# Geometrical Optics

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2021



With thanks to William Brown, OD, PhD,
who always taught the geometry first.



**References:**

Isaac Barrows Optical Lectures, 1667
Translated by H.C. Fay
Edited by A.G. Bennett
Publisher:
The Worshipful Company of Spectacle Makers
London, England; 1987
ISBN # 0-951-2217-0-1

Plane and Solid Geometry
G. A. Wentworth; 1899 revised edition

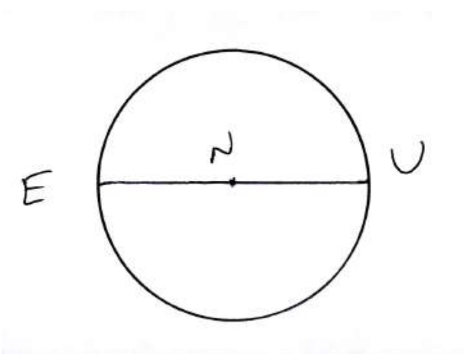
# A). Using Circles and Triangles



## 1). refraction along a line

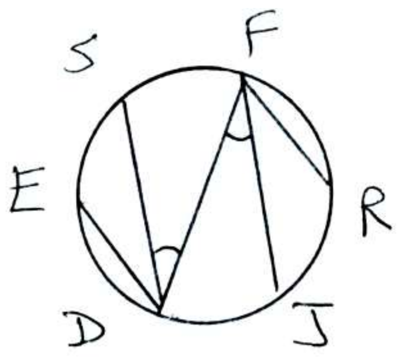


On a circle with diameter EU and center N:






Two equal arcs ~SE and ~JR can be shown to subtend equal angles by drawing any two parallel lines SD and JF. Since parallel lines intercept equal arcs across a circle,
~SF = ~JD
~SE + ~SF = ~JR + ~JD
~EF = ~RD
ED || RF, and therefore:
∠SDE = ∠JFR



Since conversely, equal angles along a circle subtend equal arcs, any angle along any circle can be defined in terms of its subtended arc and the circle's diameter.

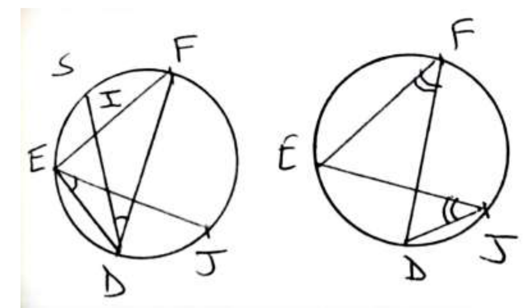
For example: ∠RFJ = ~RJ/EU



Triangles need only two equal angles to be the same shape, (or ≅).
Since equal arcs subtend equal angles along a circle:

ΔEJD ≅ ΔDFI

FD/FI = JE/JD

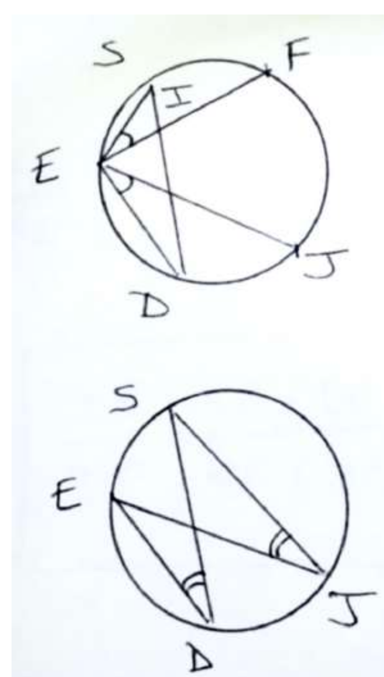





~SJ = ~FD

ΔEJS ≅ ΔEDI

EI/ED = ES/EJ





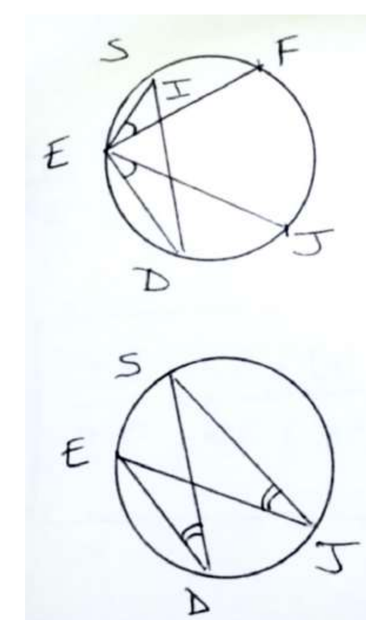

[(FD)(EI)]/[(FI)(ED)]

= [(JE)(ES)]/[(JD)(EJ)]

= SE/SF

**IE/IF = [(SE)(DE)]/[(SF)(DF)]**

**which describes an important property of any cyclic quadrilateral SEDF**



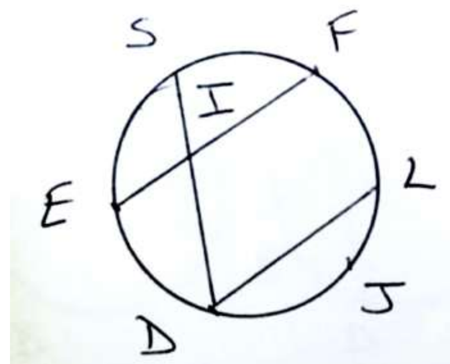
LD || FE

DE/DF =  LF/LE

IE/IF  = (SE)(LF)/(SF)(LE)

FE/FI

= {(SE)(LF) + (SF)(LE)}/(SF)(LE)



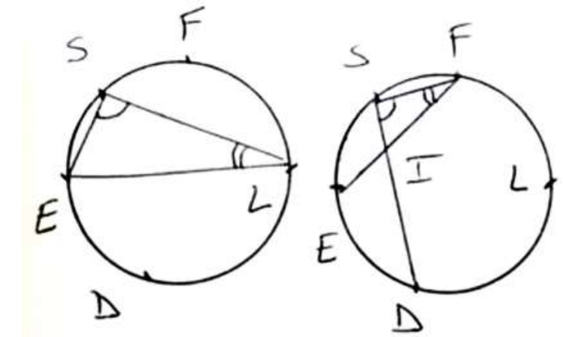



LD || FE

~EL = ~FD

ΔLSE ≅ ΔFSI

LS = {(FS)(LE)}/FI




**(FE)(LS) = (SE)(LF) + (SF)(LE)**

**which describes an important property of any cyclic quadrilateral SELF**



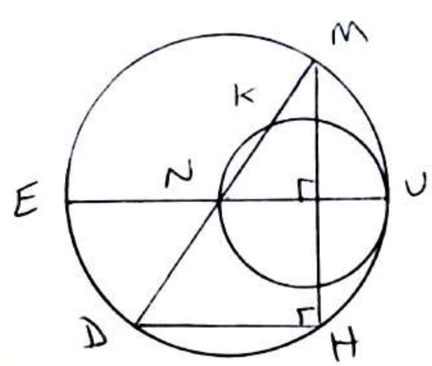
∠**KNU** = ∠MDH

∠MDH = ~MH/MD

= ~MH/UE

= 2(~UM)/UE

**=  2∠MEU**




∠KNU = ~UK/UN

= 2(~UM)/2(UN)

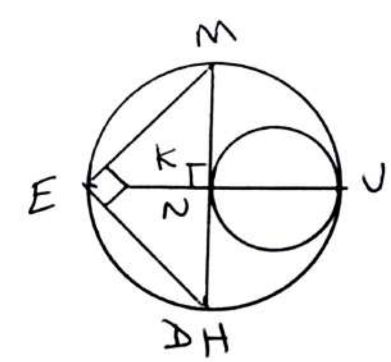
**~UK = ~UM**



Let K ⇒ N and D ⇒ H:

~UK/UN = ~MH/MD

= ~MH/UE = ∠MEH




**~UK/UN = ∠MNU**

**2(~UK)/UN = ∠MNH = π**

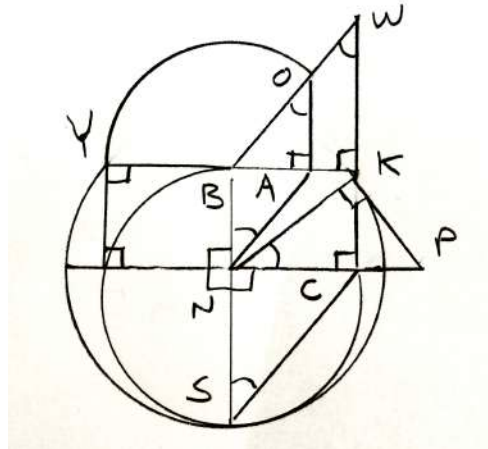
NS/NC = NC/NB
NK/NC = CN/CK

ΔNSC = ΔKWB = ΔKNP
NC = KP

ΔCKP = ΔBNA = ΔAOB
NA = KP

NC = NA = OB
NC = KB = YB

**WK** = NS **= YN**





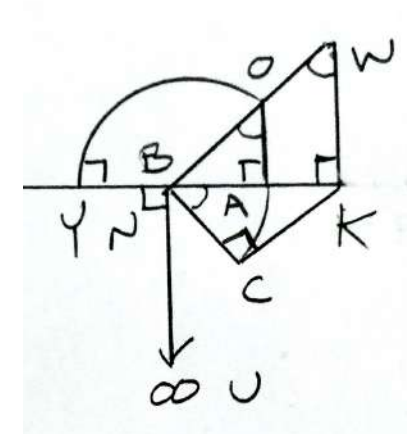

Keeping only:
NA = NC, and
ΔCNK ≅ ΔAOB ≅ ΔKWB:

**As N ⇒ B, WK ⇒ YN**

because:
WK/OA ⇒ NK/NA = NK/NC

= OB/OA = WB/WK

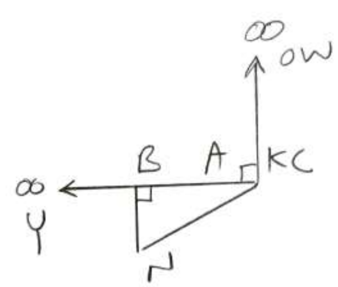
so that:
WK ⇒ OB ⇒ YN






Keeping only:
NA = NC, and
ΔCNK ≅ ΔAOB ≅ ΔKWB:

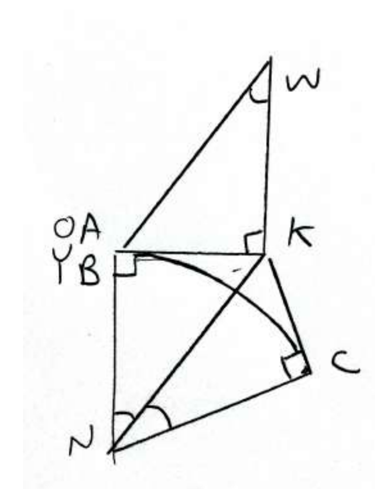
**As A ⇒ K, WK ⇒ YN**





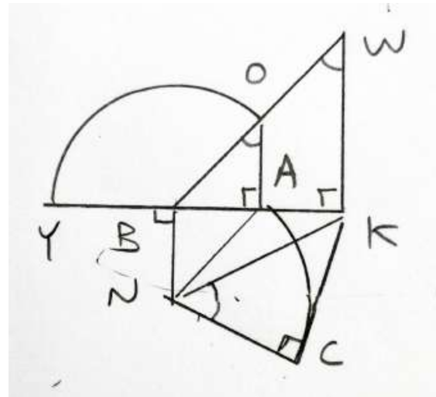

Keeping only:
NA = NC, and
ΔCNK ≅ ΔAOB ≅ ΔKWB:

**As A ⇒ B, WK ⇒ YN**



Therefore, we can presume that whenever A lies on KB of right triangle ΔKBN, if NA = NC, and ΔCNK ≅ ΔAOB ≅ ΔKWB then:

**WK = YN**



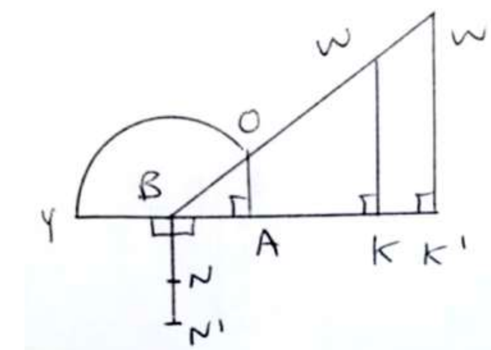

which can be shown directly using the equations:
$(CK/CN)^2 = (AB/AO)^2 = (KB/KW)^2 = (CK^2 + AB^2)/(CN^2 + AO^2)$
since: $KB^2 = KN^2 - BN^2 = KN^2 - (NC^2 - AB^2) = CK^2 + AB^2$
then: $WK^2 = CN^2 + AO^2$, which equals:
$AN^2 + AO^2 = BA^2 + BN^2 + BO^2 - BA^2 = YN^2$



OB/OA = NK/NA = N’K’/N’A

KW = YN
K’W’ = YN’

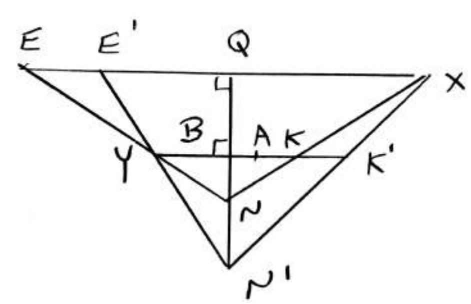
KB/YN = K’B/YN’






QX/EN = KB/YN
= K’B/YN’ = QX/E’N’

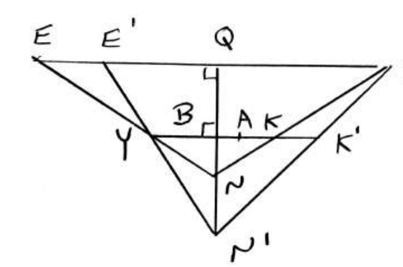
EN = E’N’




Only one N’K’X exists for NKX since only one E’N’ exists equal to EN.



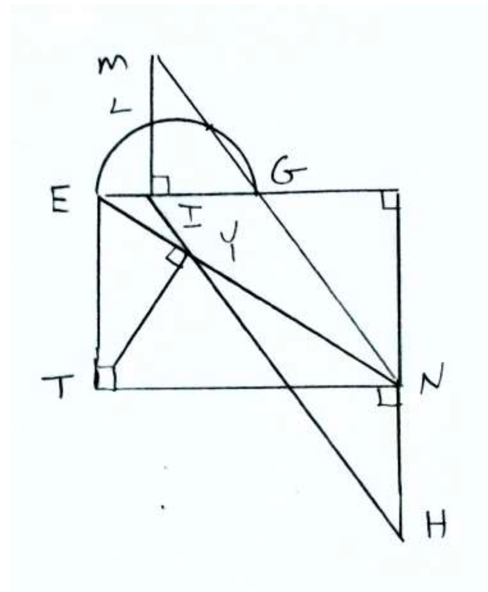
When EN is changed to become the smallest segment through Y included in the right angle EQN, E’ lies at E, and N’ lies at N. At this point, X becomes the clear image Z of object A, seen along NK. Remember that QX varies with EN because QX/EN = KB/YN = KB/KW, which is a constant.



NE || GL
TY || EL
HI || NM
HI = NM
NM > NL

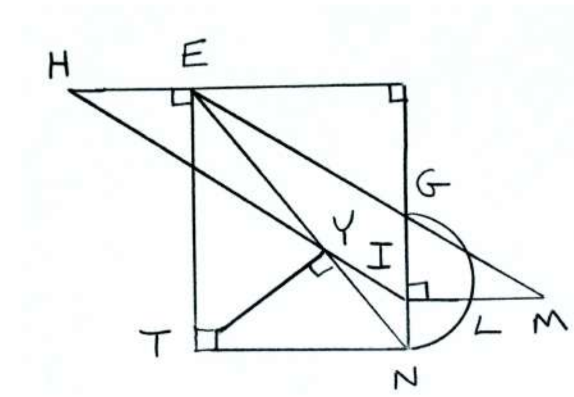
NL is the hypotenuse of right triangle NEL

NL > NE
HI > NE






NE || GL
TY || NL
HI || EM
HI = EM
EM > EL

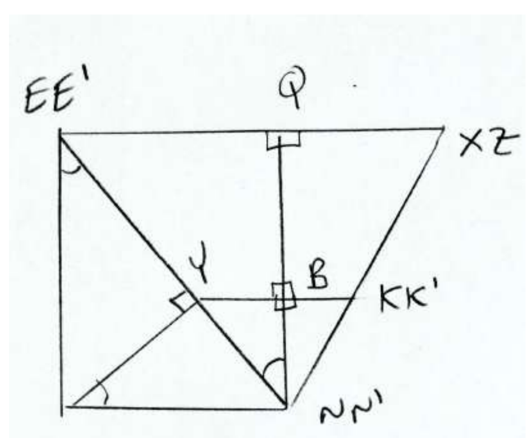



EL is the hypotenuse of right triangle ENL

EL > EN
HI > EN



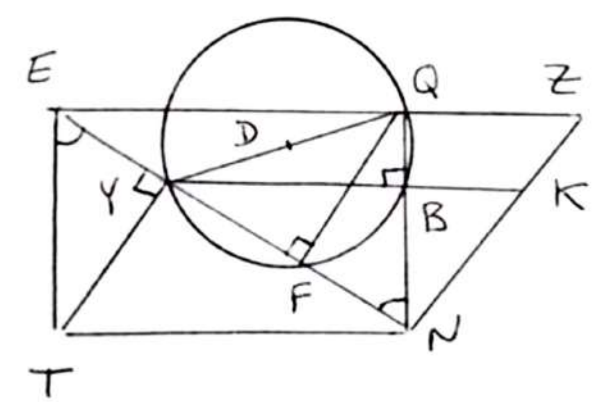
X = Z when EN is the shortest segment through Y included in right angle EQN



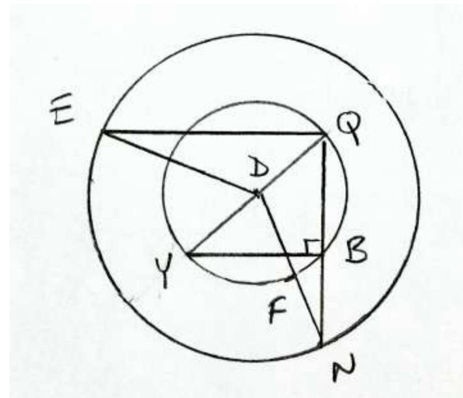



In order to find Z given ΔYBN and NK, we must find E using:

ΔYBN
≅ ΔNYT
≅ ΔNTE



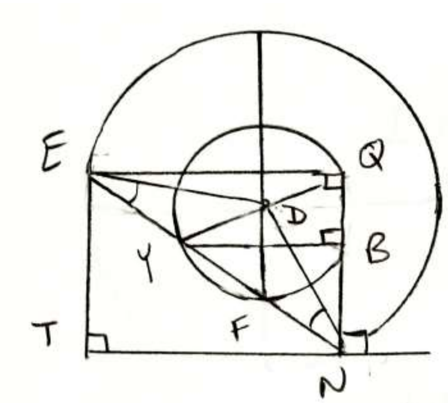
In order to find Z given $\Delta YBQ$, we must find EN so that:
right triangle
$\Delta TYE = \Delta QFN$
by drawing a circle concentric with ⊙Y(F)BQ around its center D containing arc ~EN so that YF lies on chord EN.






Not only does:
DY = DF, but also:
ED = ND and therefore
$\Delta EDY = \Delta NDF$
so EY = NF

Since $\Delta QFN$ is a right triangle, so is $\Delta TYE$. Once we have found EN, we must also find NK in order to find Z.

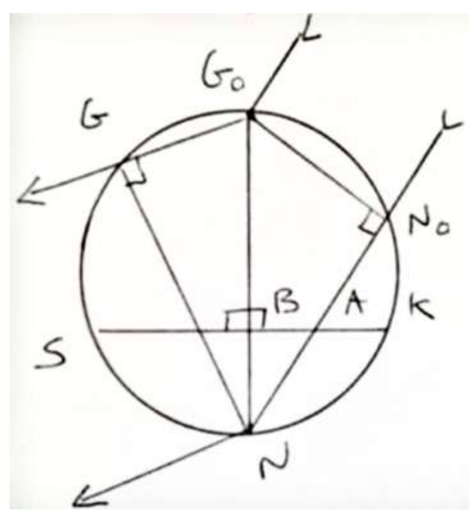





$\Delta N_oNK \cong \Delta KNA$
because:
~NS = ~NK

Wavefront $G_oN_o$ refracts into wavefront GN along $G_oN$, because it travels $G_oG$ in the same time it travels $N_oN$.

