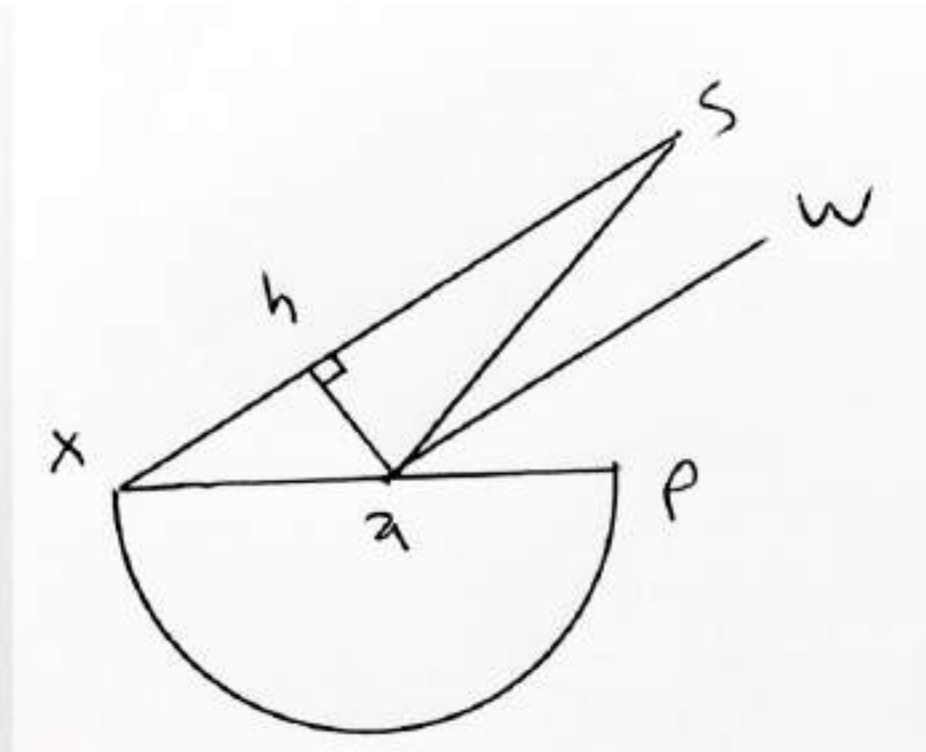
If we draw diameter XaP so:

aX = aV,   and   ∠SaP = 2 (θ + α)



$$\frac{SO^2}{SV^2} = \frac{aS}{aX} = \frac{ah/aX}{ah/aS} = \frac{\sin 2\theta}{\sin 2\alpha}$$



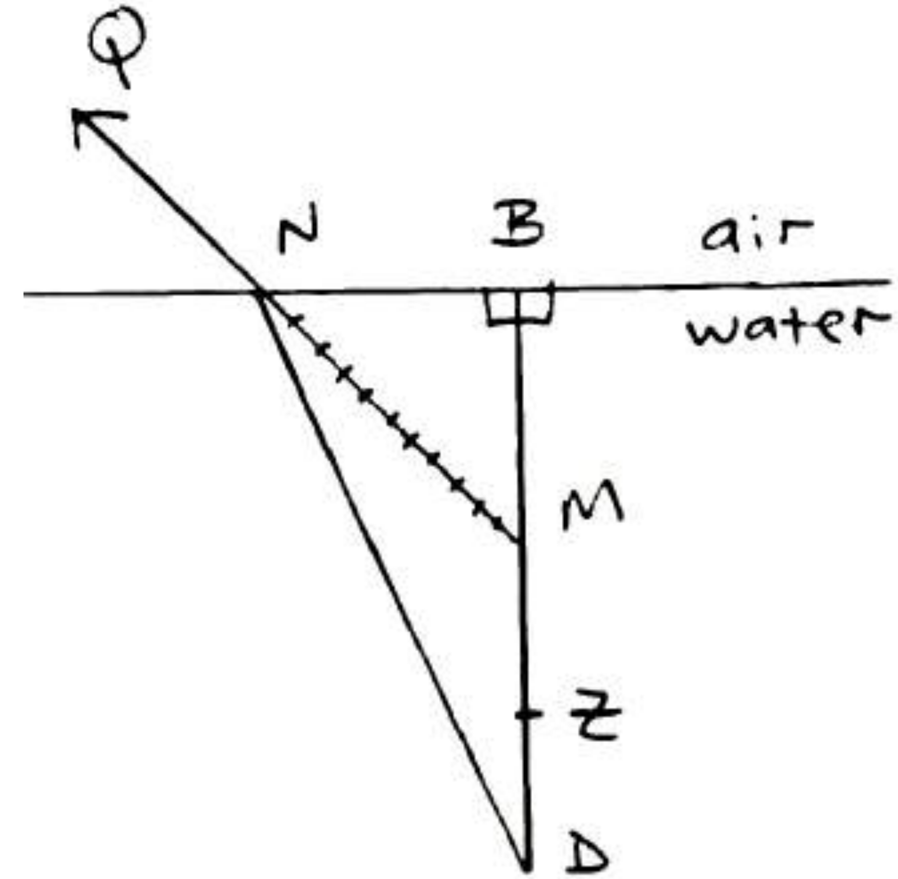


When aw || sX, we have divided the doubled angle
2 (θ + α) = ∠SaP
into 2θ = ∠WaP, and 2α = ∠WaS.

Images Seen Through Water

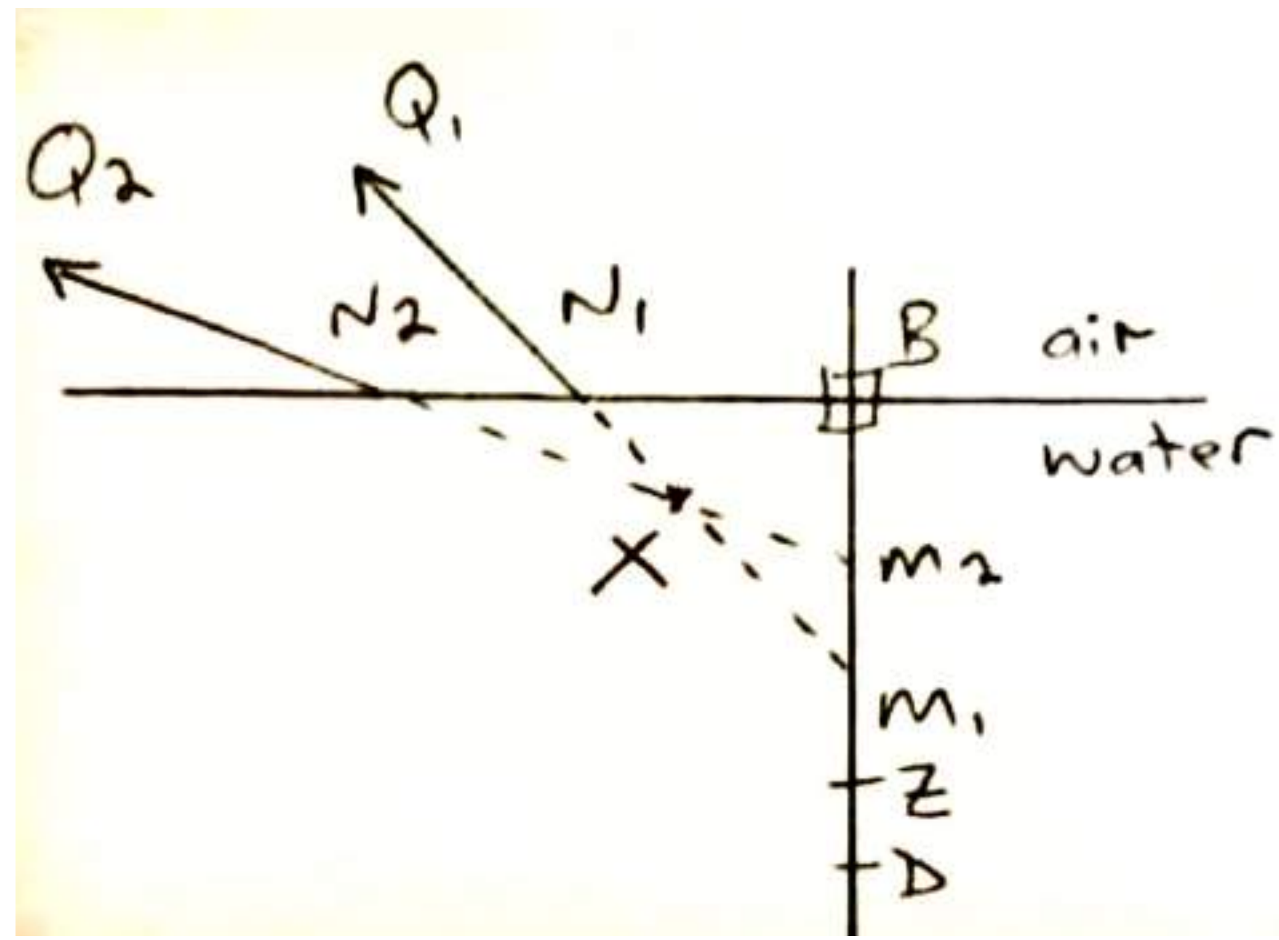
Object in water; image seen from air

If underwater object D is a perpendicular distance DB from the plane of the water surface in all radial directions, the image of object D along that perpendicular, when seen from directly above in air, is at Z, and BD/BZ = 4/3.



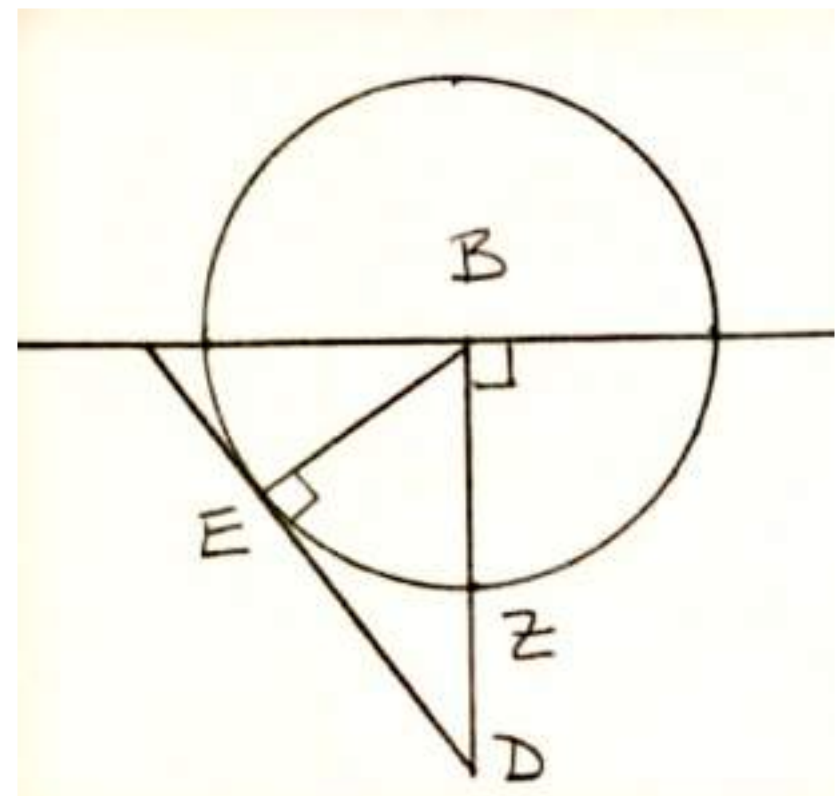
Isaac Barrow showed that the image of object D, when seen from Q *obliquely* along image ray MNQ, also lies above the object, but towards the observer relative to DB.

Isaac Barrow described a way to find all oblique image rays MNQ through a designated point X, without knowing their points of refraction (N) along the surface of the water, or their intersections (M) with the perpendicular DB.

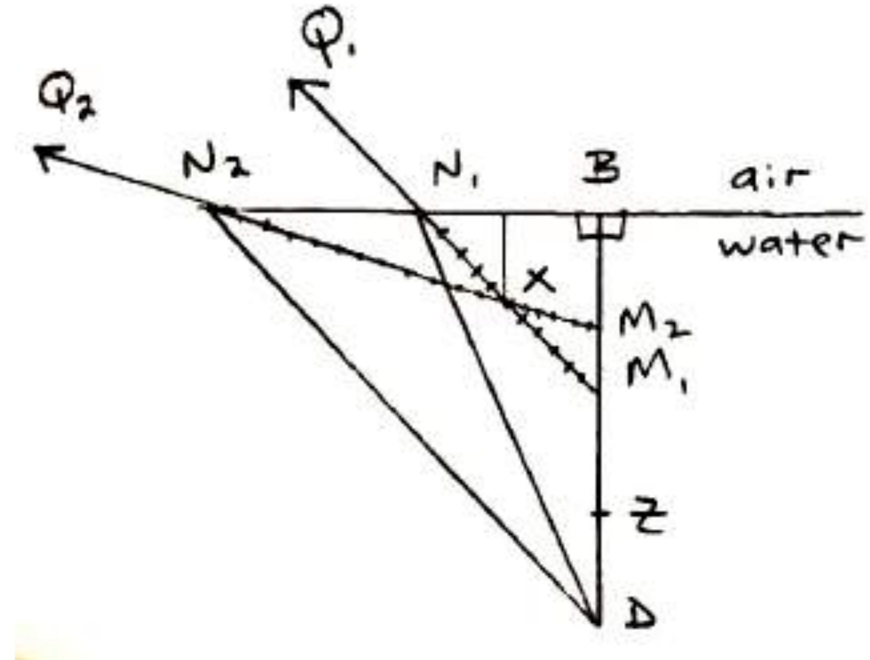


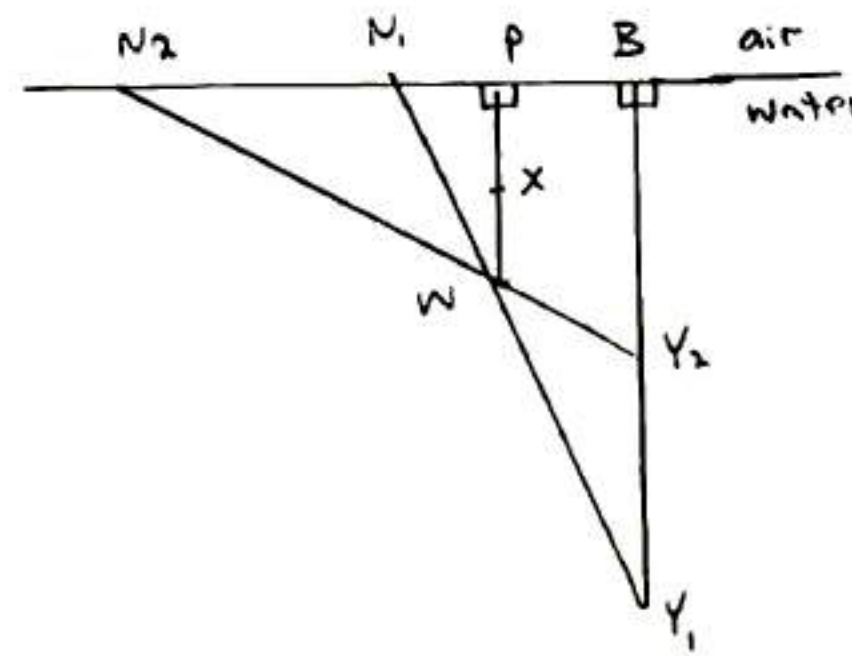
He first drew a reference right triangle created by drawing BE = BZ as shown, which created the following constant ratios for air/water refraction:

BD/BZ = BD/BE = 4/3
DB/DE = 4/√(16-9) = 1.5
ED/EB = [√(16-9)]/3 = 0.87



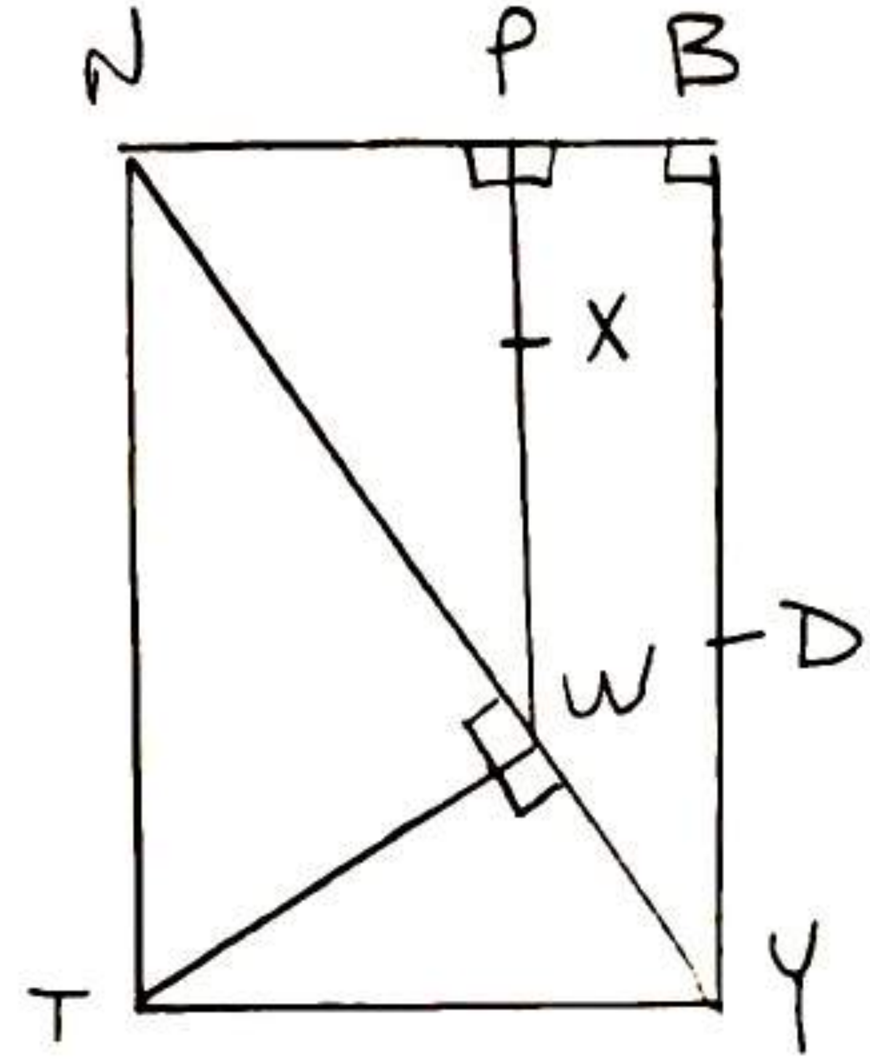
He showed that, given DB and the designated point X, if we draw the reference line segment PXW as shown, so that:
PW/PX = DB/DE = 1.5
then all image rays through X, (MXNQ) are found using:
DB/YN = ED/EB = 0.87
by drawing all possible reference lines of length YN = DB/0.87 through W, in order to locate the required positions of N.





This means that for any given DB, there can be a maximum of two image rays through the designated point X, since only two reference line segments within the right angle ∠(Y)B(N), and equaling his calculated constant YN, can fit through point W.

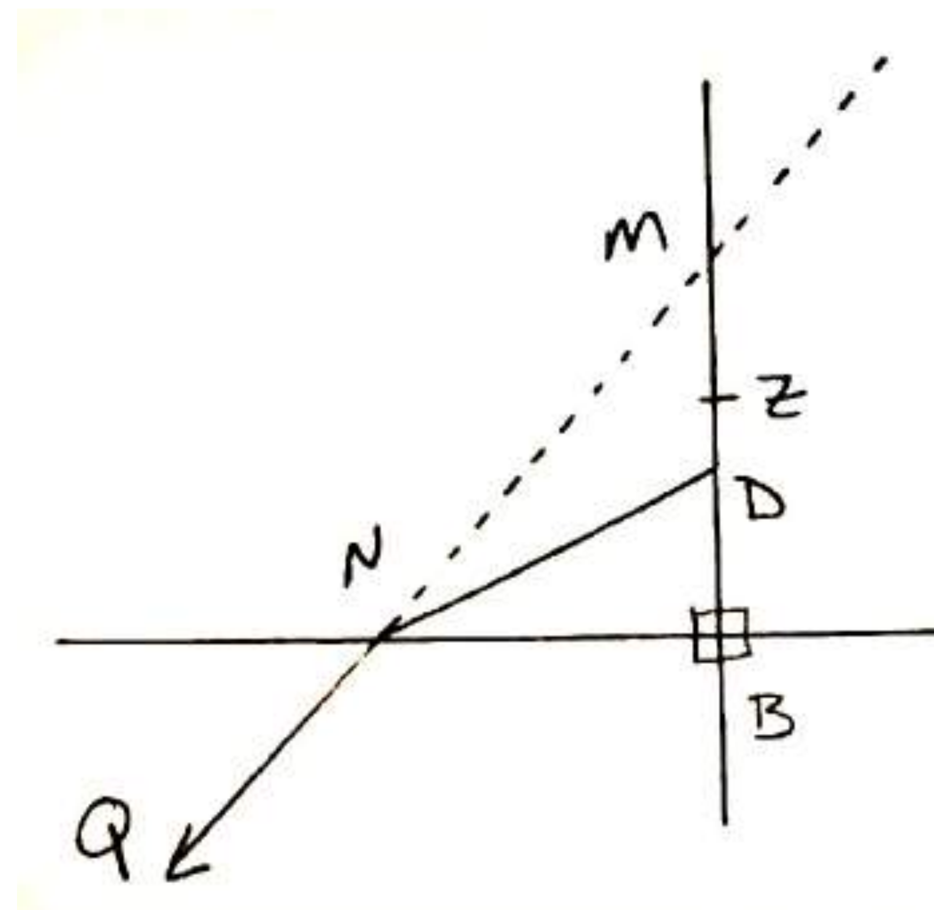
Isaac Barrow showed that YN can be drawn as the shortest segment through W bounded by the right angle ∠(Y)B(N) when right triangles ΔYBN, ΔNWT, and ΔTWY are all drawn as similar.



Keeping P constant, as we vary length YN = DB/0.87 through W to find its minimum, the position of D must vary, while PW/PX (= DB/DE) = 1.5 can remain unchanged. Therefore, when the object is in water, Isaac Barrow's analysis can find the image ray XMNQ for a designated clear image X, and an undesignated object D.

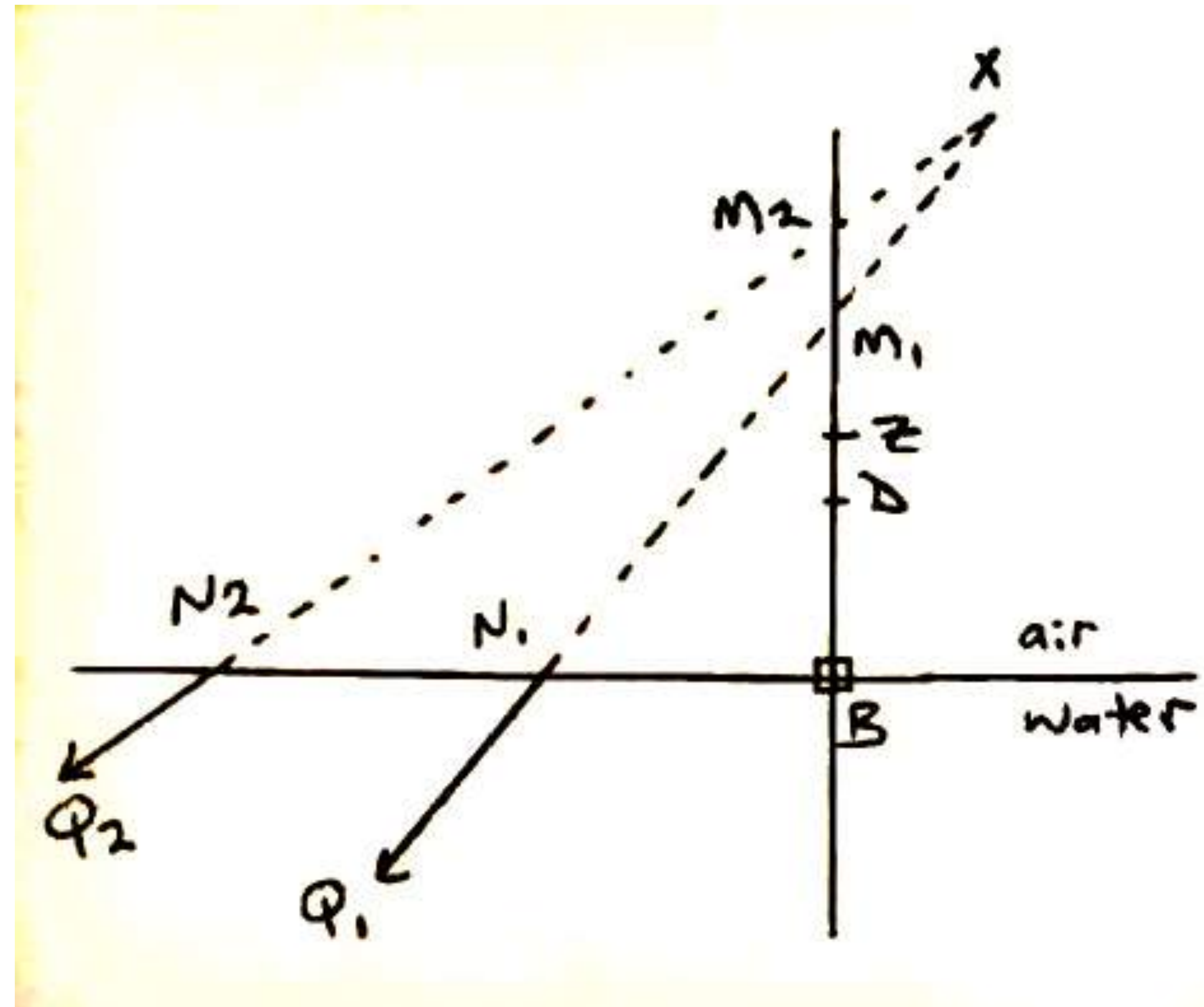
Object in air; image seen from water

If object D is in air, and at a perpendicular distance DB from the surface of water in all radial directions, the image of the object along that perpendicular when seen from underwater is at Z, and BZ/BD = 4/3.



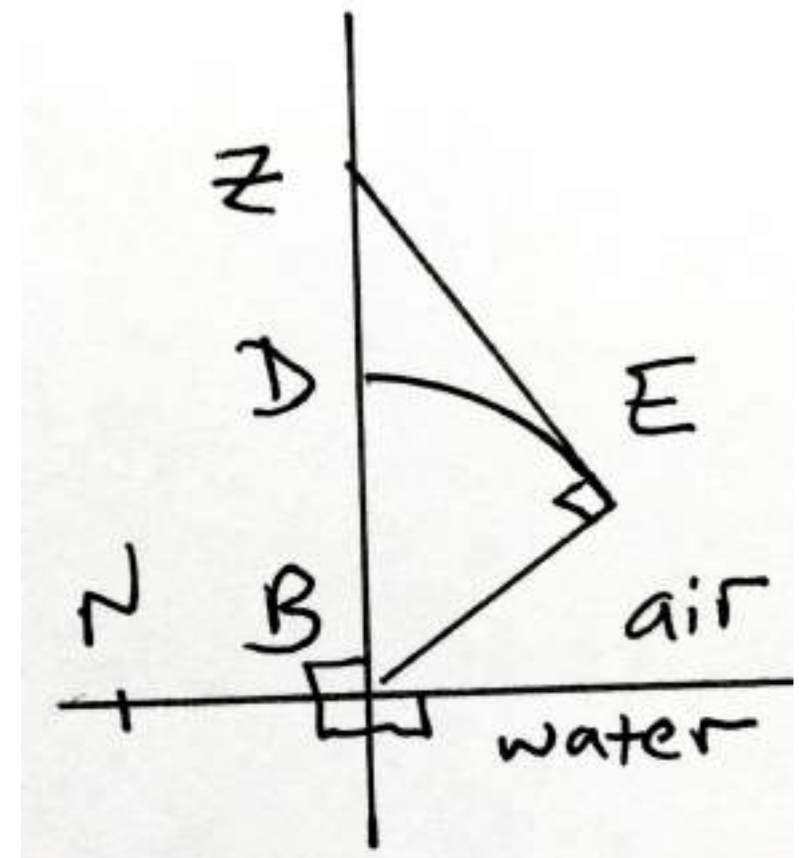
Isaac Barrow showed that the image of object D, when seen from Q *obliquely* along image ray MNQ, also lies above the object, but away from the observer relative to DB.

Isaac Barrow described a way to find all oblique image rays MNQ through a point X, without knowing their points of refraction (N) along the surface of the water, or their intersections (M) with the perpendicular DB.

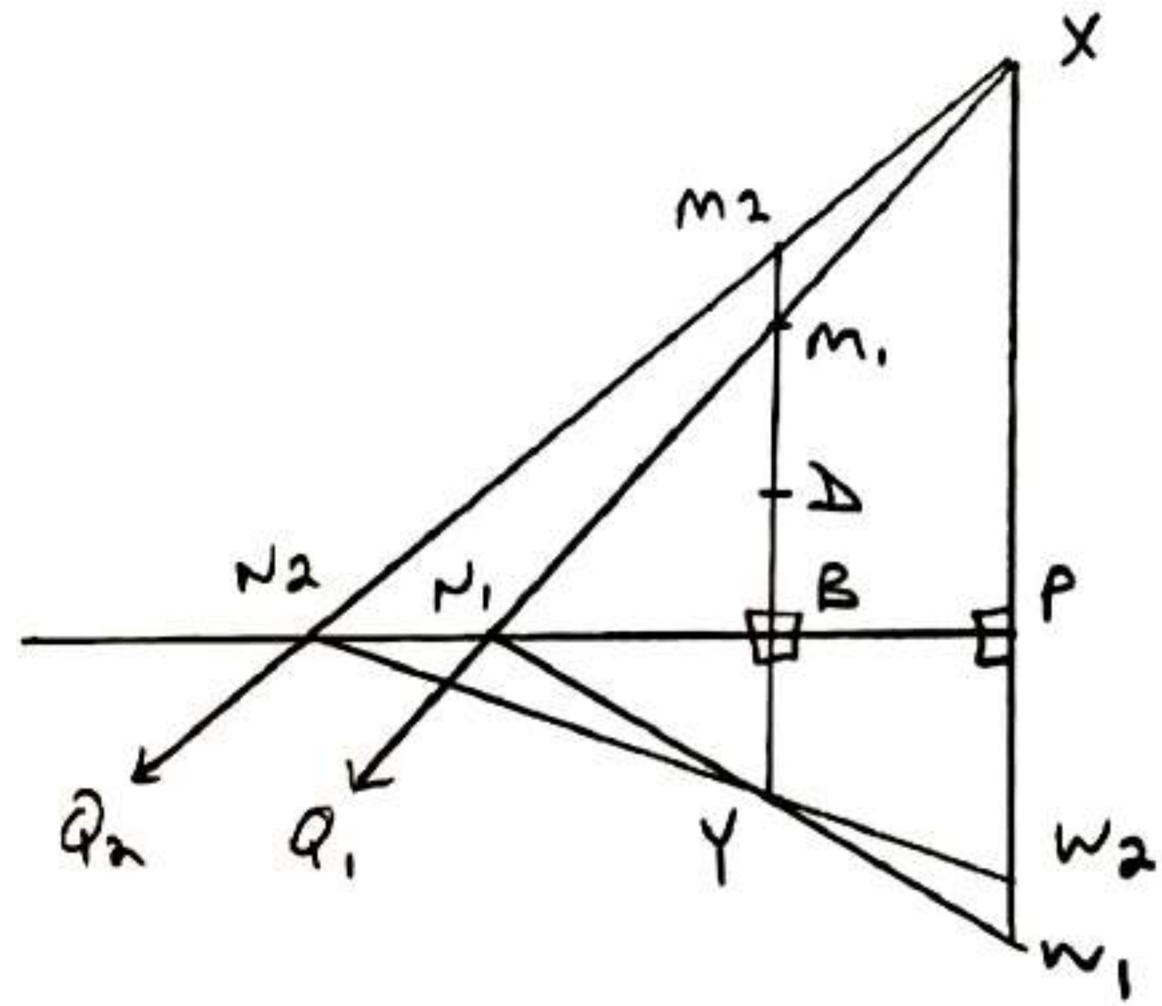


He first drew a reference right triangle created by drawing BE = BD as shown, which created the following constant ratios for air/water refraction:

BZ/BD = BZ/BE = 4/3
ZB/ZE = $4/\sqrt{(16-9)} = 1.5$
EZ/EB = $\sqrt{(16-9)}/3 = 0.87$

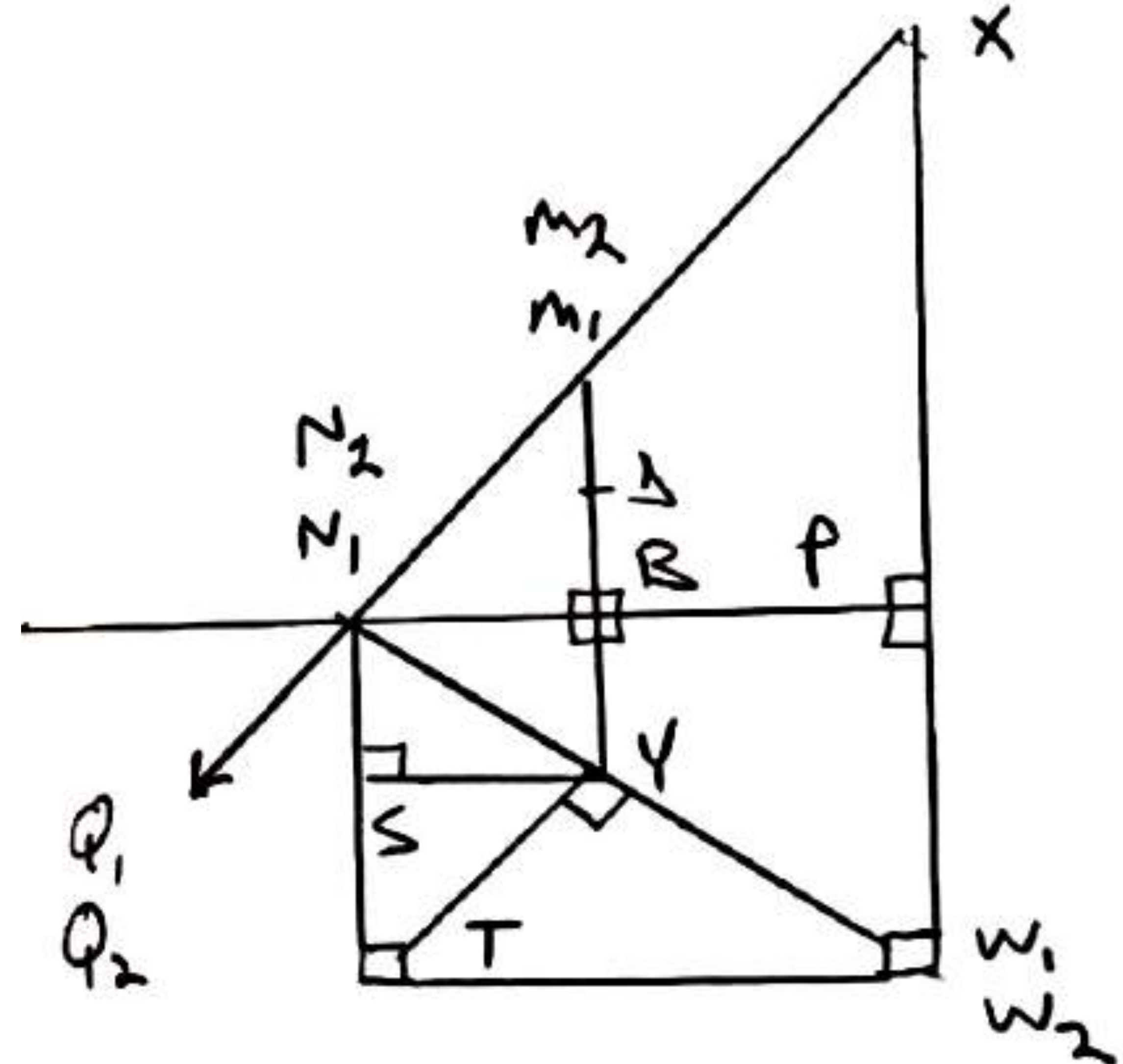


He showed that, given DB and the designated point X, if we draw
BY/BD = ZB/ZE = 1.5
then all image rays through X, (XMNQ) are found using:
XP/WN = MB/YN
= EZ/EB = 0.87
by drawing all possible reference lines of length WN = XP/0.87 through Y, in order to locate the required positions of N.