$\boldsymbol{R} = NN_o/GG_o$
$= NN_o/NK = NK/NA$

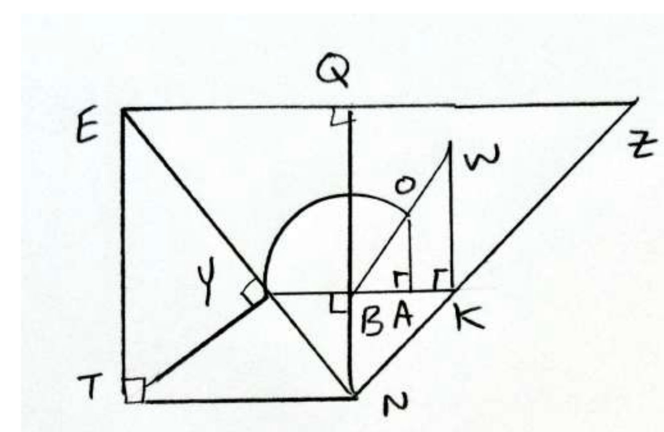




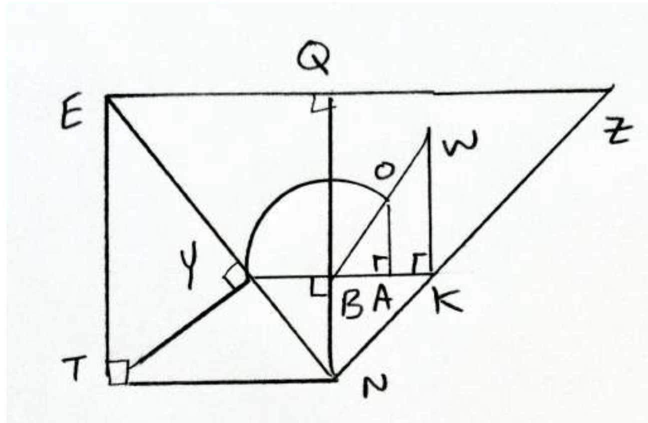

If $\boldsymbol{R} = OB/OA$,

and KW = YN:

$\boldsymbol{R} = NK/NA$




and Z is the clear image of object A refracted at N along BN



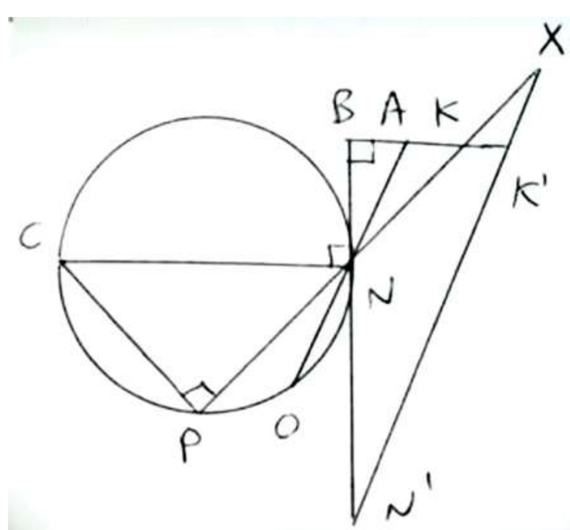

given ΔBAO:
use ΔBKW or ΔQBY to find ΔBNY
use ΔBNY to find ΔBKW or ΔQBY



# 2). refraction along a circle



ΔKNA ≅ ΔOCP
***R*** = NK/NA
= N'K'/N'A
= CO/CP

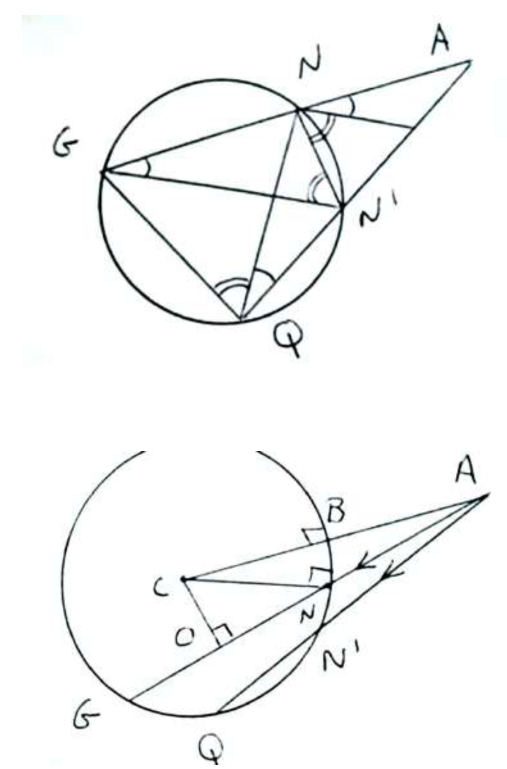





ΔANN' ≅ ΔAQG
AG/AN' = QG/NN'

(AG + AN')/2AN'
= (QG + NN')/2NN'

Real object A

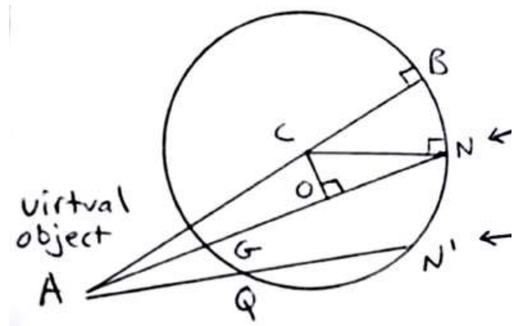


ΔANN’ ≅ ΔAQG

AG/AN’ = QG/NN’

(AG + AN’)/2AN’
= (QG + NN’)/2NN’

Virtual object A
can not be projected
on a screen due to
refraction at BN.



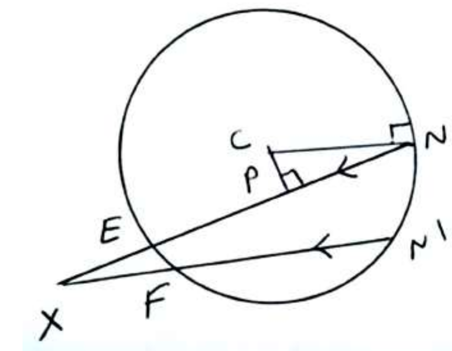



ΔXNN’ ≅ ΔXFE

XE/XN’ = EF/NN’

(XE + XN’)/2XN’
= (EF + NN’)/2NN’

Real image at (X = Z)
can be projected on a
screen.



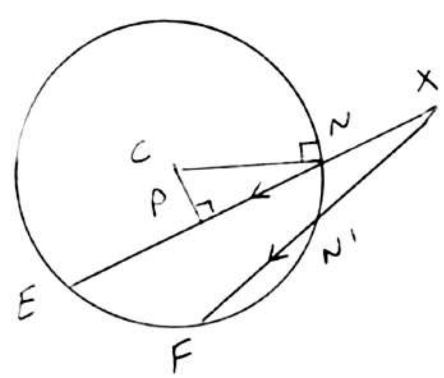



ΔXNN’ ≅ ΔXFE

XE/XN’ = EF/NN’

(XE + XN’)/2XN’
= (EF + NN’)/2NN’

Virtual image at (X = Z)
can not be projected
on a screen.






(AG + AN’)/2AN’ = (QG + NN’)/2NN’
(XE + XN’)/2XN’ = (EF + NN’)/2NN’

(QG + NN’)/(EF + NN’)
= [(AG + AN’)/2AN’][2XN’/(XE + XN’)]

As N’ ⇒ N, X ⇒ Z, and:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ (QG + NN’)/(EF + NN’)

⇒ (AO/AN)(ZN/ZP)

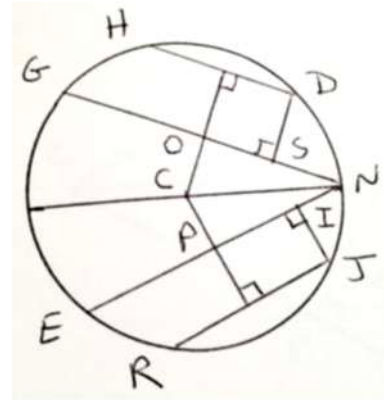
Also, when HD = QN’
and RJ = FN’

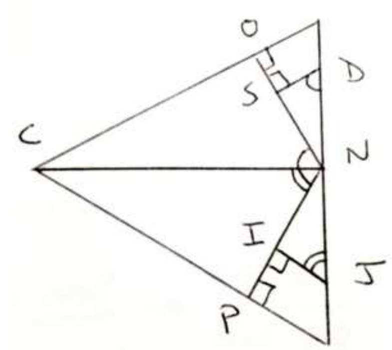
(~QG + ~NN’)/(~EF + ~NN’)
= 2(~ND)/2(~NJ) = ~ND/~NJ

As N’ ⇒ N, X ⇒ Z, and:

~DJ ⇒ line segment DJ, so:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ ND/NJ






DS/JI = CO/CP
JI/JN = NP/NC
DN/DS = NC/NO
ND/NJ = (NP/NO)(CO/CP)

As N’ ⇒ N, X ⇒ Z, and:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ (NP/NO)(CO/CP)

and therefore:
(AO/AN)(ZN/ZP) ⇒ (NP/NO)(CO/CP)

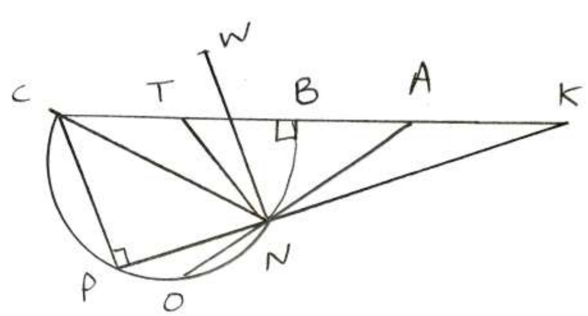


Thus **R** = CO/CP, and Z, (along both NP and CW), is the clear image of A refracted along ~BN, when:

NT||CO, so:
AO/AN = CO/NT and:

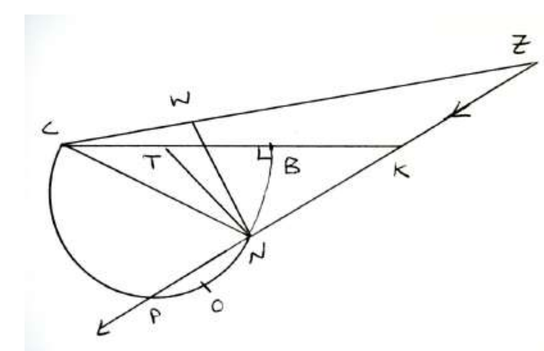
NW||CP, so:
ZN/ZP = NW/CP
and:

NW/NT = NP/NO
(ΔWNT ≅ ΔPNO)

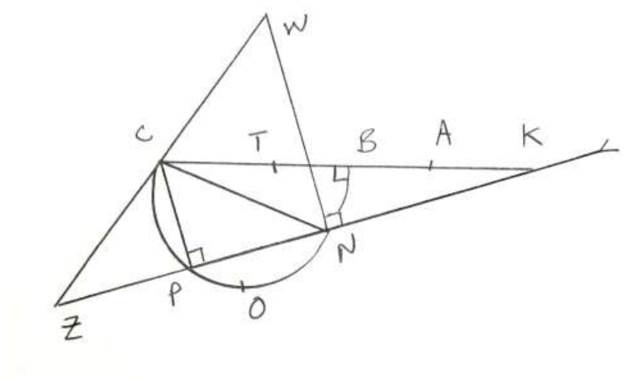





The off-axis rays from any on-axis object A, (real or virtual), can not form a virtual on-axis image at Z because NW must be less than CP for Z to be virtual; but NW must also be greater than NT.

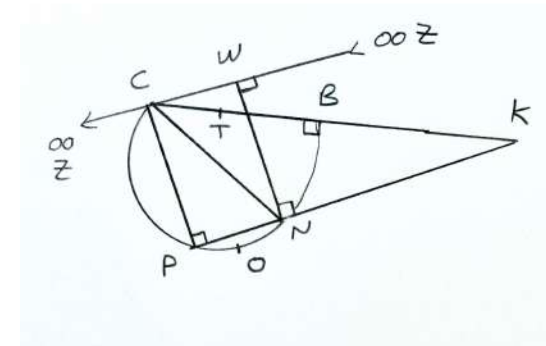


The off-axis rays from any real on-axis object A can not form a real on-axis image at Z because NW must be greater than (or equal to) CP for Z to be real; but NW must also be greater than NT.



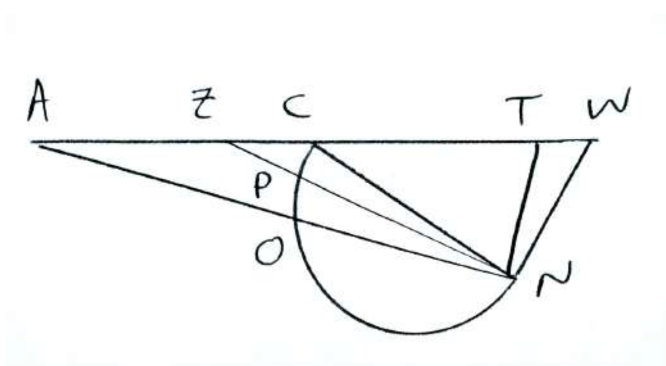



The off-axis rays from any real on-axis object A can not form a real on-axis image at Z because NW must be greater than (or equal to, as shown here) CP for Z to be real; but NW must also be greater than NT.






The off-axis rays from a virtual on-axis object A ***can*** form a real on-axis image at Z, if NW is greater than CP, and WT lies along the axis.

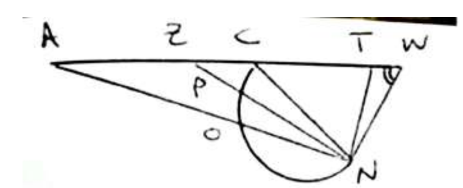


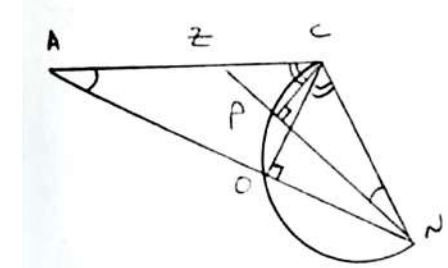



Since:
∠NWT = ∠NPO = ∠NCO
and NW||CP

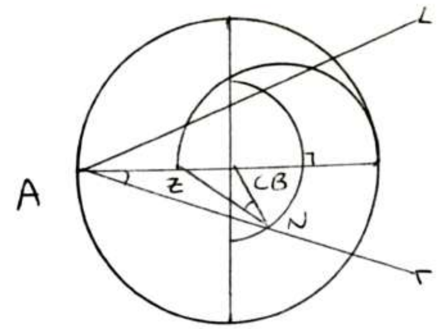
WT lies along the axis when:

ΔNCO ≅ ΔZCP



When off-axis rays from a virtual on-axis object A form a real on-axis image Z, this occurs at all points N because:




$\Delta ACN \cong \Delta NCZ$ for all N



# 3). refraction through a circle's center



Keeping:

***R*** = (CO/CP) = (NO/NP)(AO/AN)(ZN/ZP)

constant as:
N $\Rightarrow$ B:

(BC/BC)(AC/AB)(ZB/ZC) $\Rightarrow$ ***R***



Refraction through a circle's center occurs when N lies at B, so that an object's ray from A to N lies along ABC, and an image ray lies along BCZ. The locations of the object A and image Z along the optic axis BC are described by the equation:

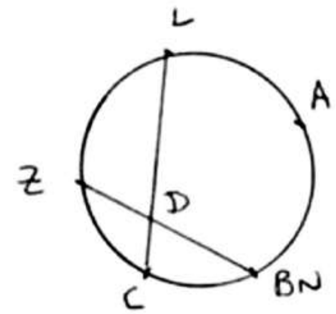
***R*** = CO/CP = (AC/AB)(ZB/ZC)

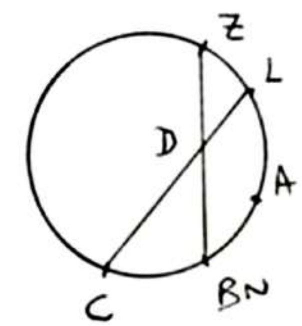
If we draw A and Z along the optic axis BC ***as if*** it were a circle, and draw CDL so that AL || ZB:

ΔACB ≅ ΔZCD, and:

(AC/AB)(ZB/ZC) =
(ZC/ZD)(ZB/ZC) =
(ZB/ZD)
so as the reference circle's radius ⇒ ∞

(ZB/ZD) ⇒ ***R***



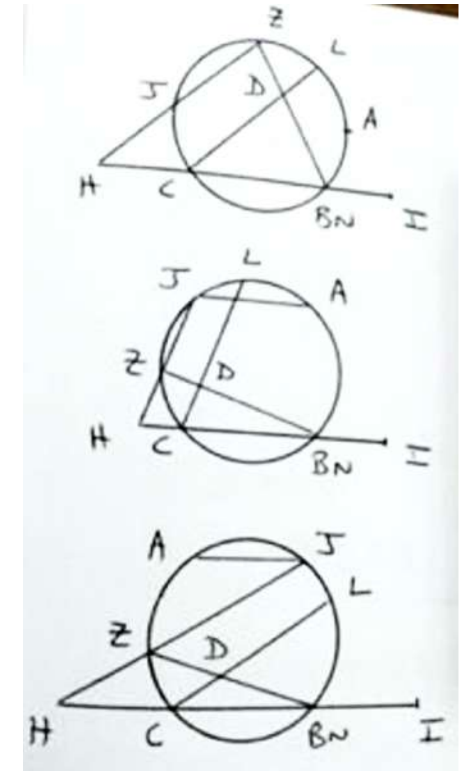



AL ll ZB
AZ = BL
~AZ = ~BL

HZ ll CL
ZC = LJ
~ZC = ~LJ

~AZ + ~ZC = ~AZC
~BL + ~LJ = ~BLJ

~AZC = ~BLJ
AJ ll CB

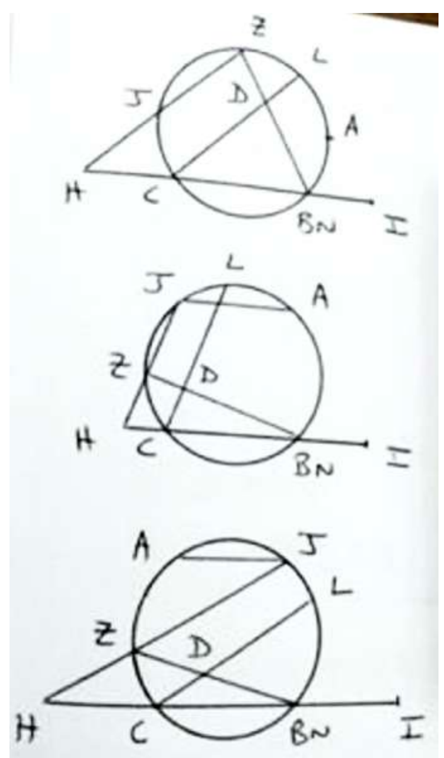





HZ ll CL
ZB/ZD = HB/HC
ΔHBZ ≅ ΔHJC
when ΔHJC = ΔIAB:
HC = IB, and:
IB/IA = HZ/HB

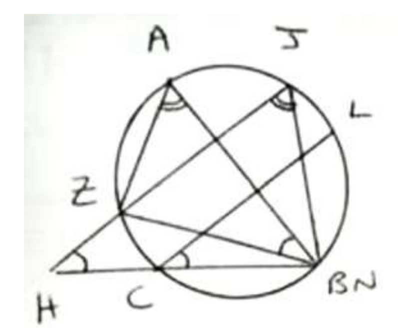
This results in
**Newton's Equation**
as the reference circle's radius ⇒ ∞:

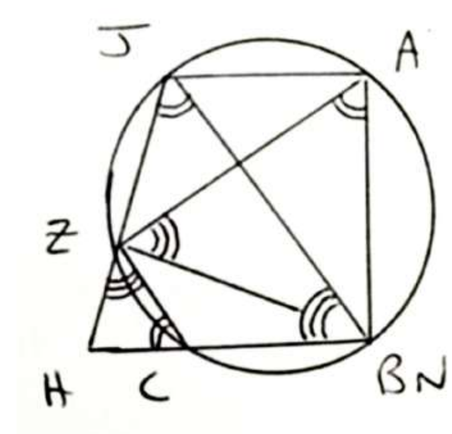
**(AI)(ZH) = (BI)(BH)**

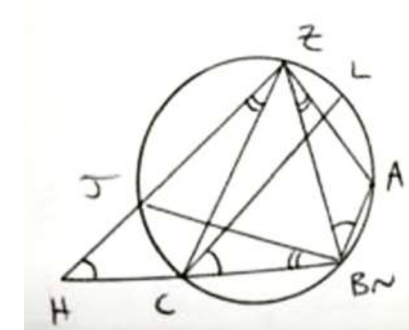





ΔHCZ ≅ ΔHJB ≅ ΔBAZ
(HC/HZ) = (BA/BZ)
[1/(HZ)(BA)] = [1/(HC)(BZ)]



as the reference circle’s radius ⇒ ∞:

[1/(HZ)(BA)] = [1/(HC)(BZ)] ⇒ ***R***/(HB)(BZ)

and the resulting possible sums occur:

HZ= HB + BZ
HB = HZ + BZ
BZ = HZ + HB

which, when multiplied by the above three factors, form the **conjugate foci equations**.



The conjugate foci equations allow for the effect of axial refraction at a circle to be expressed as the term:

(1/HC) = (***R***/HB)

which is then additive with object vergence, defined as (1/BA); or image vergence, defined as (***R***/BZ).