This means that for any given DB, there can be a maximum of two image rays through the designated point X, since only two reference line segments within the right angle ∠(W)P(N), and equaling his calculated constant WN, can fit through point Y.

Isaac Barrow showed that WN can be drawn as the shortest segment through Y bounded by the right angle ∠(W)P(N) when right triangles ΔWPN, ΔNYT, and ΔWYT are all drawn as similar.

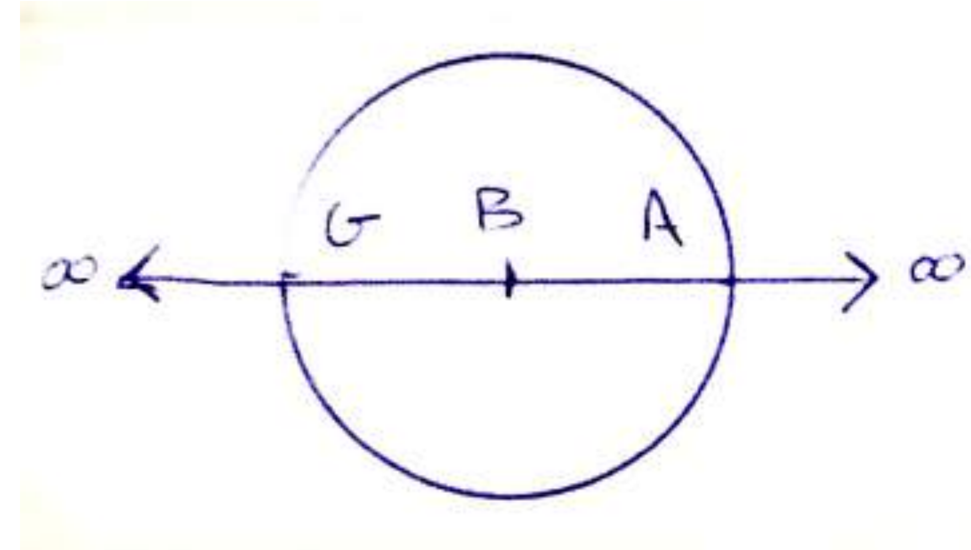


Keeping P constant, as we vary length WN = XP/0.87 through Y to find its minimum, the position of X must vary, while BY = DB(1.5) can remain unchanged. Therefore, when the object is in air, Isaac Barrow’s analysis can find the image ray XMNQ for a designated object D, and an undesignated clear image X.

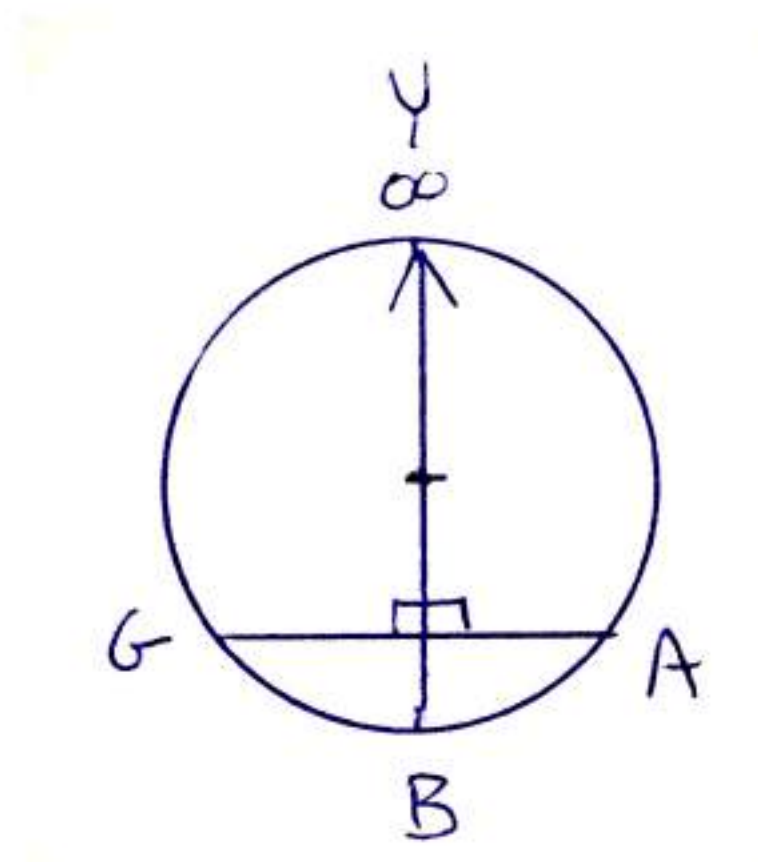
Using conic sections

The complex geometry supporting Isaac Barrow's discussion of images seen through water will be explained in the context of refraction through a flat glass surface using conic sections.

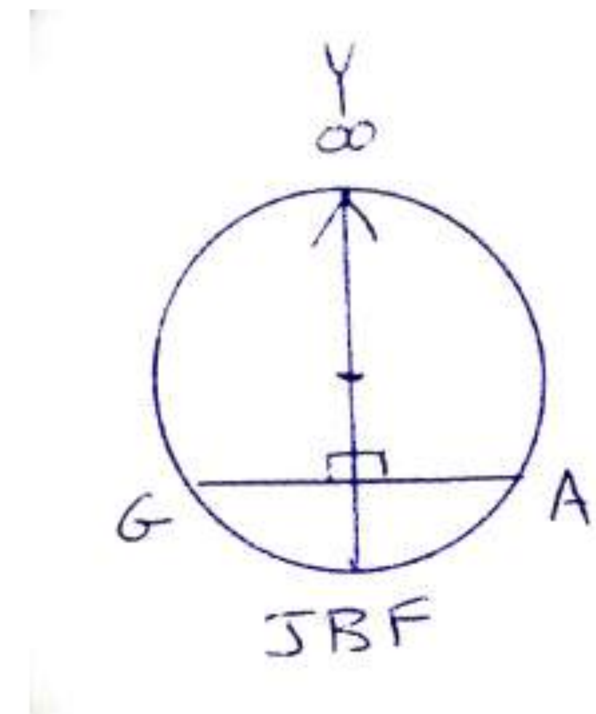
If we consider a circle with center B and diameter GBA with an “axis” infinitely long through GBA:



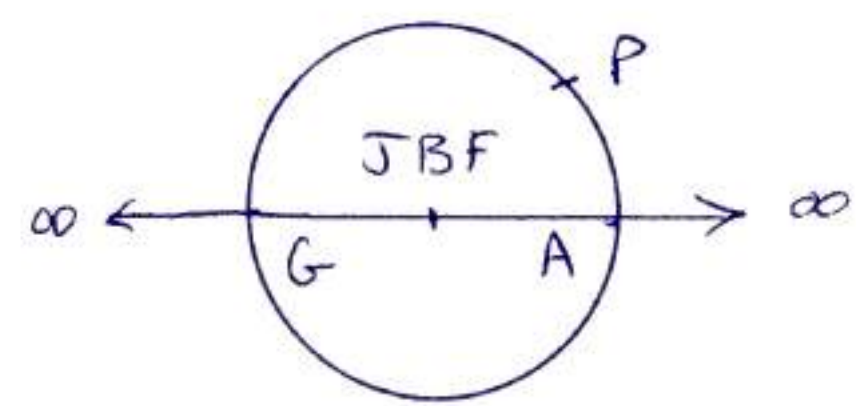
we can represent GBA along a circle of infinite diameter BY, and draw BG = BA.



If we call points J & F, (both of which in this case lie at B), the “focal points” of the finite circle, we can consider the shape of the finite circle with diameter GBA to equal its “eccentricity” = e = BF/BA = 0.



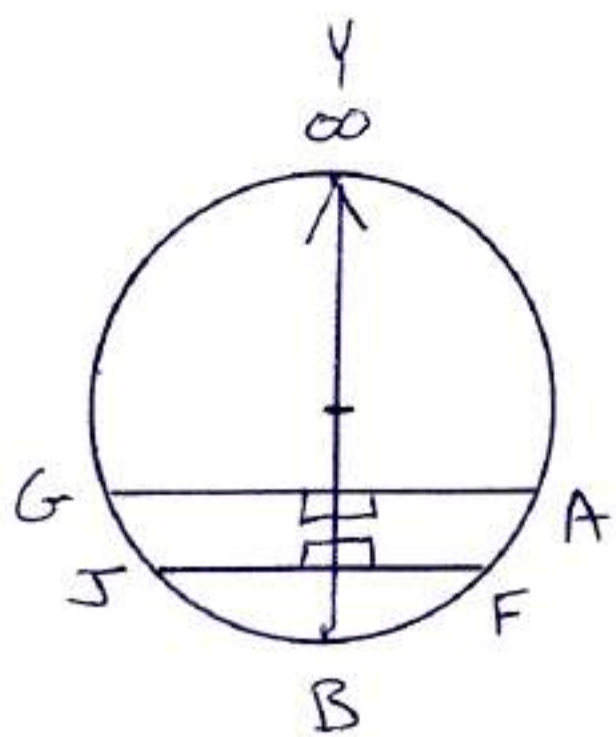
We will have drawn a finite circle where AJ + AF = AG along its diameter and “axis” GJBFA, if it is also true that:

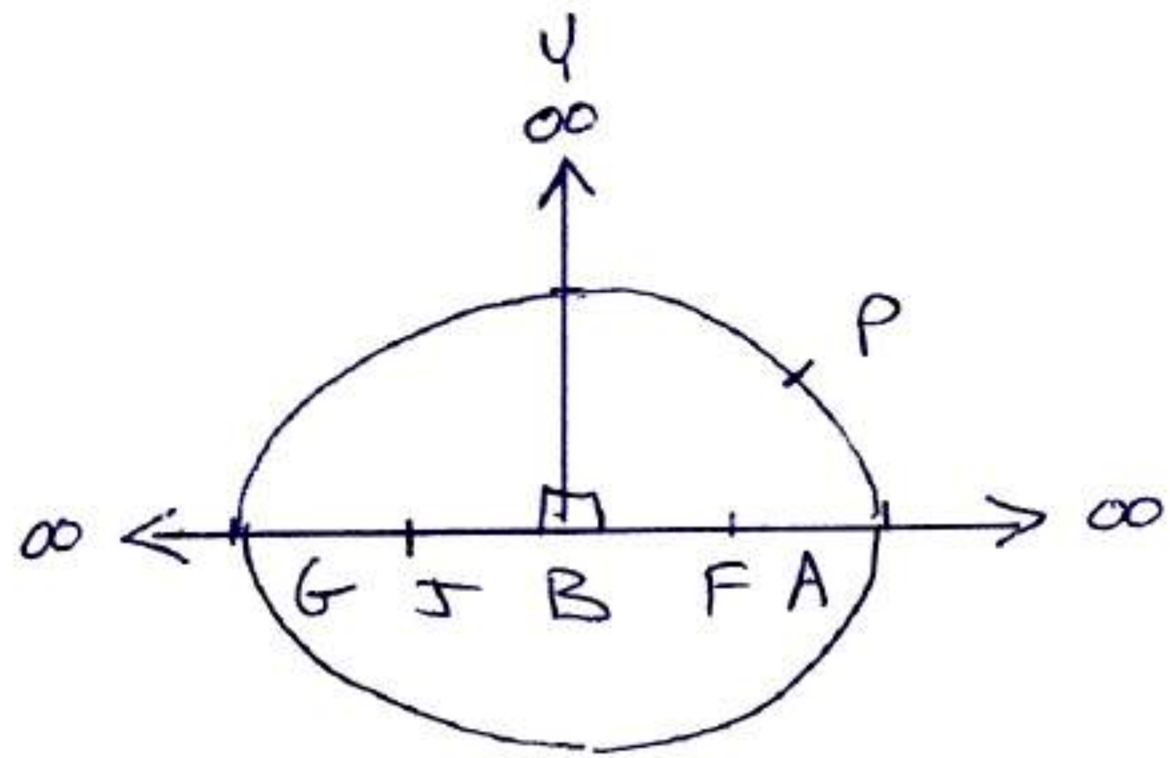


PJ + PF = AG

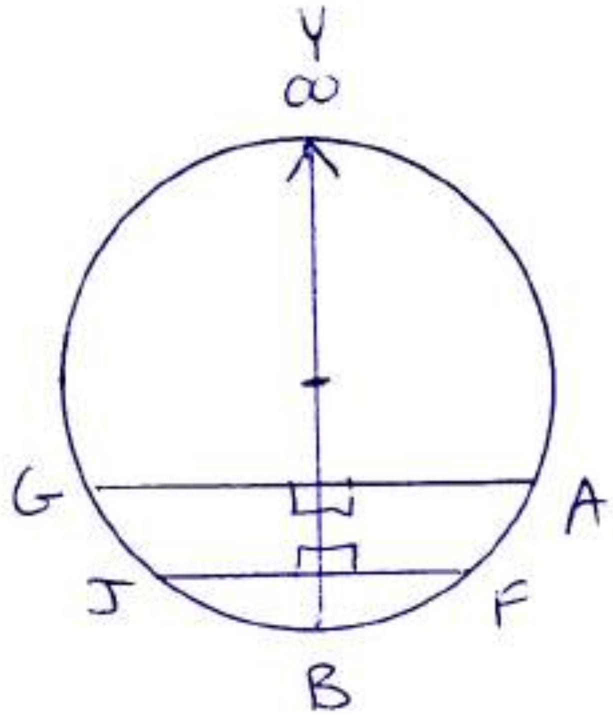
If we draw: $0 < e = BF/BA < 1$

we will have drawn a finite **ellipse** where AJ + AF = AG along its "major axis" GJBFA, if it is also true that PJ + PF = AG.