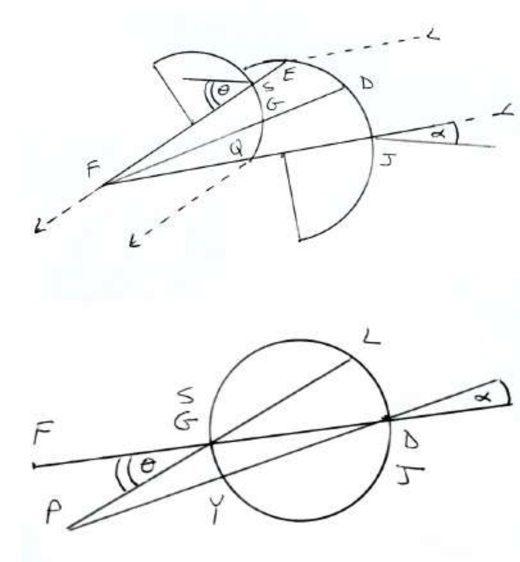


# 4). afocal angular magnification/minification



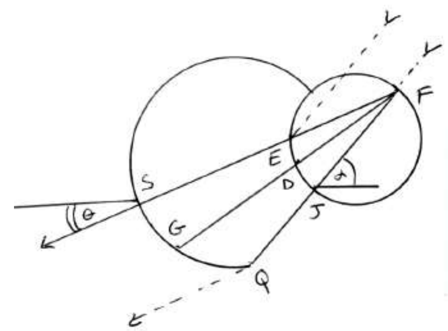
## Afocal Angular Magnification

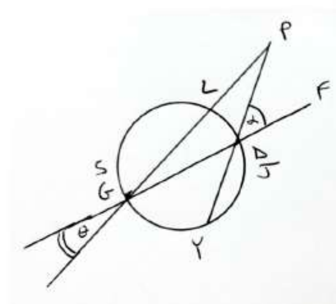
When distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis DGF as shown;
as ∠JFD = ∠SFG, and both approach zero:



## Afocal Angular Minification

Or when distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis FDG, as shown; as ∠JFD = ∠SFG, and both approach zero:






$\theta/\alpha \Rightarrow$ (~LD/GD)/(~YG/GD) as P $\Rightarrow$ F

$\theta/\alpha \Rightarrow$ (FD/FG) as P $\Rightarrow$ F

so that **afocal *axial* angular magnification/minification** equals:

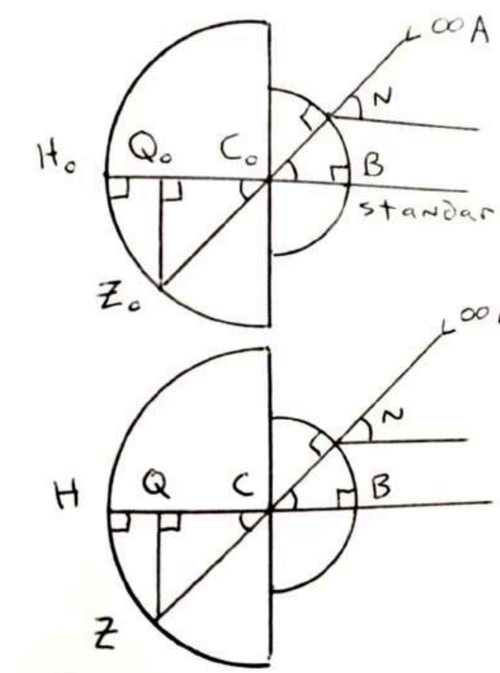
FD/FG



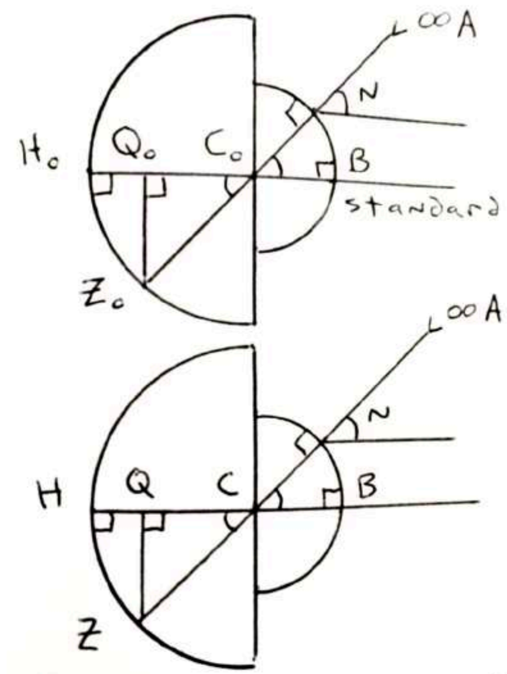
# 5). retinal image size magnification



The top diagram illustrates a standard single-surfaced eye with a distant object A, and resulting retinal image size $H_o Z_o$.



The bottom diagram illustrates any single-surfaced eye with a distant object A, and resulting retinal image size HZ.






As N ⇒ B, the retinal image size magnification, $ZH/Z_oH_o$, (relative to an arbitrary standard which factors out with subsequent comparisons), then approaches its <u>*axial*</u> value:

$ZQ/Z_oQ_o = ZC/Z_oC_o = HC/H_oC_o$

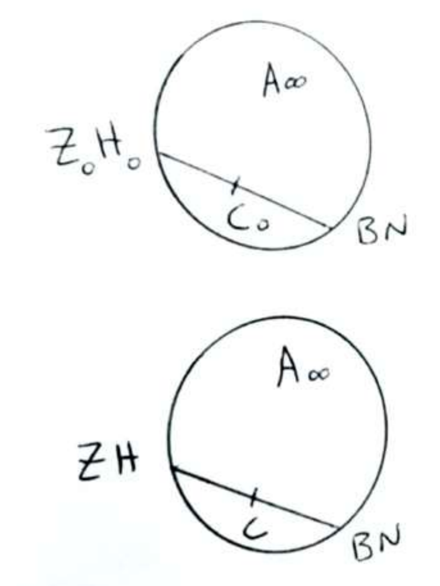
$= (BH/\boldsymbol{R})/(BH_o/\boldsymbol{R}) = BH/BH_o$



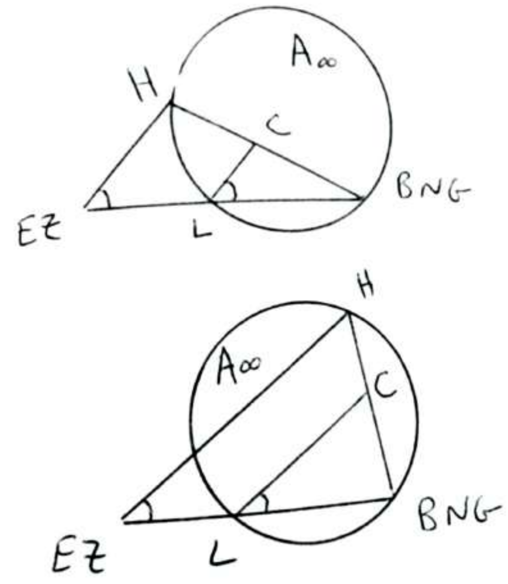
# 6). axial magnification of distance correction



Once again representing the optic axis BCZ as a circle of infinite radius, the distant object A is focused by the curve of radius BC towards the axial object Z, (which lies at the retina H when there is no distance refractive error).

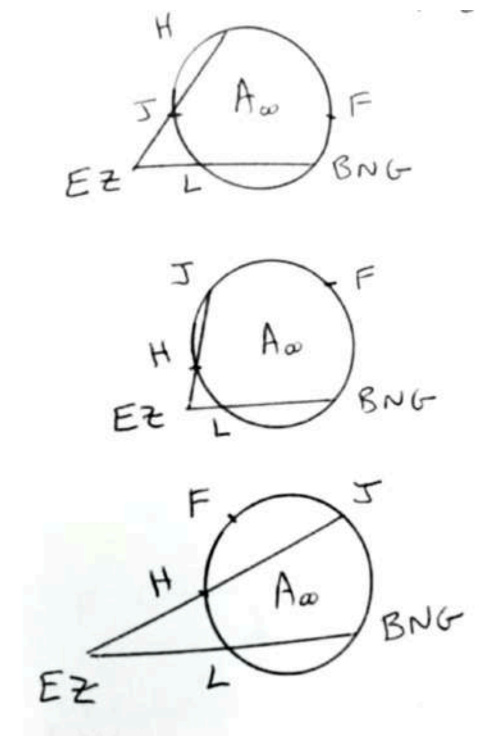


additional refraction at G (at B) will create distance refractive error and a combined single refractive surface of radius BL.



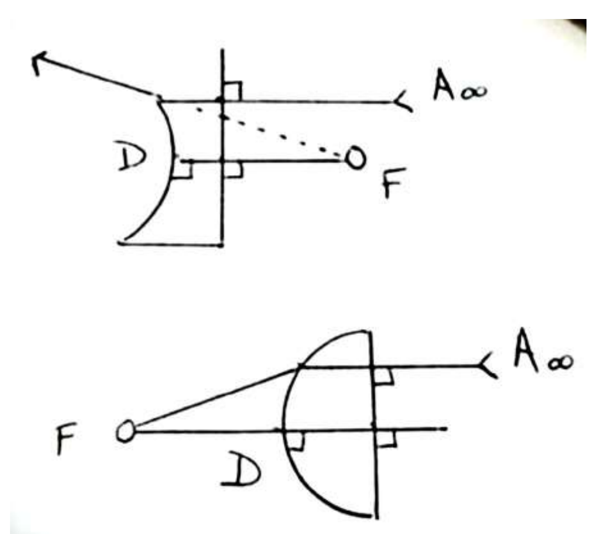



A distance correction must focus the distant object A towards the focal point F of the refractive error G, so that JF || BE, in order to move Z back to H.