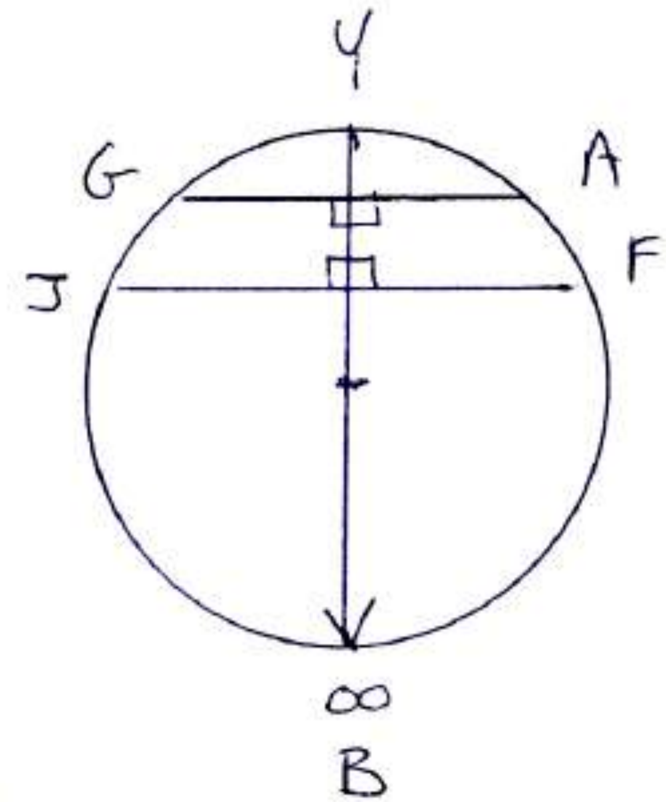




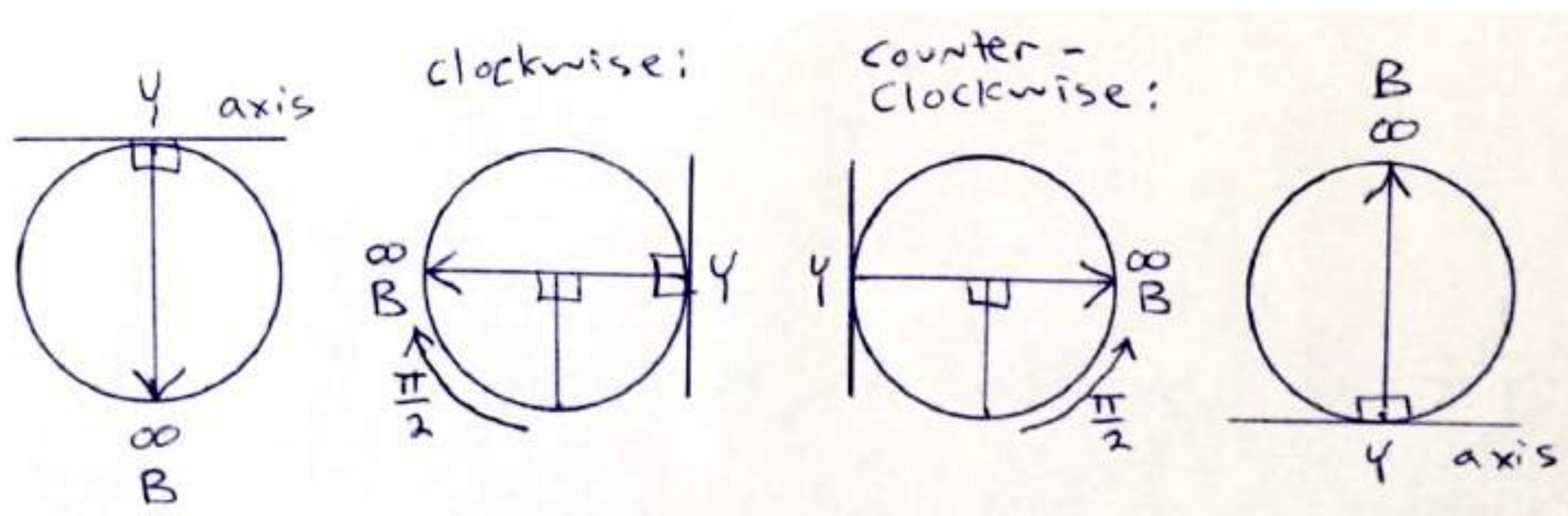
As:



becomes:

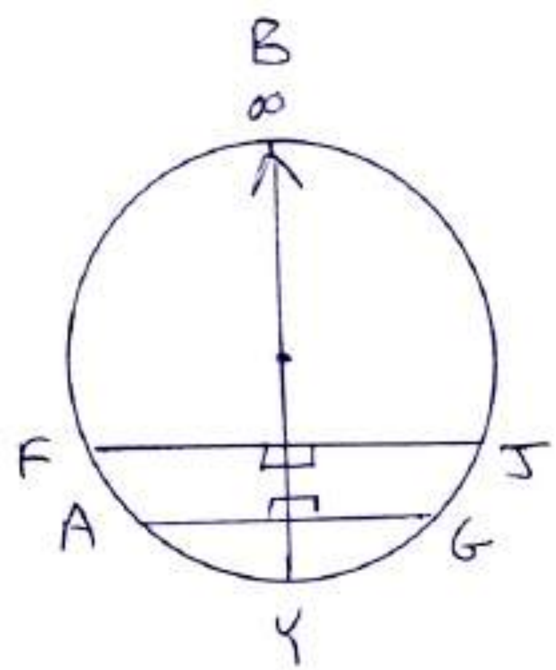


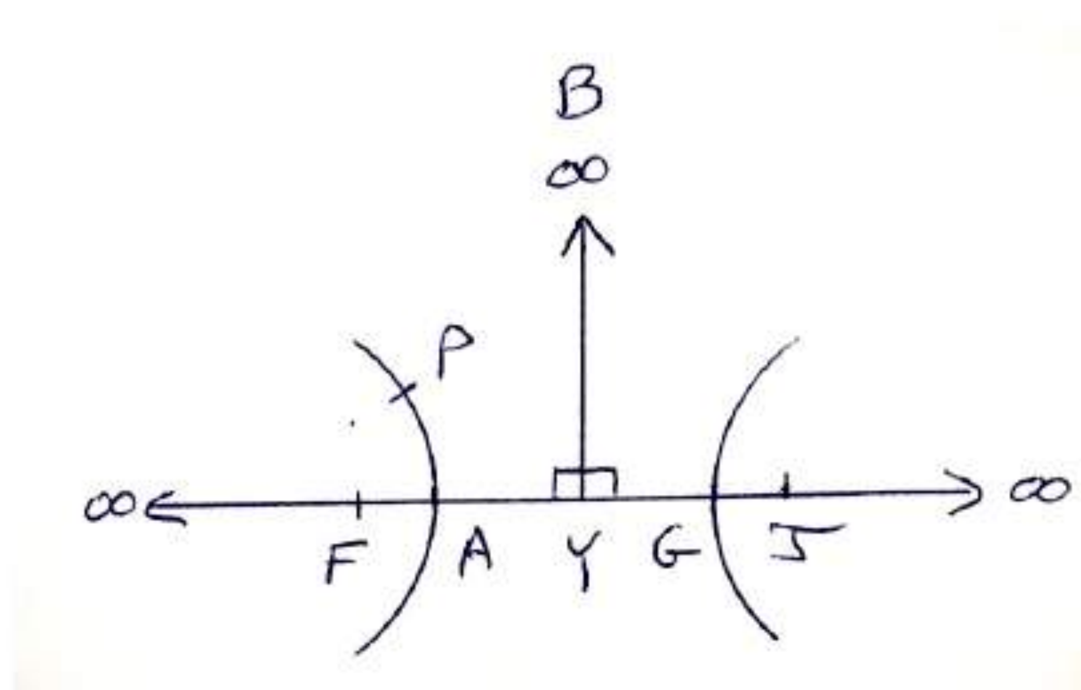
and rotates:



If we draw: 0 < e = YF/YA > 1

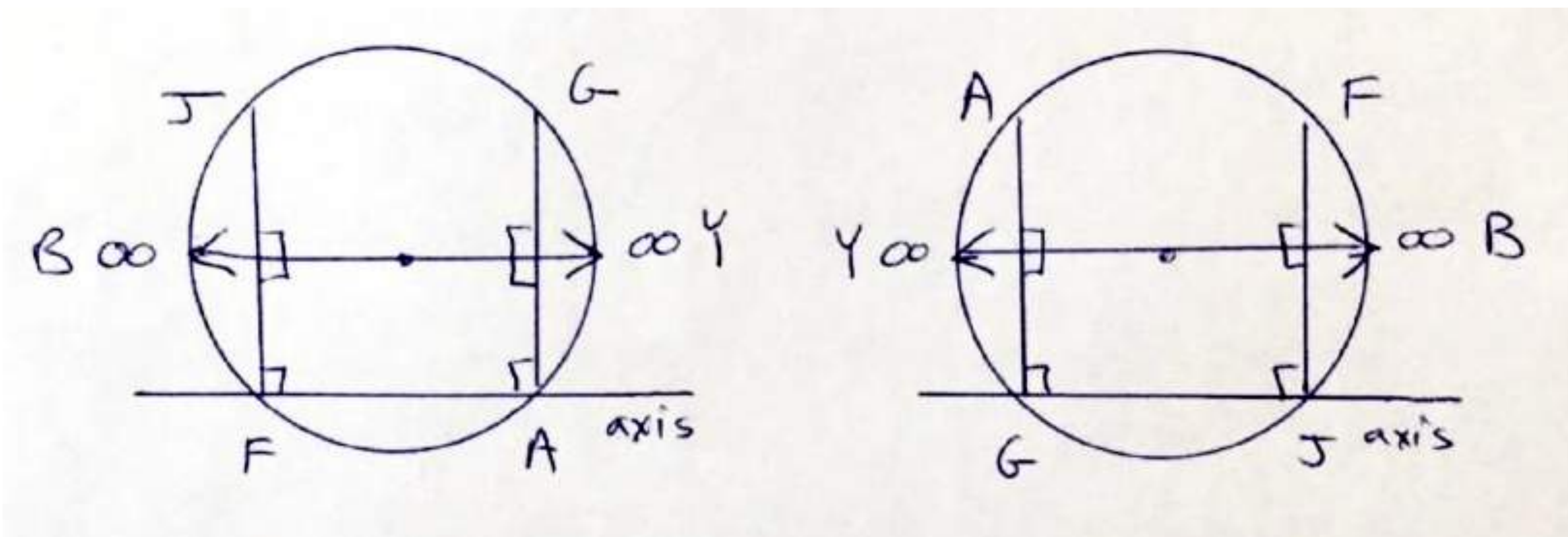
we will have drawn a **hyperbola** where AJ - AF = AG along its “transverse axis” FAYGJ, if it is also true that PJ - PF = AG.



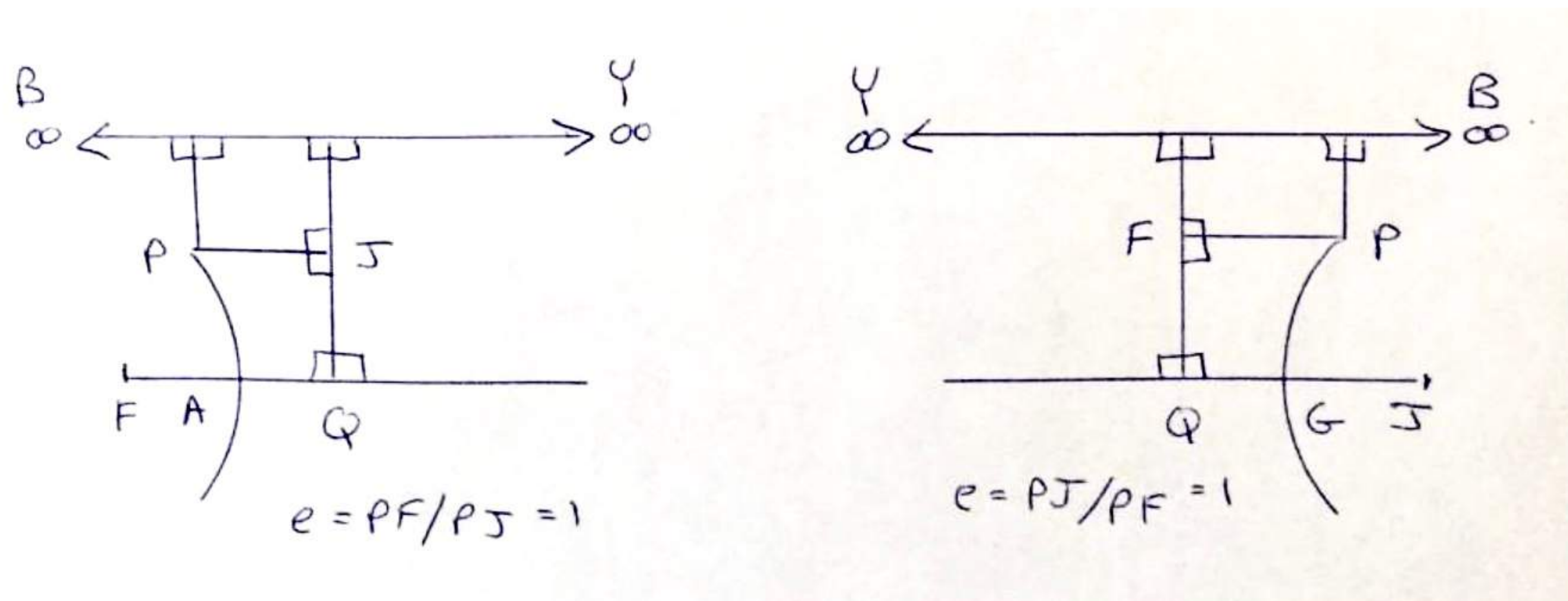


When the infinitely large reference circle only rotates by π/2 radians in either direction, it no longer remains a circle equally divided by an infinitely long upward ray with its base on an axis, because reference points B and Y are ***both*** infinitely far. However, due to the halfway rotation of the reference circle, we can presume these curves resulting from clockwise and counter-clockwise rotation have an eccentricity halfway between that of an ellipse ($e < 1$), and that of an hyperbola ($e > 1$).

These resulting curves are defined as a parabolas (e = 1), and like the circle (e = 0), they represent a special case with a singular shape, or eccentricity.



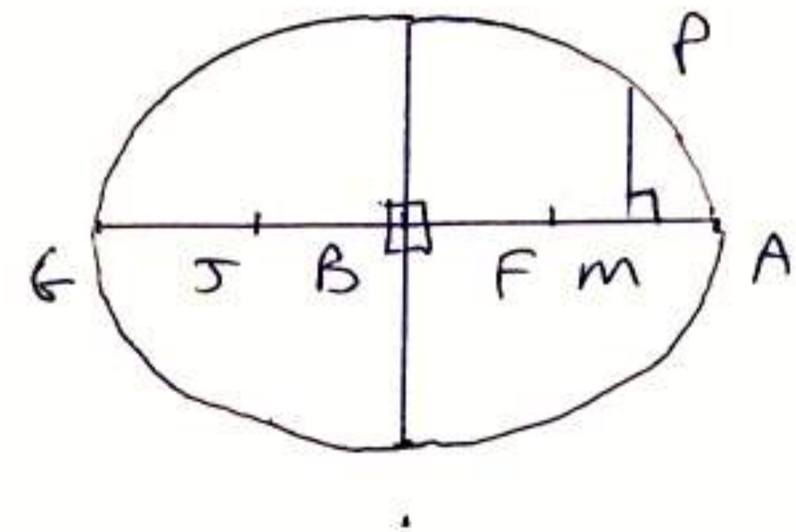
If we draw AF = GJ we will have drawn parabolas along their respective “axes” AF or GJ, if it is also true that PF = PJ. Since parabolas represent the eccentricity as an ellipse transforms into an hyperbola, (or visa versa), QA = QG.



Ellipse

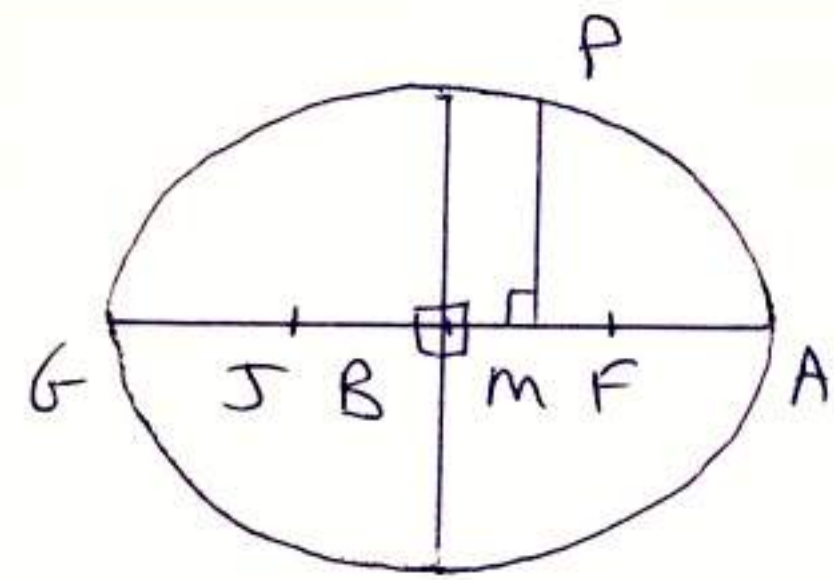
2(BF) = MJ - MF

2(BM) = MJ + MF



2(BF) = MJ + MF

2(BM) = MJ - MF



$$PJ^2 - FP^2 = (MJ^2 + MP^2) - (MF^2 + MP^2)$$

$$(PJ + FP)(PJ - FP) = (MJ + MF)(MJ - MF)$$

$$AG(PJ - FP) = 2(BM)\,2(BF)$$

$$PJ - FP = [2(BM)\,2(BF)]/2(BA)$$

$$\text{eccentricity} = e = BF/BA$$

$$PJ - FP = 2(BM)e$$

Since:

$$FP + PJ = AG = 2(BA)$$

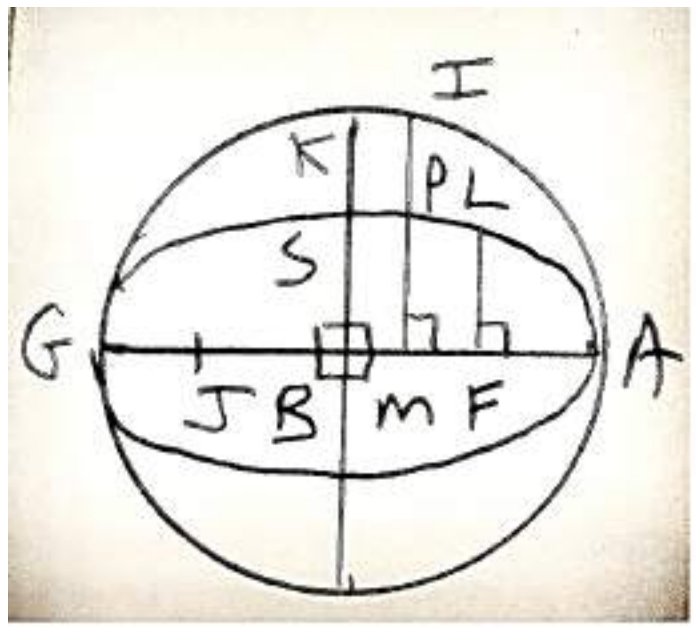
$$(FP + PJ) + (PJ - FP) = 2(PJ) = 2(BA) + 2(BM)e$$