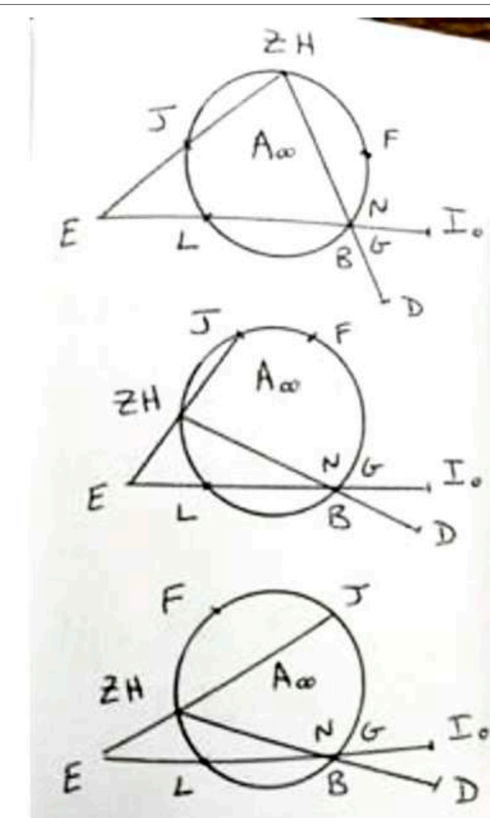

The distance correction at D:






Since the distance correction at D moves Z to H, rays leaving G after this correction must be afocal, resulting in afocal axial angular magnification equaling:

FD/FG (= FD/FB)

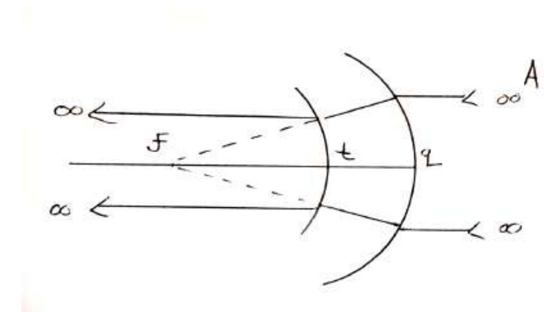


The (total) axial magnification of distance correction equals:

$M = (BH/BH_o)(FD/FB)$



When the front surface of a spectacle lens that corrects distance refractive error is not flat, it is convex; and adds an additional “shape” factor, (fq/ft), to the afocal axial magnification of distance correction. (Point “t” lies at D, and FD/FB remains the “power” factor of the afocal axial magnification of distance correction).





$\Delta EBH \cong \Delta EJL$

If E is at $H_o$, the distance refractive error is completely due to an axial length that is not standard.

If $\Delta EJL \cong \Delta I_oFB$, then:

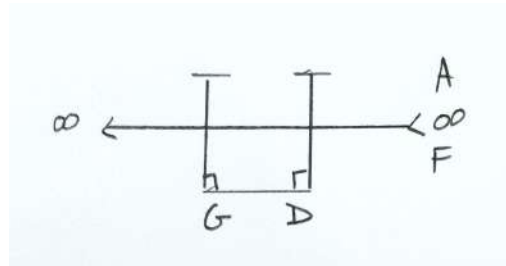
$M = (FB/FI_o)(FD/FB) = FD/FI_o$

There is then no (total) axial magnification of distance correction if the correction D lies at $I_o$, the front focal point of the standard eye.



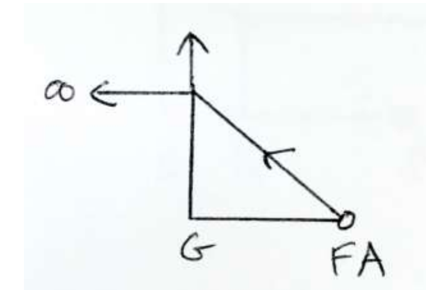
# 7). axial magnification of near correction

There is no afocal axial angular magnification FD/FB when object A is at distance with an emetropic eye.
(The refractive error at G, (at B), is zero; and the focal point F of that refractive error lies at infinity).



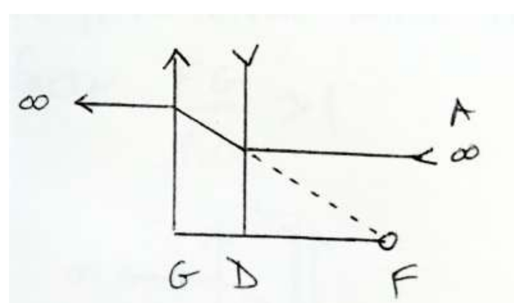



There is also no afocal axial angular magnification when object A is at the front focal point of an uncorrected myopic eye.
(The system is not afocal, and involves only one refracting element).






As discussed, a distance myopic correction at D creates afocal axial angular minification:

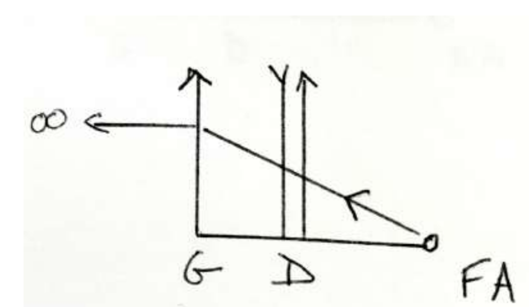



FD/FG < 1

and this is relative to either the myopic eye with object A at its front focal point F, or the emetropic eye with object A at distance.



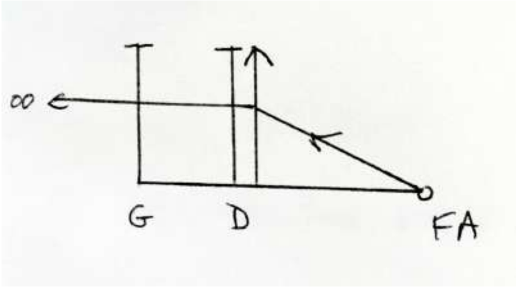
Removing the myopic distance correction at D with a converging lens at D removes this afocal axial angular magnification with the factor:




FG/FD > 1

and this magnification of near correction is relative to the distance corrected myope.

If additional converging power is added to the converging lens so that the near focal point is in focus for an *emetropic* eye, which we then consider to be the reference eye, the magnification of near correction is still that which is removed with the factor:



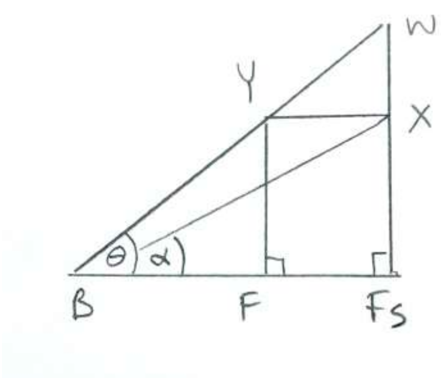

FG/FD > 1



# 8). object angular subtense magnification

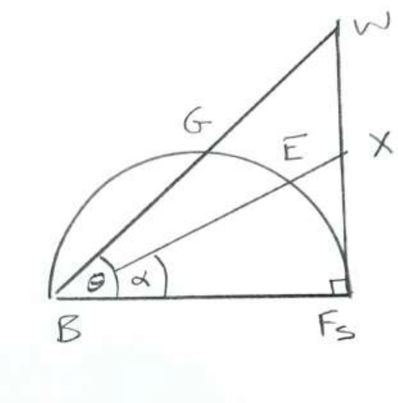


When an object at a standard distance Fs is moved to F:






The object angular subtense magnification equals:




$\theta/\alpha = (\sim GFs/BFs)/(\sim EFs/BFs)$

as XFs ⇒ 0

the object angular subtense magnification approaches its axial value:

$\theta/\alpha \Rightarrow WFs/XFs = WFs/YF = BFs/BF$

which equals the *axial*
object angular subtense magnification.



The ratio describing axial object angular subtense magnification:

BFs/BF

when multiplied by the ratio describing near magnification due to a single converging lens producing parallel light for an emmetropic eye:

FB/FD

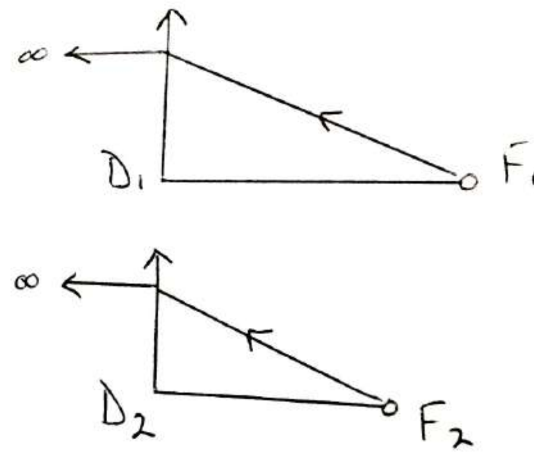


produces a ratio which factors out the object's actual distance to the eye, confirming that when a converging lens is used with its front focal point at the object, so parallel light leaves the converging lens from the object, the image size is the same regardless of the object-to-eye distance.



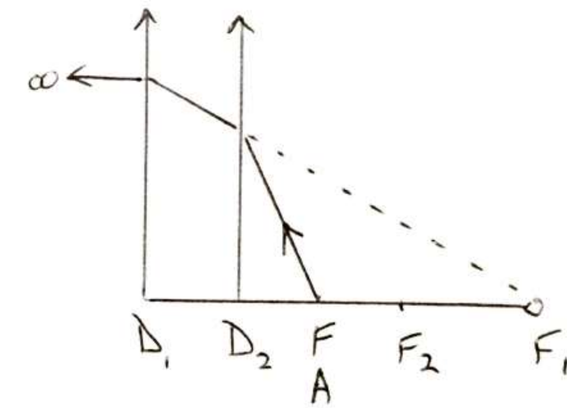
# 9). stand magnifier magnification

When the converging lens at D is split into two converging lenses:






with the same combined focus F:






the ratio describing axial near magnification due to a single converging lens producing parallel light for an emmetropic eye:

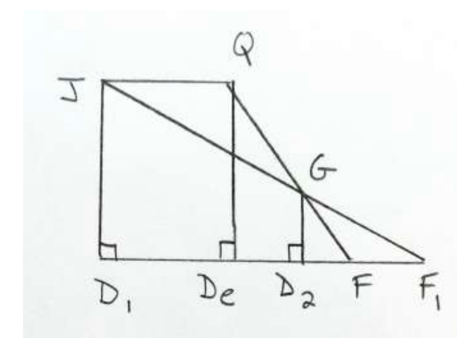
FB/FD

must be expressed ***as if*** all convergence occurred at a single unknown axial point De:

FB/FDe



De can be located using triangles.




$D_2G/D_2F = DeQ/DeF$

$D_2G/D_2F_1 = D_1J/D_1F_1$

$D_2F(DeQ/DeF) = D_2F_1(D_1J/D_1F_1)$

$DeQ/DeF = (D_2F_1/D_2F)(D_1J/D_1F_1)$

$1/DeF = (D_2F_1/D_2F)(1/D_1F_1)$

$FB/FDe = (D_2F_1/D_2F)(FB/D_1F_1)$

Multiplying the axial object subtense magnification by the axial magnification of near correction (relative to the same eye without refractive error) produces:

$$BFs/FDe = (D_2F_1/D_2F)(BFs/D_1F_1)$$



The converging lens $D_2$ creates a virtual image $F_1$ of an object at F. When considering a stand magnifier with lens $D_2$, constant stand height $D_2F$, and reading spectacle add or ocular accommodation $D_1$, the stand magnifier's (constant) enlargement of the object at F equals:

$$E = D_2F_1/D_2F$$

The stand magnifier's axial magnification is its (constant) enlargement factor E, multiplied by what would be produced by $D_1$ alone, if the object A were at $F_1$.