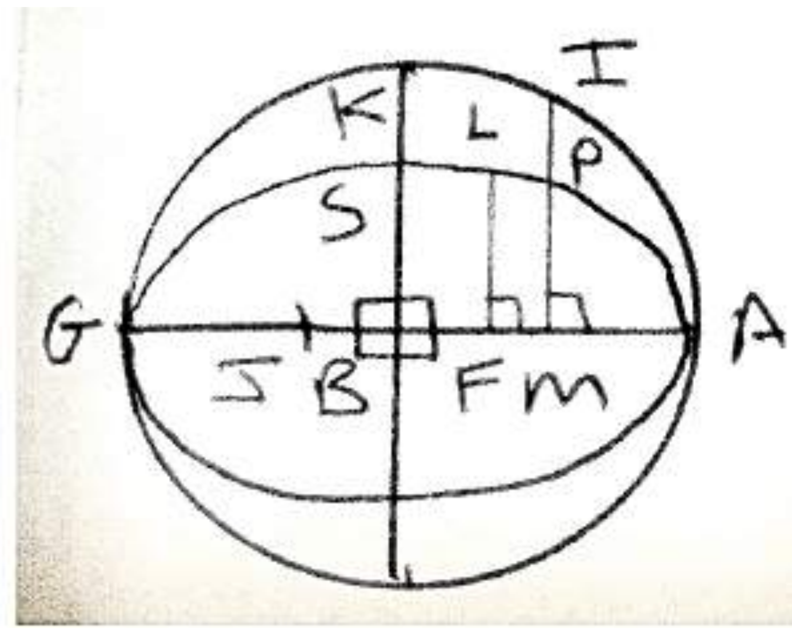
$$(FP + PJ) - (PJ - FP) = 2(FP) = 2(BA) - 2(BM)e$$

$$\mathbf{PJ = BA + (BM)e}$$

$$\mathbf{PF = BA - (BM)e}$$

$FM = BF - BM$ $FM = BM - BF$





$$FM^2 = BF^2 + BM^2 - 2(BF)BM$$

$$e = BF/BA = FB/FS$$

$$BA^2 = BF^2 + BS^2$$

$PF^2 = [BA - (BM)e]^2$

$PF^2 = BA^2 + (BM)^2e^2 - 2(BM)BF$

$PM^2 = PF^2 - FM^2$

$PM^2 = [BA^2 + (BM)^2e^2 - 2(BM)BF]$

$- [BF^2 + BM^2 - 2(BF)BM]$

$PM^2 = BS^2 + BM^2(e^2 - 1)$

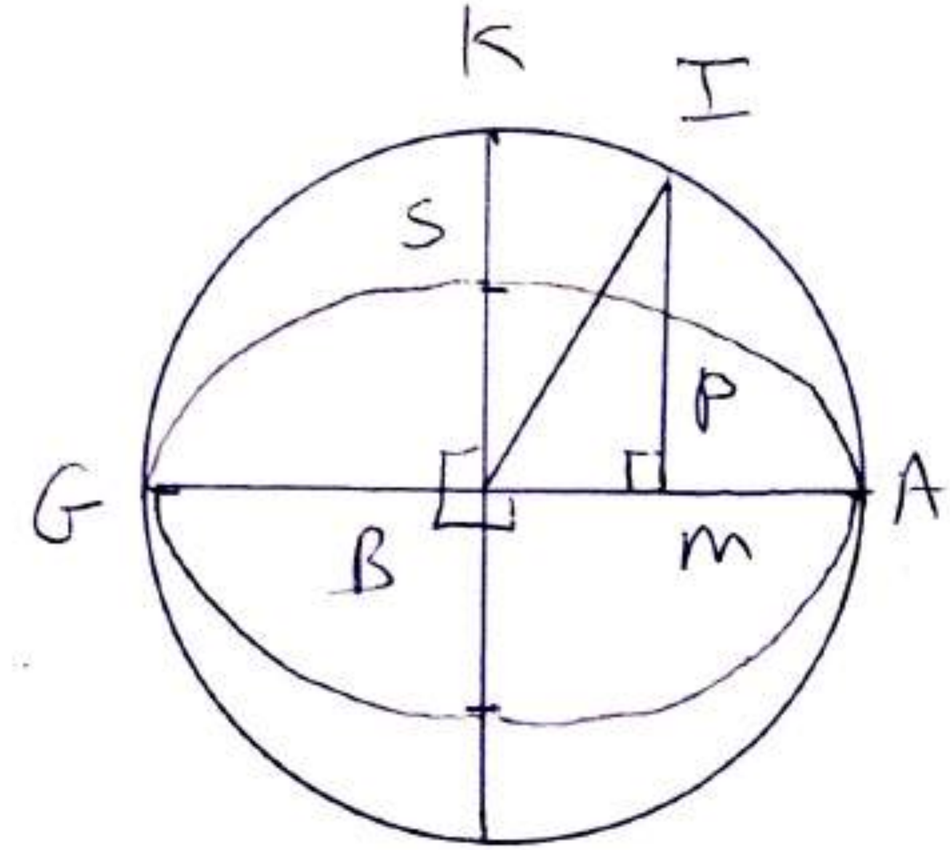
$PM^2 = BS^2 - BM^2(1-e^2)$

$(PM)^2BA^2 = (BS)^2BA^2 - BM^2[BA^2 - BF^2]$

$(PM)^2BA^2 = BS^2[BA^2 - BM^2]$

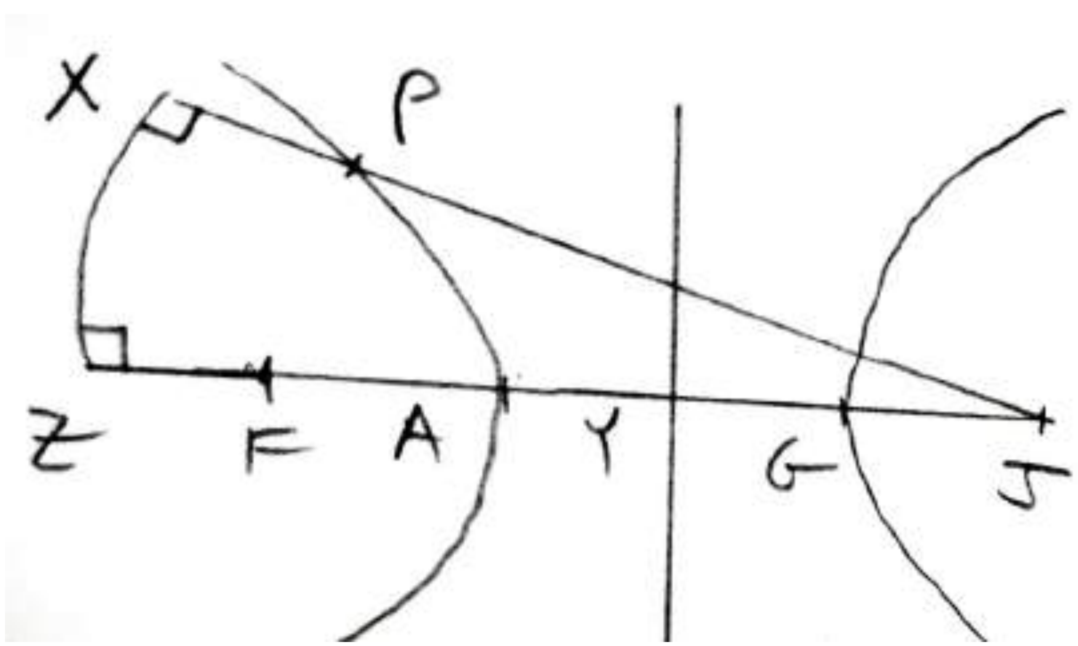
$(MP/MI)^2 = (BS/BA)^2$

$MP/MI = BS/BK$



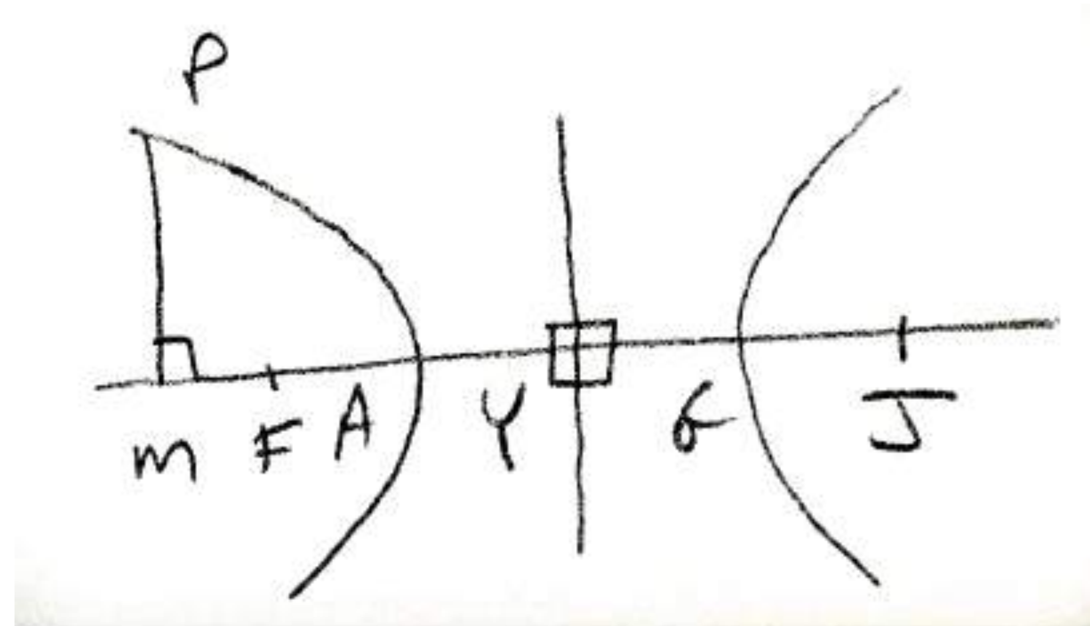
Hyperbola

Draw the hyperbola arm ~AP by making:
JZ = JX, and:
ZJ - AG = XP + FP
So: XJ - XP = FP + AG
and PJ - FP = AG



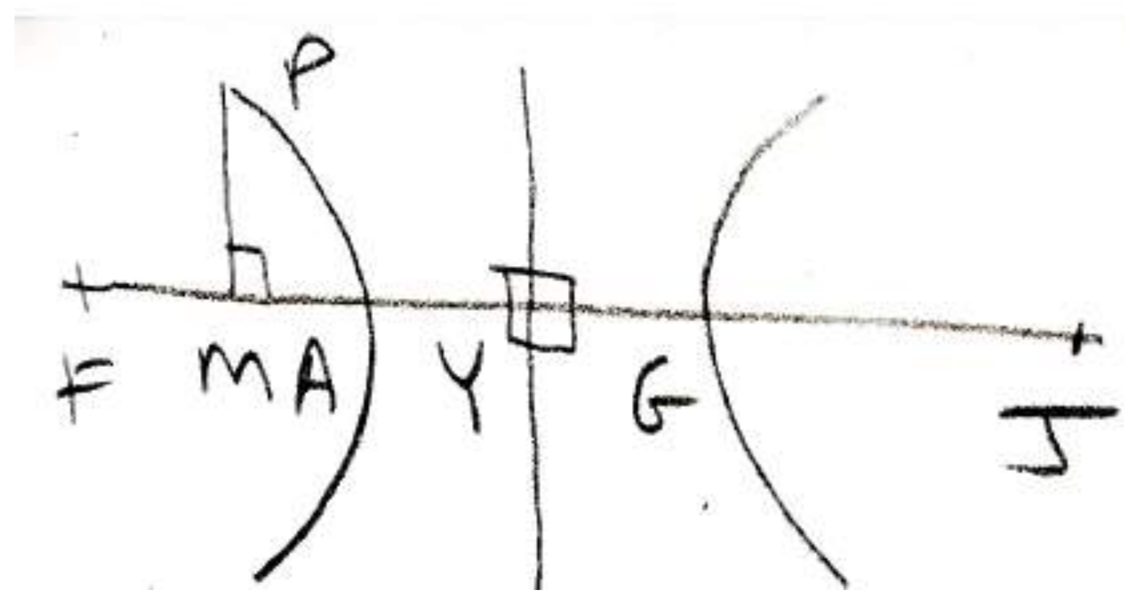
MJ - MF = 2(YF)

MJ + MF = 2(YM)



MJ - MF = 2(YM)

MJ + MF = 2(YF)



$$PJ^2 - FP^2 = (MP^2 + MJ^2) - (MP^2 + MF^2)$$

$$(PJ + FP)\ (PJ - FP) = (MJ + MF)\ (MJ - MF)$$

$$(PJ + FP)AG = 2(YM)\ 2(YF)$$

$$PJ + PF = [2(YM)\ 2(YF)]/2(YA)$$

eccentricity = e = YF/YA

$$PJ + PF = 2(YM)e$$

Since: $PJ - PF = AG = 2(YA)$

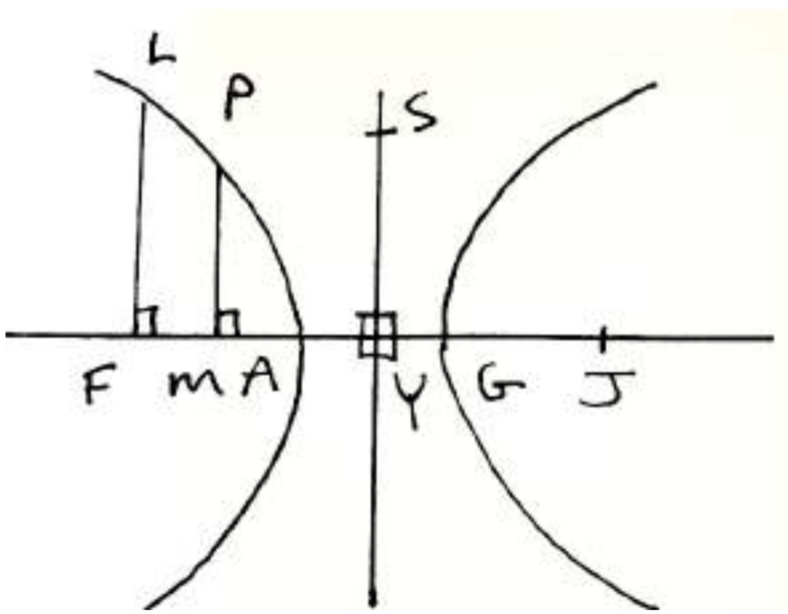
$(PJ + PF) + (PJ - PF) = 2(PJ) = 2(YM)e + 2(YA)$

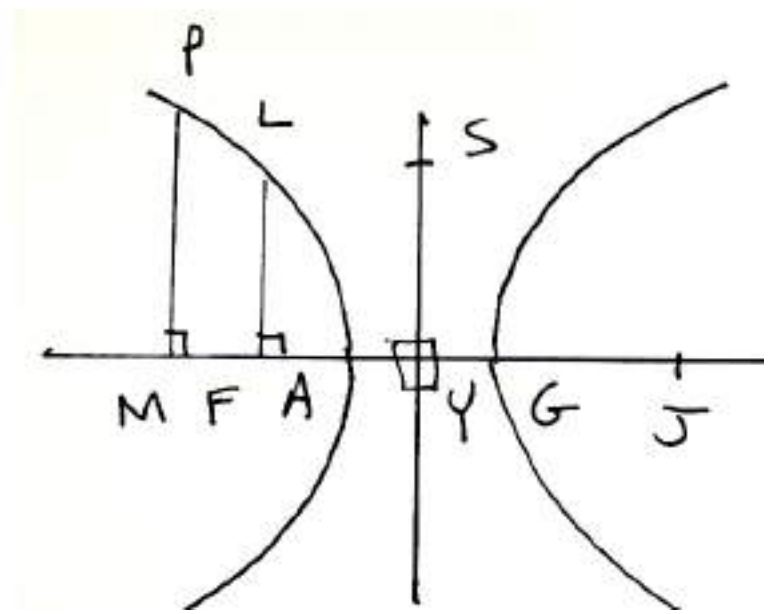
$(PJ + PF) - (PJ - PF) = 2(PF) = 2(YM)e - 2(YA)$

$PJ = (YM)e + YA$

$PF = (YM)e - YA$

$FM = YF - YM$ $\qquad$ $FM = YM - YF$





$$FM^2 = YF^2 + YM^2 - 2(YF)YM$$

$$e = YF/YA = AS/AY$$

$$YF^2 = YA^2 + YS^2$$

$$PF^2 = [(YM)e - YA]^2$$

$$PF^2 = YM^2e^2 + YA^2 - 2(YM)YF$$

$$PM^2 = PF^2 - FM^2$$

$$PM^2 = [YM^2e^2 + YA^2 - 2(YM)YF]$$

$$- [YF^2 + YM^2 - 2(YF)YM]$$

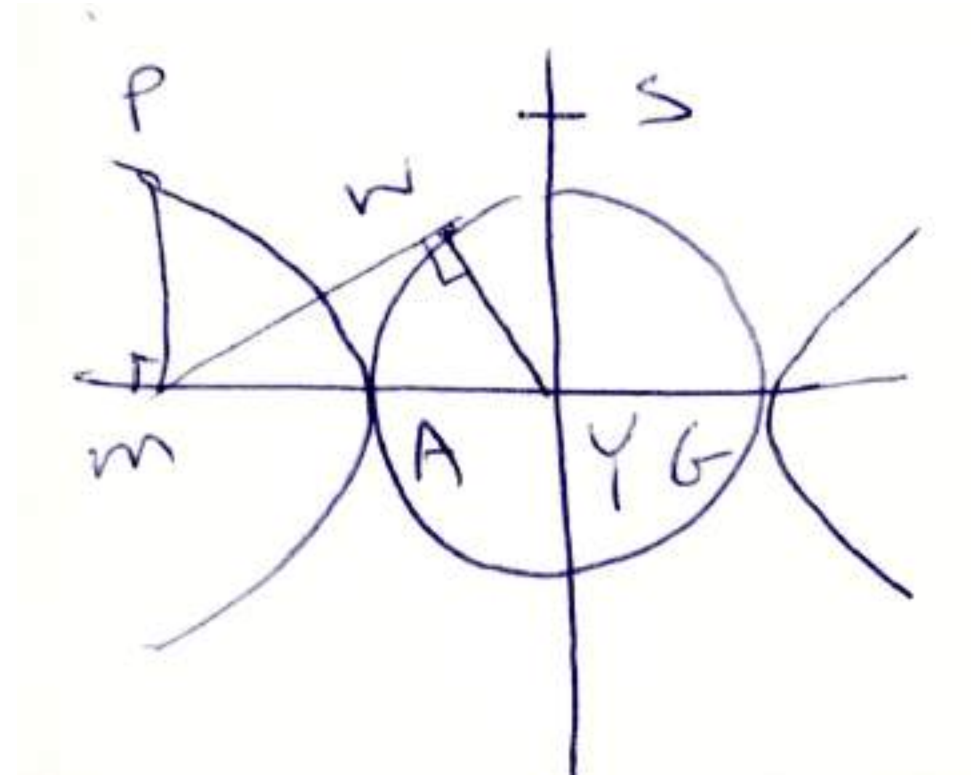
$$PM^2 = YM^2(e^2 - 1) - YS^2$$

$$PM^2\ YA^2 = YM^2[YF^2 - YA^2] - YS^2\ YA^2$$

$$PM^2\ YA^2 = YS^2(YM^2 - YA^2)$$

$$(MP/MW)^2 = (YS/YA)^2$$

$$MP/MW = YS/YA$$



$$\Delta MWA \cong \Delta MGW$$

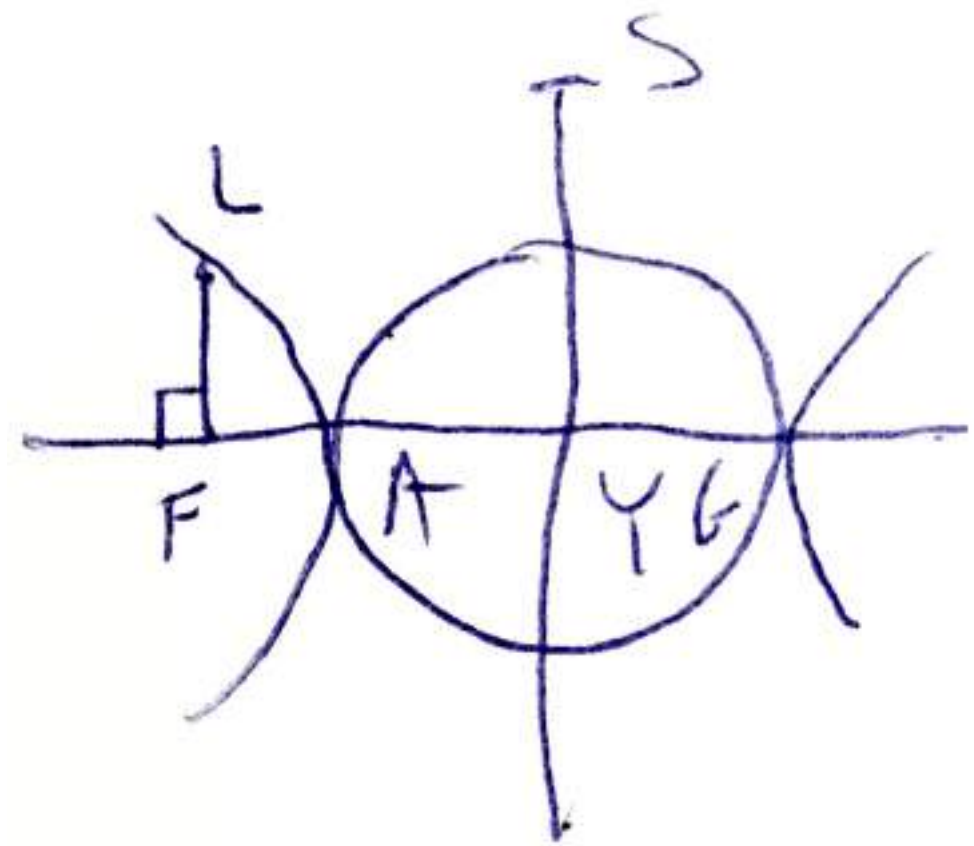
$$MW^2 = (MA)MG$$

$MP^2/(MA)MG = (YS/YA)^2 = FL^2/(FA)FG$

$(FA)FG = (YF - YA)(YF + YA)$

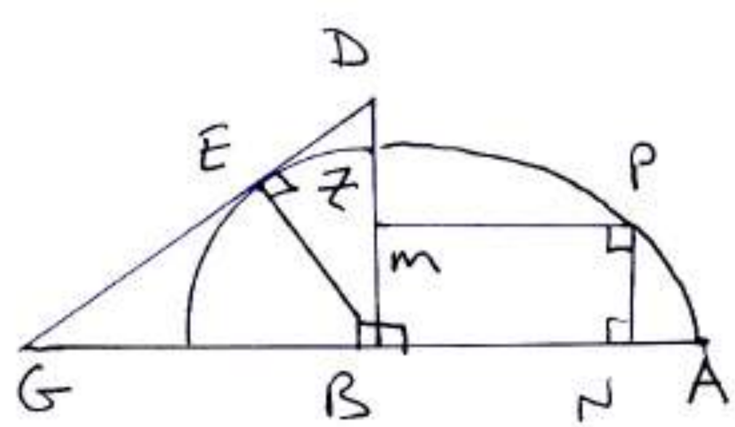
$(FA)FG = YF^2 - YA^2 = YS^2$

$FL/YS = YS/YA$

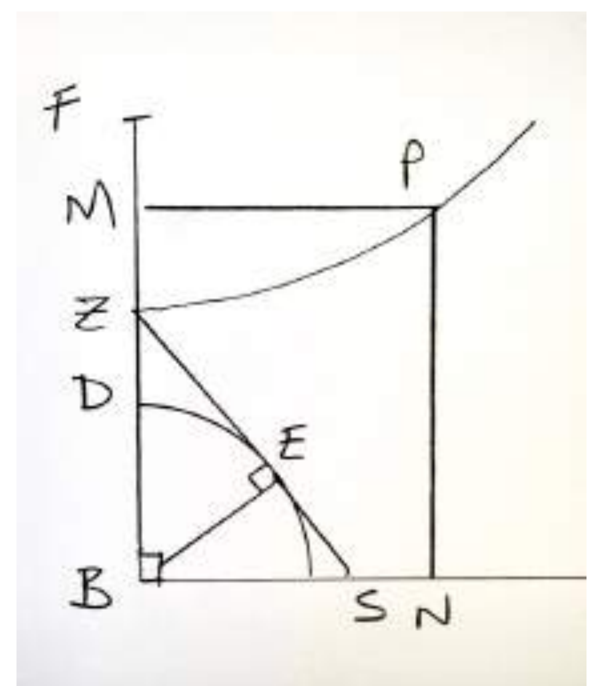


The following discussion will be presented in two columns for clarity. The left column represents the object in glass, and the right column represents the object in air.

Given refraction along line GBNA, object D in glass, and image Z seen along BZD, a non-perpendicular image ray NM can be found using the reference semi-ellipse GZPA:

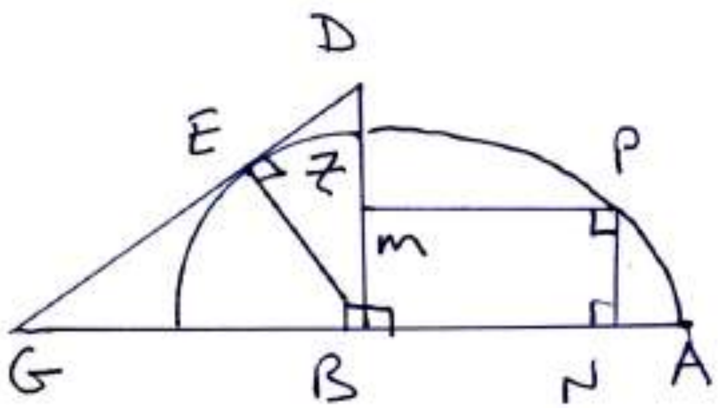


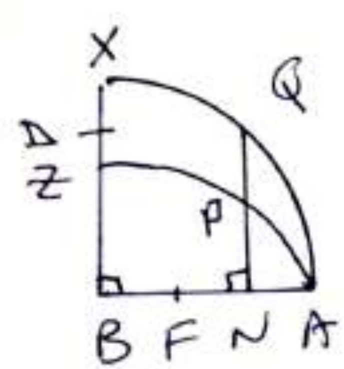
Given refraction along line BSN, object D in air, and image Z seen along BDZ, a non-perpendicular image ray NM can be found using the reference hyperbola arm ZP:

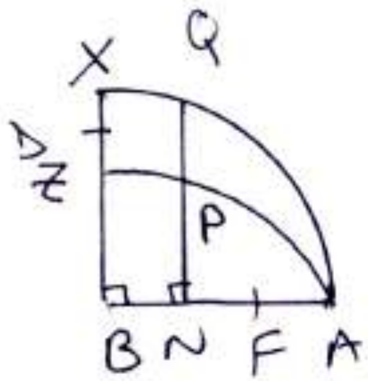


(with vertex designated as B instead of Y for consistency)

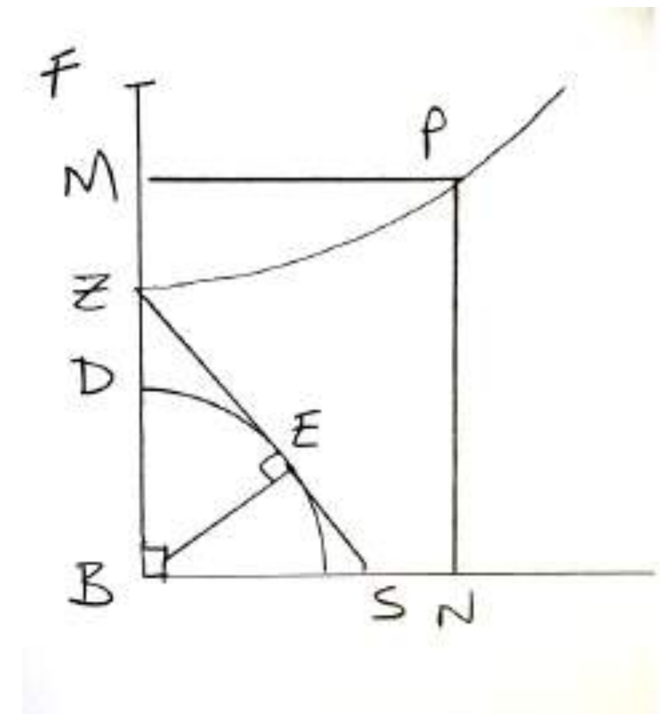
because:
e = BF/BA = FB/FZ
and: NQ/NP = BX/BZ