# B). Using Conic Sections



## 1). crossed cylinders

It is useful to know the meridian of maximum axial refraction when combining the effects of two cylindrical refracting surfaces at an oblique axis. To do this, we need to first describe how their axial radii of curvature change with various meridional cross sections. Meridional cross sections of cylindrical surfaces are ellipses until they become parallel lines along the cylinder axis.

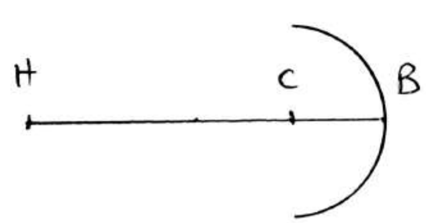


However, assuming a cylinder is parabolic rather than spherical, and that meridional cross sections are parabolic until they rotate into a single line parallel to the cylinder axis, allows for an approximation of the axial radii of curvature of these meridional cross sections. When these axial radii of curvature are expressed in forms that are additive in terms of refraction, we can then find the maximum sum of those expressions in terms of the meridional axis.



With any axial radius of curvature CB, and index of refraction ***R***, the axial image of a distant object lies at H when:

$$\boldsymbol{R} = HB/HC$$





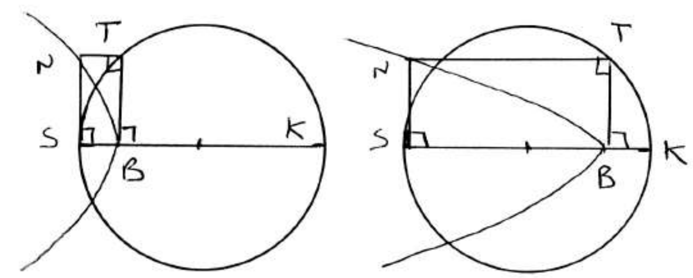

The axial refractive effects of compound refractive surfaces at B are additive only as their refractive "powers," which equal:

$$\boldsymbol{R}/HB = 1/HC = [(HB - HC)/HC]/CB = (\boldsymbol{R} - 1)/CB$$

All parabolas have the same shape, in the same way that all circles have the same shape. However, while circles have a single (internal) determining constant, the radius of curvature, parabolas have both a determining constant internal and external to the curve, and can be defined by either.



For example, a parabola's external determining constant equals BK when:

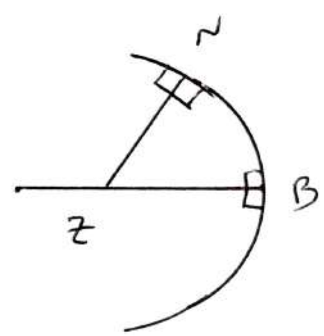



$$\frac{SB}{BT} = \frac{BT}{BK}$$

[2(SN) equals the sagitta corresponding to the sagittal depth SB].

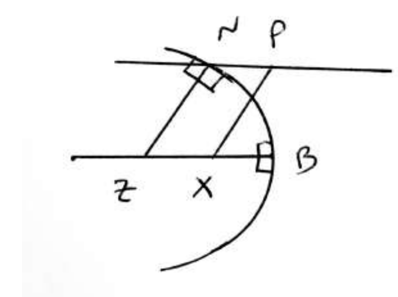


We can set up the necessary off-axis conditions to determine a parabola's axial center of curvature in terms of its internal determining constant XB, by involving ZN in the geometric solution for XB.






In order to keep the determining geometrical relationships axial as N ⇒ B, they should also depend on line NP being parallel to the axis, and XP being parallel to ZN.



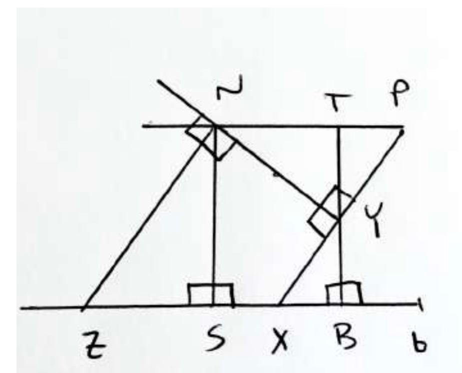

We know X lies between Z and B, since parabolas flatten in their periphery.

Since as N ⇒ B, Z ⇒ C by definition, and

since XP = ZN, P will remain external to the curve, and X can therefore not be its axial center of curvature, but must instead lie somewhere along CB.



In order to maintain ZN perpendicular to the parabola at N as N ⇒ B, the same geometrical relationships must exist that allow for that when N lies at B.




In other words:

YP = YX and
Bb = BX so
CB = 2(XB)



Since:

$$\frac{TN}{TB} = \frac{TN}{2(TY)} = \frac{YB}{2(XB)} = \frac{YB}{CB} = \frac{TB}{2(CB)}$$

We know the external determining constant BK equals 2(CB), and the internal determining constant XB equals (CB)/2.



Axial refracting power equals (***R***- 1)/CB

Since for a parabola:

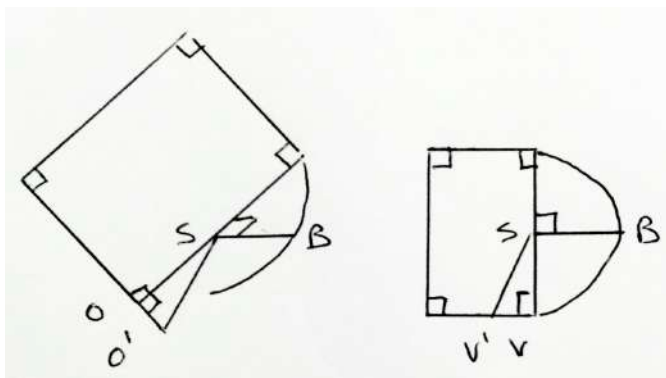
SB/SN = SB/TB = TB/[2(CB)]

If ***R*** = 1.5

The axial refracting power of a parabola equals:

1/[2(CB)] = $SB/SN^2$ = 1/BK

When 2(SO) equals the minimum sagitta of an oblique parabolic cylinder, and when with equal sagittal depth SB, 2(SV) equals the minimum sagitta of a more highly curved parabolic cylinder with a horizontal axis:

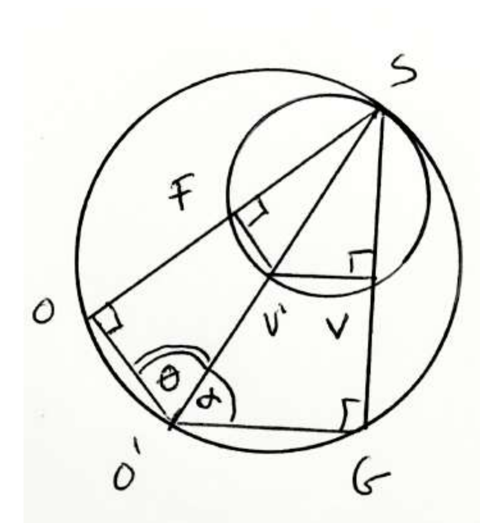





Keeping ΔOSV constant, as we rotate circle SOG with variable diameter SV'O' around point S:

∠OO'G is constant because ∠OSG is constant,

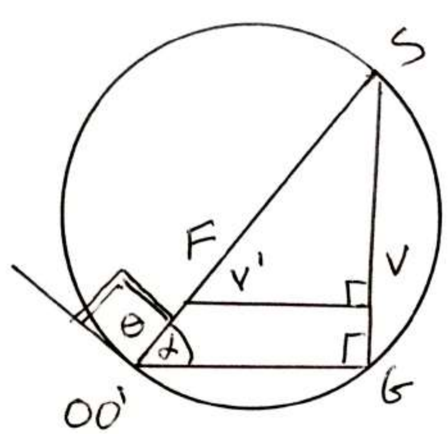
so $\Delta\theta = -\Delta\alpha$






As O' ⇒ O

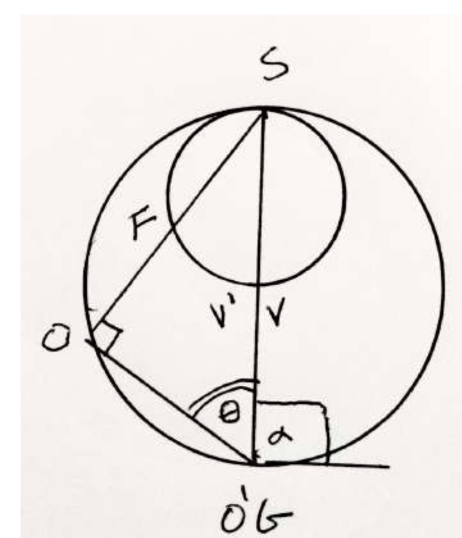
SV' increases more than SO' decreases






As V' ⇒ V

SO' increases more than SV' decreases



Since the sum (SO' + SV') increases when either:

O' ⇒ O,   or V' ⇒ V

there must be a specific SV'O' within ΔOSV producing a minimum sum (SO' + SV'),
which must be near where small rotations produce only minimal changes in (SO' + SV').



Since as when one term of the sum (SO' + SV') increases, the other always decreases, this process can be taken to its limits to determine the meridian with minimum (SO' + SV') using:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta(SO') = \lim_{\Delta\alpha \Rightarrow 0} \Delta(SV')$$



However, the combined effects of refraction are additive only as refractive powers,
which, when ***R*** = 1.5, equal:

$SB/(SO')^2$ and $SB/(SV')^2$



Therefore, the meridian with the maximum combined effects of this refraction can be found using:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta [SB/(SO')^2] = \lim_{\Delta\alpha \Rightarrow 0} \Delta [SB/(SV')^2]$$

To solve this equation, all variables must be expressed in terms of the variables approaching zero, so:

$$\underset{\Delta\theta \Rightarrow 0}{\text{Limit}}\ \Delta\{[SB(SO/SO')^2]/SO^2\} = \underset{\Delta\alpha \Rightarrow 0}{\text{Limit}}\ \Delta\{[SB(SV/SV')^2]/SV^2\}$$

$$\underset{\Delta\theta \Rightarrow 0}{\text{Limit}}\ \Delta\{[(SB)\sin^2\theta]/SO^2\} = \underset{\Delta\alpha \Rightarrow 0}{\text{Limit}}\ \Delta\{[(SB)\sin^2\alpha]/SV^2\}$$

$$(SB/SO^2)\ \underset{\Delta\theta \Rightarrow 0}{\text{Limit}}\ \{\Delta\sin^2\theta\} = (SB/SV^2)\ \underset{\Delta\alpha \Rightarrow 0}{\text{Limit}}\ \{\Delta\sin^2\alpha\}$$



$$\{\text{Limit as } \Delta\theta \Rightarrow 0 \text{ of } [\Delta\sin^2\theta]\}/\{\text{Limit as } \Delta\alpha \Rightarrow 0 \text{ of } [\Delta\sin^2\alpha]\}$$

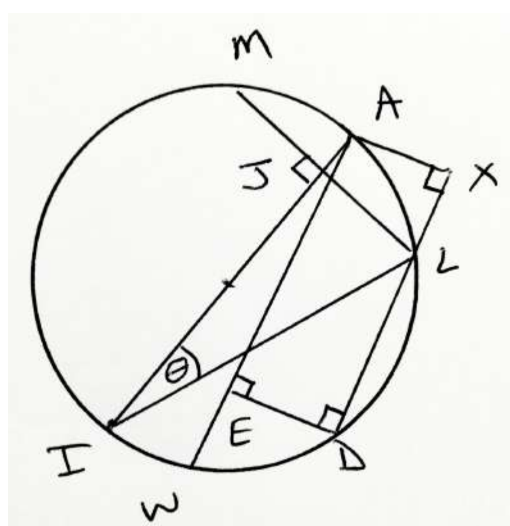
$$= [SO^2/SV^2]$$



Solve for

$$\underset{\Delta\theta \Rightarrow 0}{\text{Limit}}\ \Delta \sin^2\theta$$

on the reference circle:




AW ≥ LD || AW

$\angle ALD = \sim AID/AI$

$\geq \sim AI/AI = \pi$

Establish the necessary functions of θ in terms of line segments and chords.