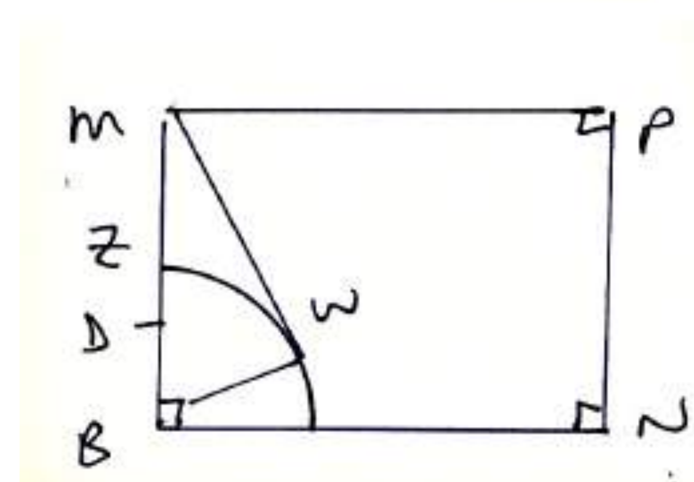






because:
e = BF/BZ = ZS/ZB
and: MW/MP = BZ/BS





$NQ/NP = BX/BZ$

$BZ^2/NP^2 = BA^2/(BA^2 - BN^2)$

$(BZ^2 - NP^2)/NP^2 = BN^2/(BA^2 - BN^2)$

$(BZ^2 - NP^2)/BN^2 = NP^2/(BA^2 - BN^2)$

$= NP^2/NQ^2 = BZ^2/BG^2 = BE^2/BG^2$
$= ED^2/BD^2 = (BD^2 - BZ^2)/BD^2$

$(BZ^2 - NP^2)/BN^2 = (BD^2 - BZ^2)/BD^2$

$MW/MP = BZ/BS$

$MW^2/MP^2 = (MB^2 - ZB^2)/BN^2$

$BZ^2/BS^2 = EZ^2/EB^2$
$= (ZB^2 - DB^2)/DB^2$

$(MB^2 - ZB^2)/BN^2$
$= (ZB^2 - DB^2)/DB^2$

$(NP^2 - BZ^2)/BN^2 = (BZ^2 - BD^2)/BD^2$

$(MN^2 - BZ^2)/BN^2 = BZ^2/BD^2$

$(MN^2 - BZ^2)/BZ^2 = BN^2/BD^2$

$MN^2/BZ^2 = (BN^2 + BD^2)/BD^2$

$MN^2/DN^2 = BZ^2/BD^2$

$MN/DN = BZ/BD$

$(MB^2 - ZB^2 + BN^2)/BN^2$
$= BZ^2/BD^2$

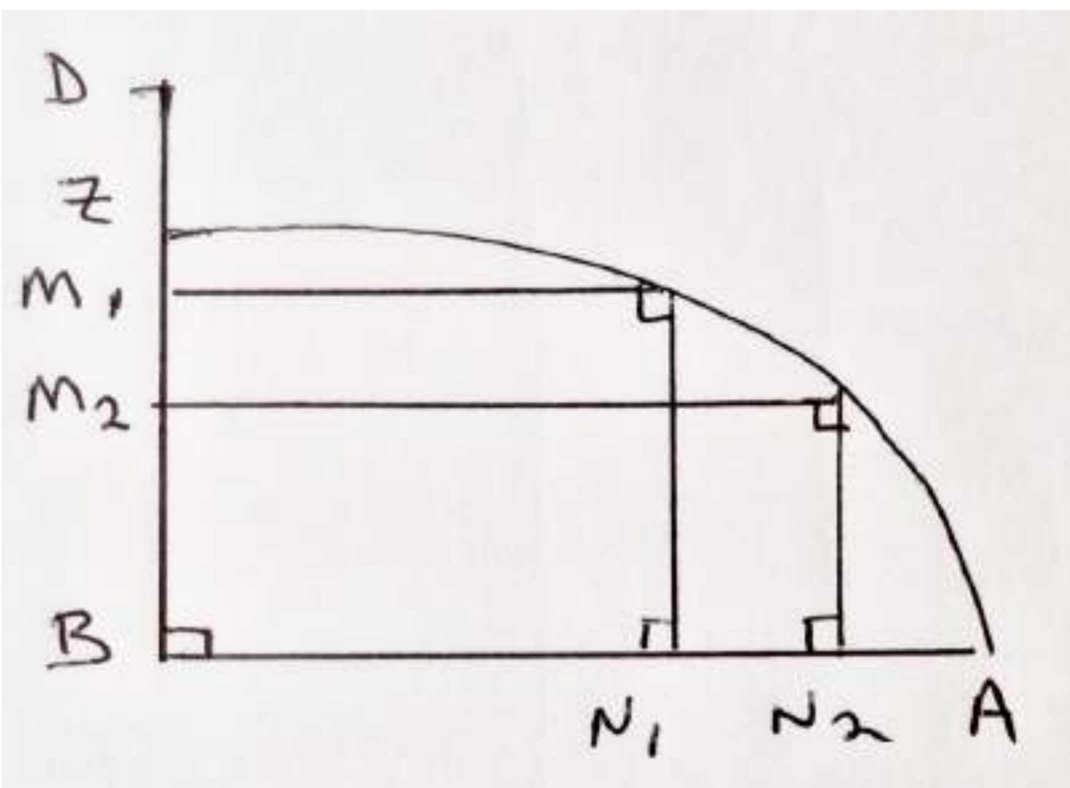
$(MN^2 - BZ^2)/BZ^2$
$= BN^2/BD^2$

$MN^2/ZB^2 = DN^2/DB^2$

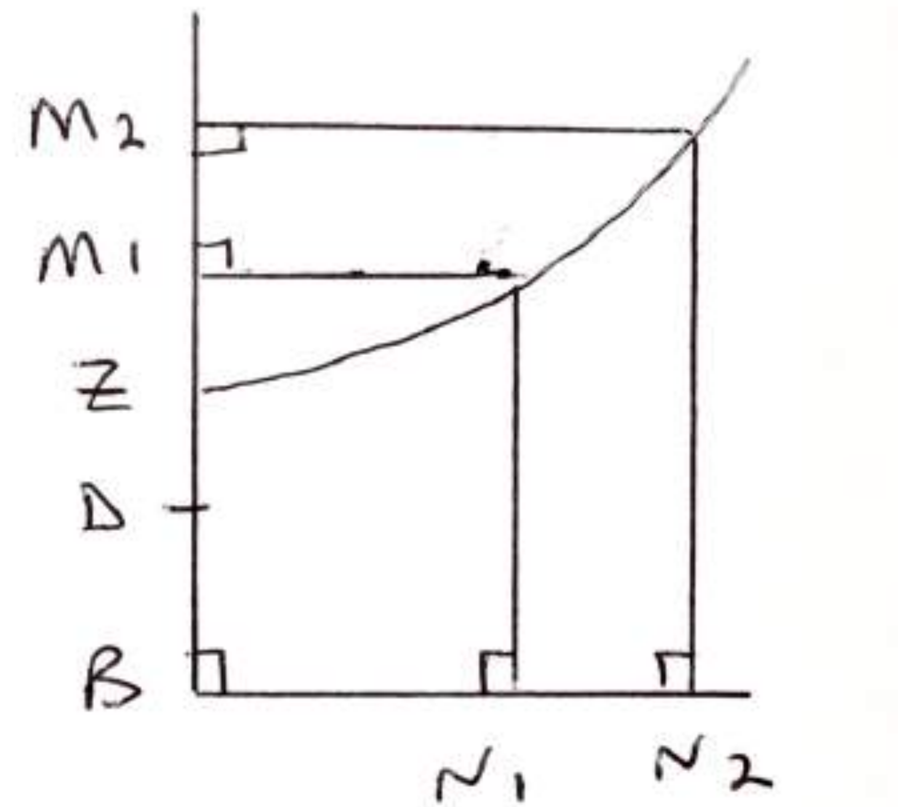
$MN^2/DN^2 = BZ^2/BD^2$

$MN/DN = BZ/BD$

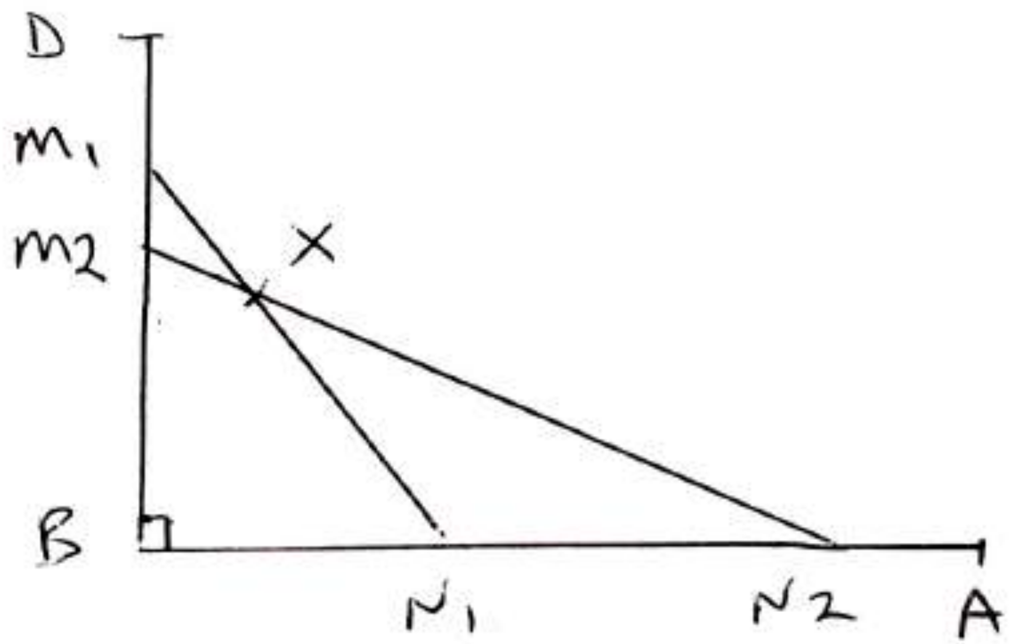
$\mathbb{R} = BD/BZ$
$\mathbb{R} = N_1D/N_1M_1$
$\mathbb{R} = N_2D/N_2M_2$



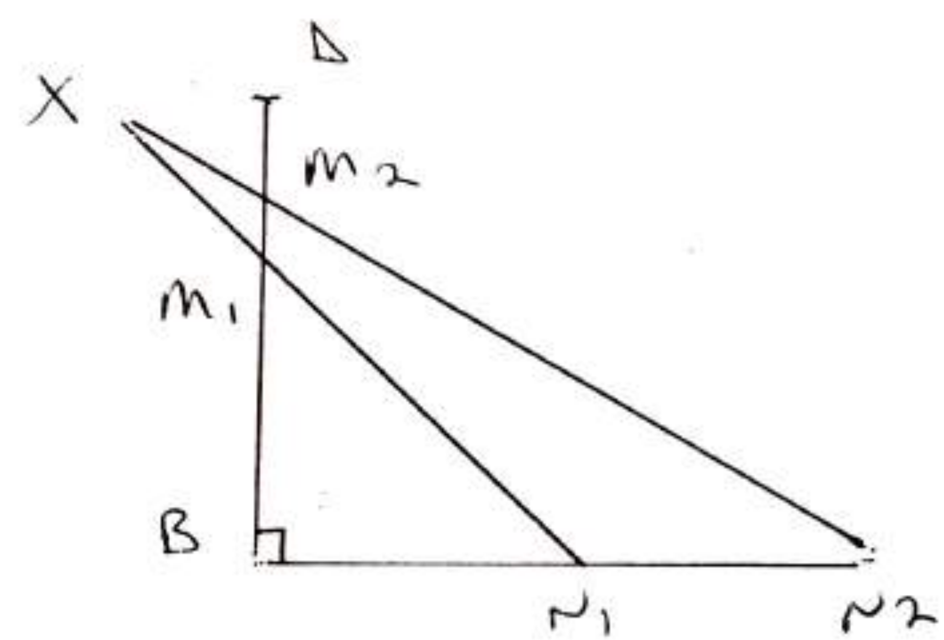
$\mathbb{R} = BZ/BD$
$\mathbb{R} = N_1M_1/N_1D$
$\mathbb{R} = N_2M_2/N_2D$

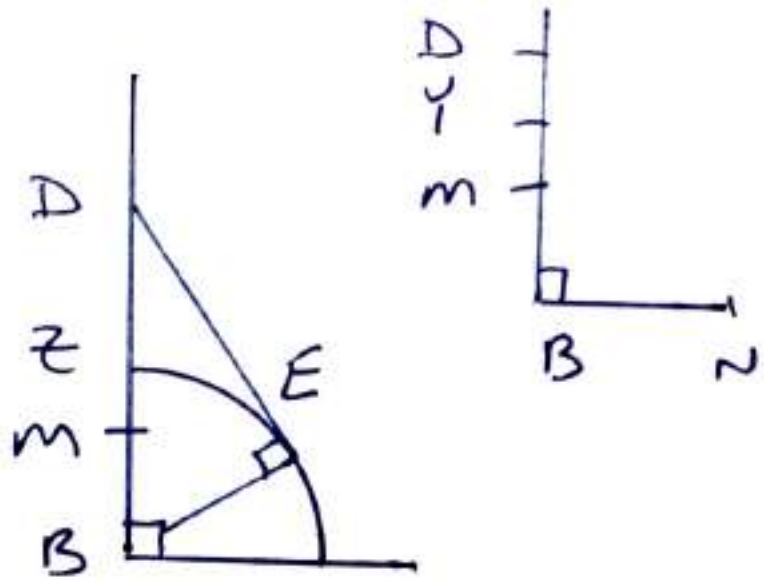


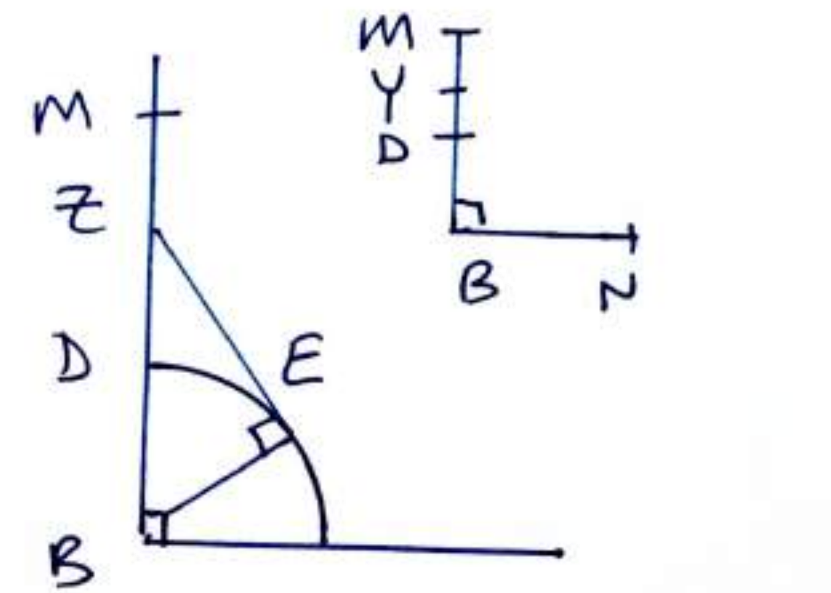
$BM_1 > BM_2$ and N_1M_1 crosses N_2M_2 at X within the right angle $\angle DBA$.



$BM_2 > BM_1$ and N_1M_1 crosses N_2M_2 at X outside the right angle $\angle DBN_2$.







ℝ = DB/BZ = ND/NM

if: BY/MB= DB/DE
then: DB/YN = ED/EB
because:

ℝ = BZ/DB= NM/ND

if: BY/DB= ZB/EZ
then: MB/YN= EZ/EB
because:

$$MB^2 = MN^2 - BN^2$$

$$MB^2 = MN^2 - YN^2 + BY^2$$

$$BY^2/(MN^2 - YN^2 + BY^2)$$
$$= DB^2/(DB^2 - BZ^2)$$
$$= DN^2/(DN^2 - MN^2)$$

$$BY^2/(YN^2 - MN^2) = DN^2/MN^2$$

$$BY^2 = YN^2 - BN^2$$

$$BY^2/DB^2 = BZ^2/(BZ^2 - EB^2)$$

$$BY^2/(BY^2 - DB^2) = BZ^2/DB^2$$

$$BZ^2/DB^2 = MN^2/DN^2$$

$$BY^2/MN^2 = (BY^2 - DB^2)/DN^2$$

$$(BY^2 + MN^2)/MN^2$$
$$= (BY^2 - DB^2 + DN^2)/DN^2$$

$$BY^2 = YN^2 - DN^2 + DB^2$$

$$(YN^2 - DN^2 + DB^2)/(YN^2 - MN^2) = DN^2/MN^2$$

$$(YN^2 + DB^2)/YN^2 = DN^2/MN^2$$

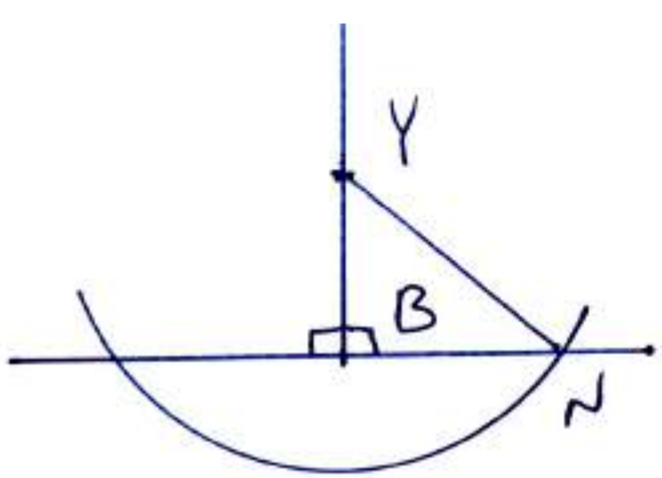
$$DB^2/YN^2 = (DN^2 - MN^2)/MN^2 = (DB^2 - BZ^2)/DB^2 = ED^2/EB^2$$

$$(BY^2 + MN^2)/(BY^2 + BN^2) = MN^2/DN^2$$

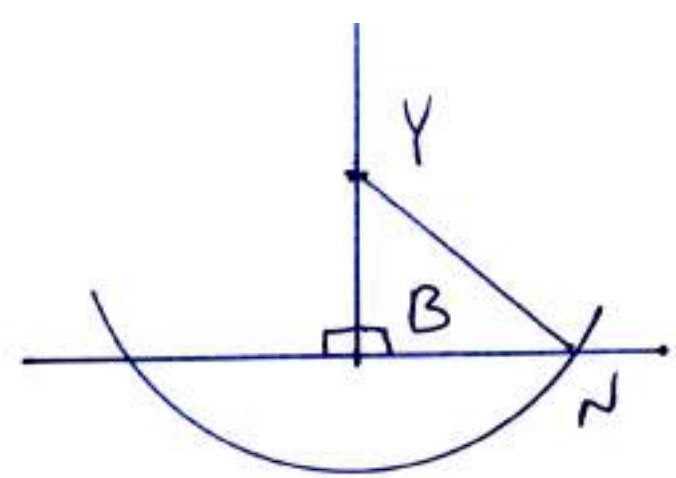
$$(MN^2 - BN^2)/NY^2 = (MN^2 - DN^2)/DN^2$$

$$MB^2/YN^2 = (BZ^2 - DB^2)/DB^2 = EZ^2/EB^2$$

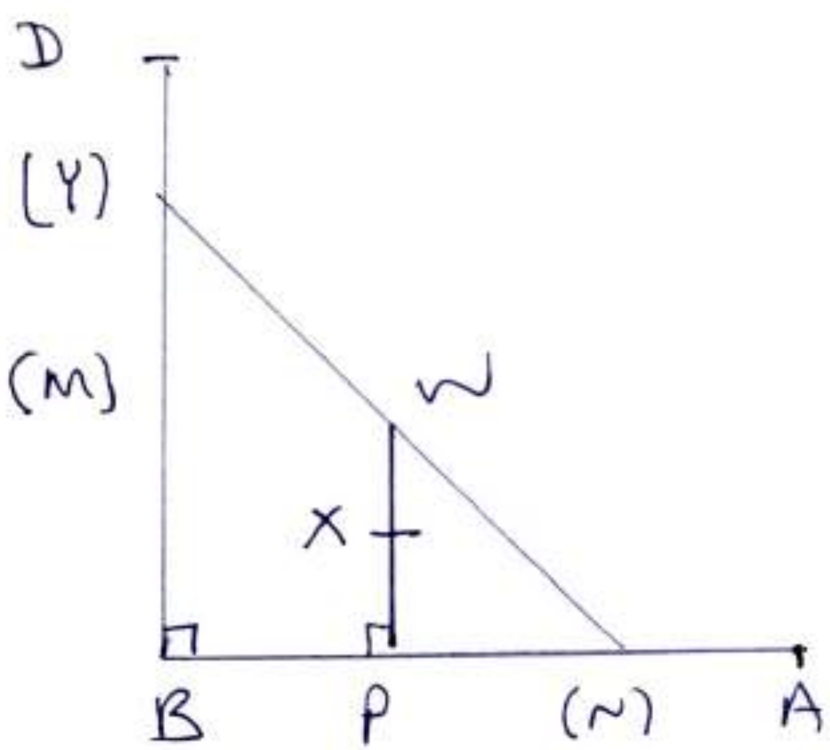
When given point M, after calculating BY with known BM, (as well as known DB/DE); we can use known DB, (as well as known ED/EB), to calculate YN and use that as a radius about Y to find N:



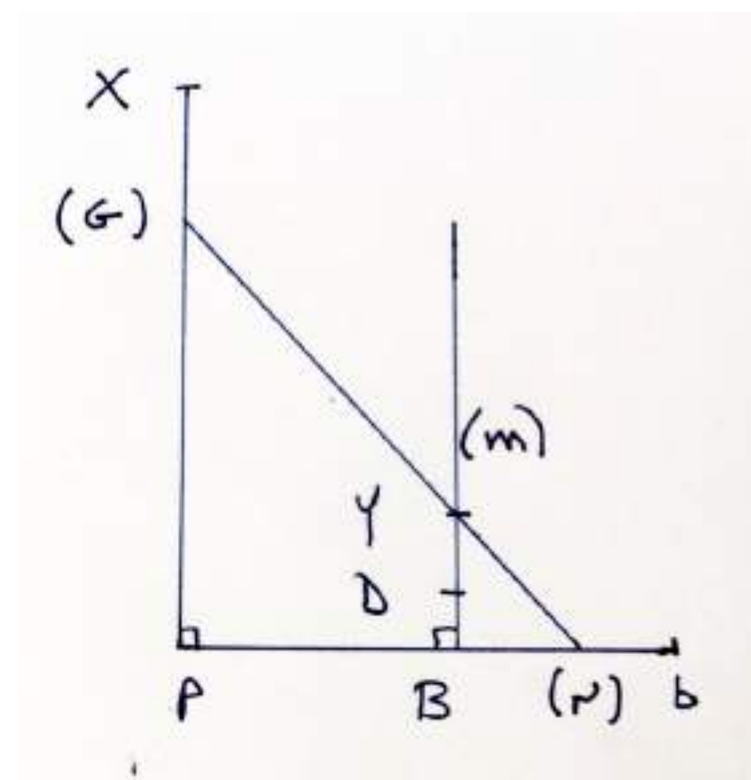
When given point M, after calculating BY with known DB, (as well as known ZB/ZE); we can use known MB, (as well as known EZ/EB), to calculate YN and use that as a radius about Y to find N:



Since M must be known to find N, this gives no advantage over the previously described reference ellipse. However, it provides a way to find N on image ray MX(N) without knowing M.



Since M must be known to find N, this gives no advantage over the previously described reference hyperbola arm. However, it provides a way to find N on image ray XM(N) without knowing M.



To find an image ray through a given point X, first calculate PW with known PX and DB/DE using:
PW/PX = (BY/MB) = DB/DE

Since DB and ED/EB are also known, find the length of YWN using:
DB/YN = ED/EB

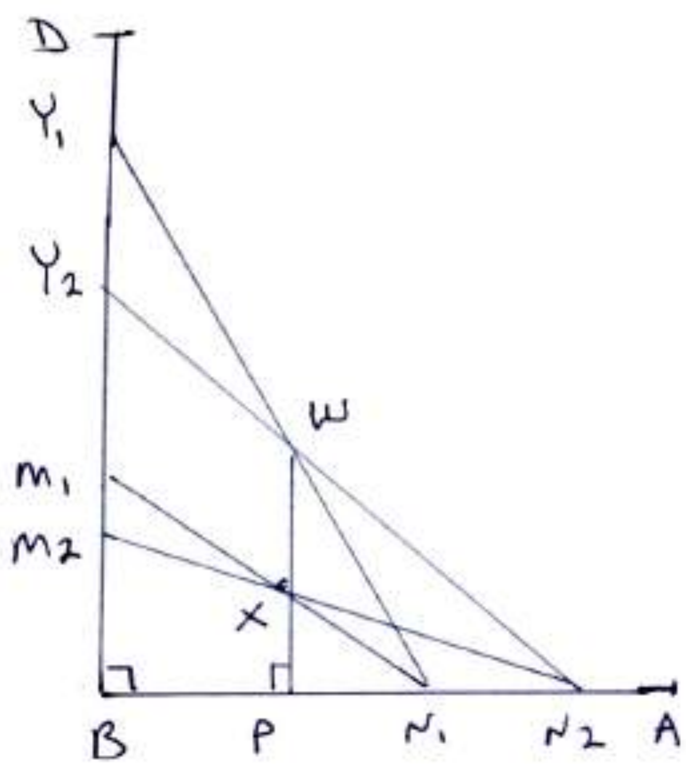
We can then find (N) by inserting the calculated length YWN within the right angle ∠DBA through W.

To find an image ray through a given point X, first calculate BY with known DB and ZB/ZE using:
BY/DB = ZB/EZ

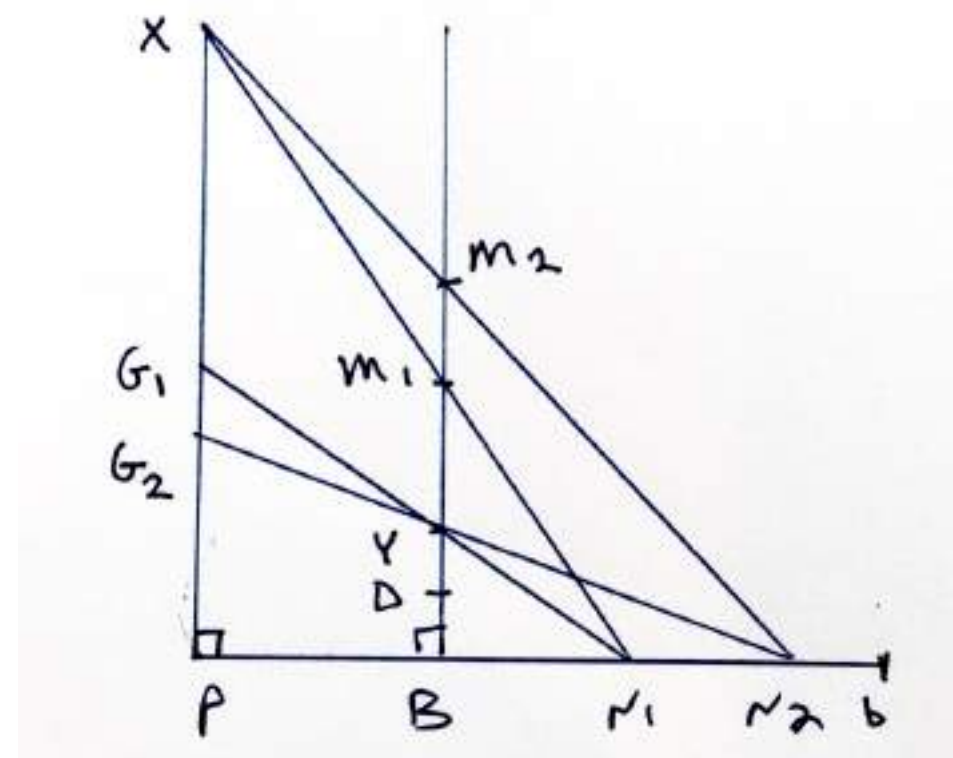
Since PX and EZ/EB are also known, find the length of GYN using:
PX/GYN = (MB/YN) = EZ/EB

We can then find (N) by inserting the calculated length GYN within the right angle ∠XPb through Y.

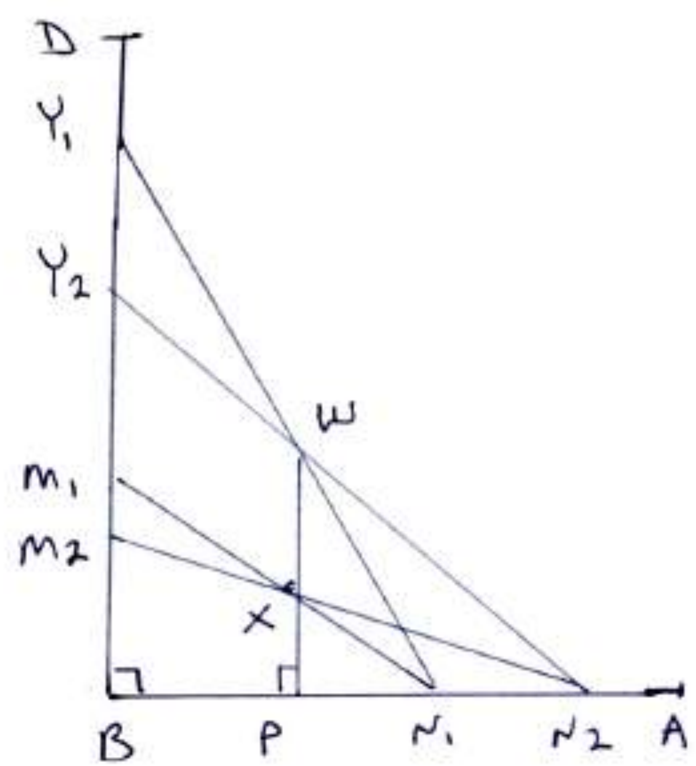
For any given calculated value of YN, a maximum of two line segments ($Y_1N_1 = Y_2N_2$) fit though W within the right angle ∠DBA.



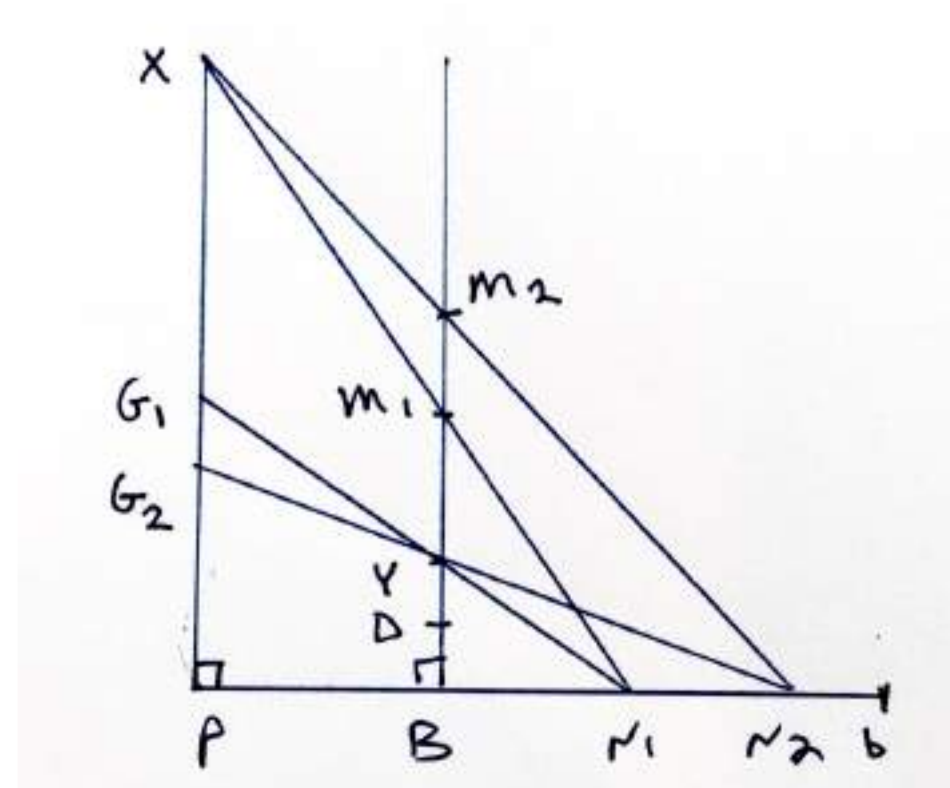
For any given calculated value of GN, a maximum of two line segments ($G_1N_1 = G_2N_2$) fit though Y within the right angle ∠XPb.



These two line segments are drawn to find both N_1 and N_2 for the image rays through X.



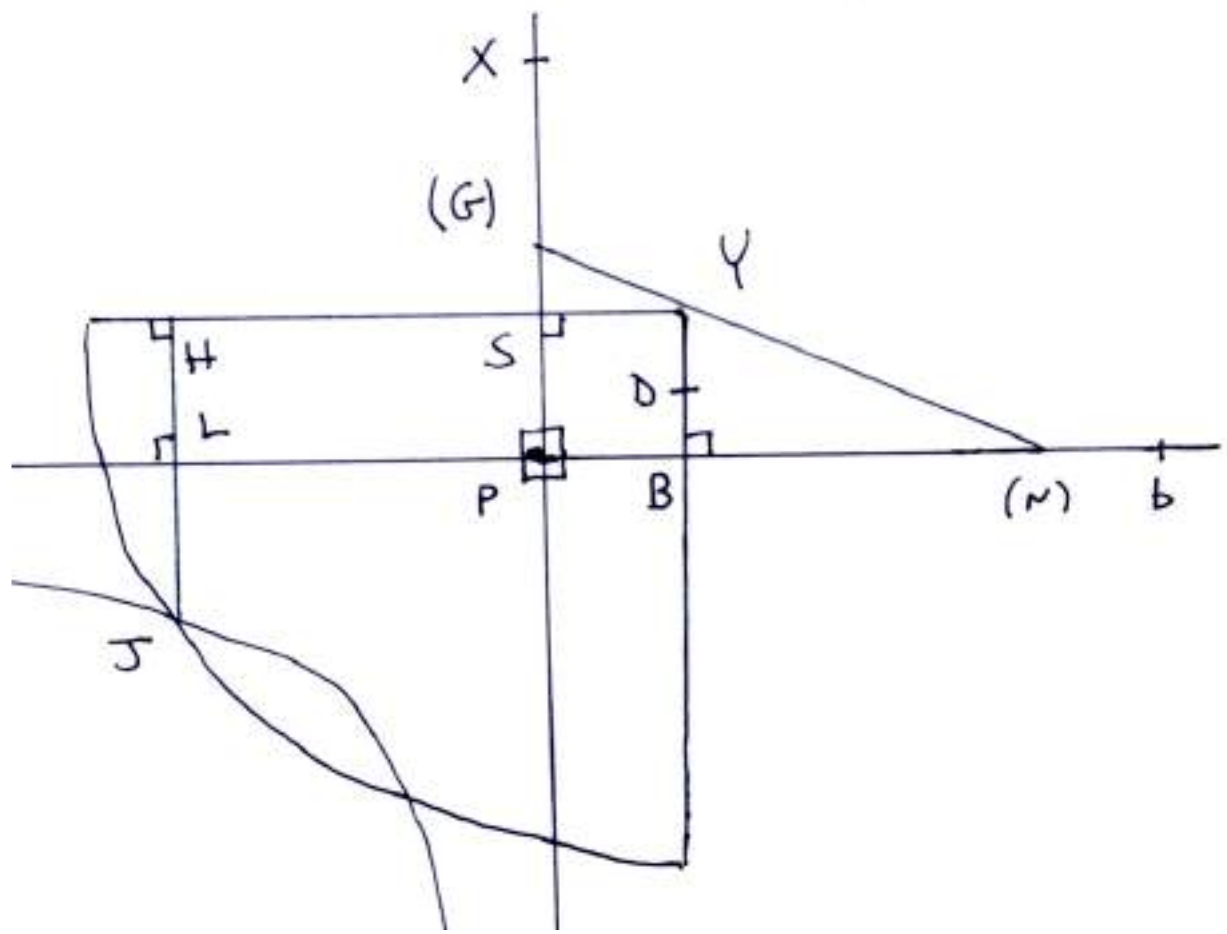
These two line segments are drawn to find both N_1 and N_2 for the image rays through X.



The clear image of X occurs when YN through its specified point W is its minimum possible length, so that N_1 lies at N_2. Since both BY/MB = PW/XP and DB/YN are constants, YN can be varied while keeping the image location XP constant, but not the object location DB.

The clear image of X occurs when GN through its specified point Y is its minimum possible length, so that N_1 lies at N_2. Since both MB/YN = XP/GN and BY/DB are constants, GN can be varied while keeping the object location DB constant, but not the image location XP.

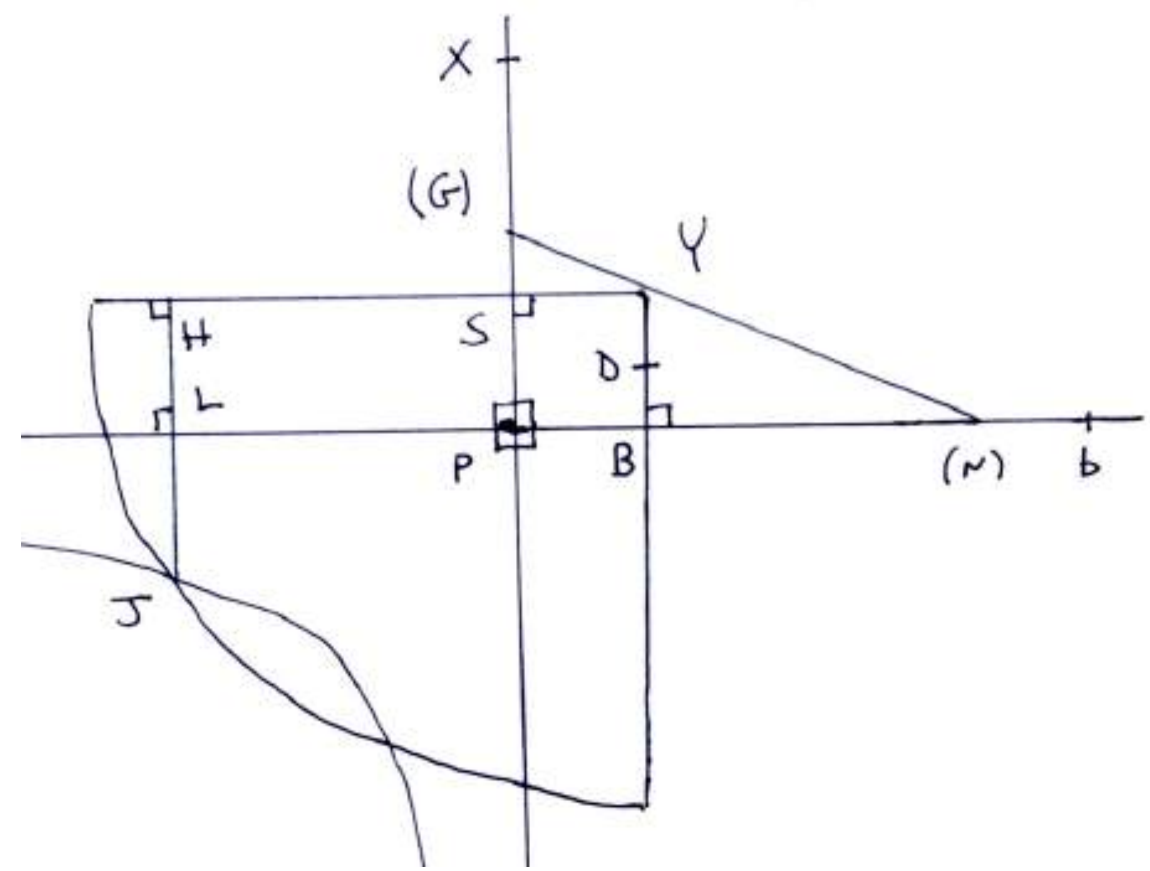
Expanding on the right side column representing the object in air, (where GN can be varied while keeping the object location DB constant, but not the image location XP), consider Y to be on a reference hyperbola defined by: (LP)LJ = (BP)BY, and draw its opposite arm:



We know LP/BY = BP/LJ.
If we construct BN = LP, then BN/BY = BP/LJ.
But SY/SG = BN/BY = BP/LJ
and since SY = BP:

SG = LJ
SG + SP = LJ + HL
PG = HJ

and since by
construction BN = LP:
PN = LB = HY
ΔNPG = ΔYHJ
GN = YJ



The reference radius length YJ intersects the reference hyperbola at a maximum of two possible points J_1 and J_2. Both G_1YN_1 and G_2YN_2 can be drawn by constructing BN = LP for each point J.

A clear image of object D occurs when N_1 and N_2 overlap, or when the reference radius length YJ = GN intersects the reference hyperbola at a single point J. The required GN for this condition gives the required location of N, as well as the location of the clear image at X, (remember that PX varies with GN).