$\theta = \sim\frac{AL}{AI}$ ; $\sin^2\theta = \frac{AL^2}{AI}$

$\Delta\theta = \sim\frac{LD}{AI}$ ; $\sin^2\Delta\theta = \frac{LD^2}{AI}$

$(\theta + \Delta\theta) = \sim\frac{ALD}{AI}$ ; $\sin^2(\theta + \Delta\theta) = \frac{AD^2}{AI}$

$\cos\theta = \frac{IL}{AI}$ ; $\cos(\theta + \Delta\theta) = \frac{DI}{AI}$

$\sin\theta = \frac{AL}{AI} = \frac{JL}{IL}$ ; $\sin\theta\cos\theta = \frac{JL}{IL}\ \frac{IL}{AI}$

$2(\sin\theta\cos\theta) = \frac{ML}{AI} = \sin 2\theta$

Then consider the following property of the cyclic quadrilateral circle ALDW: AD(LW) = AL(DW) + LD(AW)

$$\Delta DIA \cong \Delta EWD = \Delta XLA \; ; \; AD^2 = AL^2 + LD(AW)$$

$$AW = LD + 2(AL)\frac{LX}{LA} \; ; \; AW = LD + 2(AL)\frac{ID}{IA}$$

$$AD^2 - AL^2 = LD^2 + 2(LD)(AL)\frac{ID}{IA}$$



$$AI\,[\sin^2(\theta + \Delta\theta) - \sin^2\theta] =$$

$$AI\,[\sin^2\Delta\theta] + 2(LD)(AL)\cos(\theta + \Delta\theta) =$$

$$AI\,[\sin^2\Delta\theta] + 2(LD)\,[(AI)\sin\theta]\cos(\theta + \Delta\theta)$$

Divide both sides by AI:

$$\sin^2(\theta + \Delta\theta) - \sin^2\theta = \sin^2\Delta\theta + 2(LD)\sin\theta\cos(\theta + \Delta\theta)$$

$$\lim_{\Delta\theta \Rightarrow 0} \frac{\Delta(\sin^2\theta)}{\sim LD} = 2\sin\theta(\cos\theta) = \sin 2\theta$$



Therefore, the meridian with the maximum combined effects of refraction can be found using:

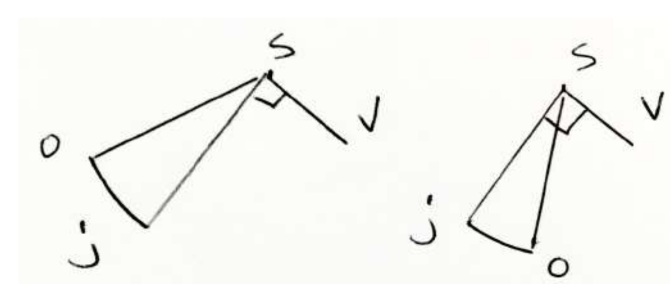
$$\frac{\sin 2\theta}{\sin 2\alpha} = \frac{SO^2}{SV^2}$$

The first step to solve this problem is to divide SV into SaV so that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$



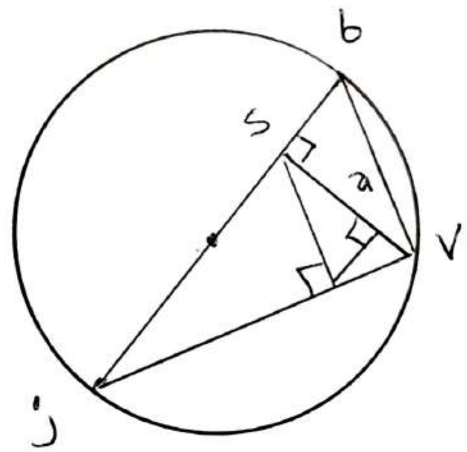
Make SO = Sj ⊥ SV to construct:



$$\frac{Sj}{SV} = \frac{SV}{Sb} \quad ; \quad \frac{Sj^2}{SV^2} = \frac{Sj}{Sb} = \frac{SO^2}{SV^2}$$

Similar triangles show that:

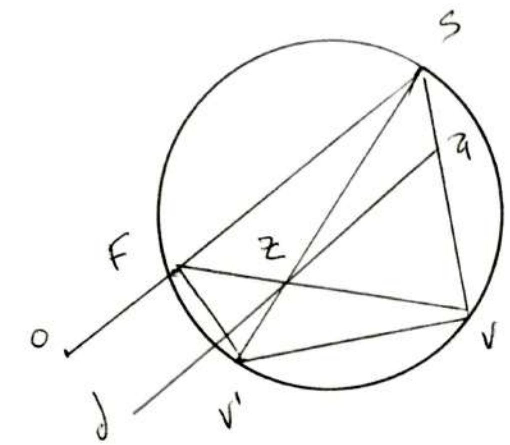
$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$





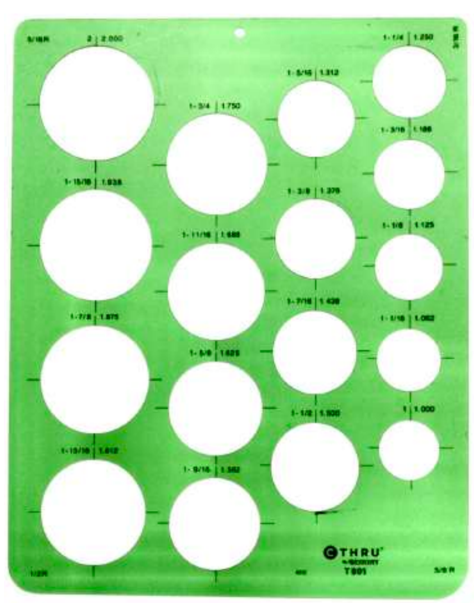

Draw ad || SO

Choose a circle through S and V with a variable diameter SV' so that FZV lies on a common chord.



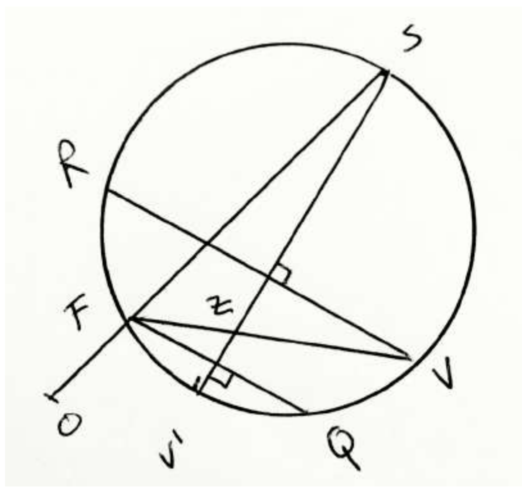



The easiest way to do this involves a template of various circles, each with the location of their diameters already marked.





SV' is the meridian with the maximum combined effects of refraction because:



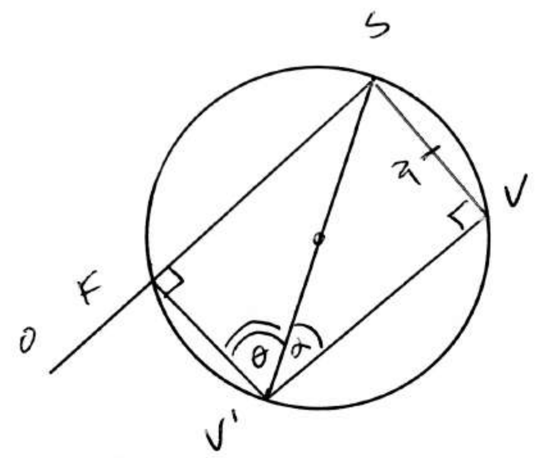

$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{FZ}{ZV} = \frac{FQ/2}{RV/2} = \frac{FQ}{RV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

## Double-angle Method:

Given constant $\Delta OSV$:
$\angle FSV$ is constant
$\angle FSV + (\theta + \alpha) = \pi$
$(\theta + \alpha)$ Is constant

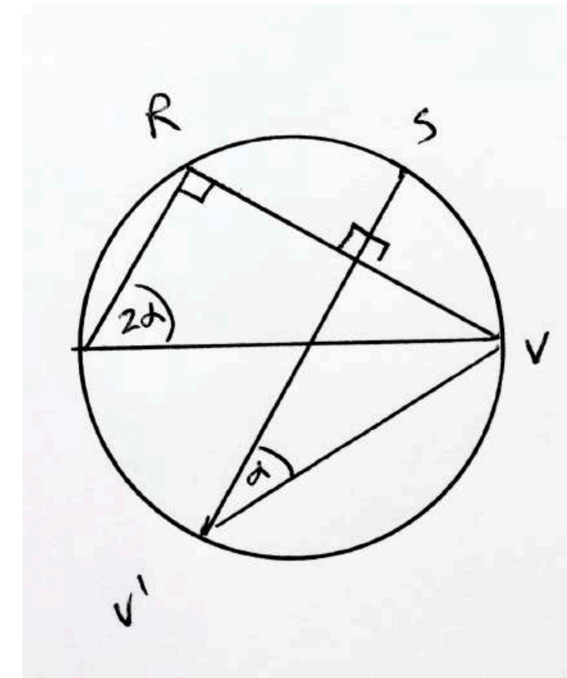
We have already shown how to find single angles $\theta + \alpha$ so that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{\sin 2\theta}{\sin 2\alpha}$$





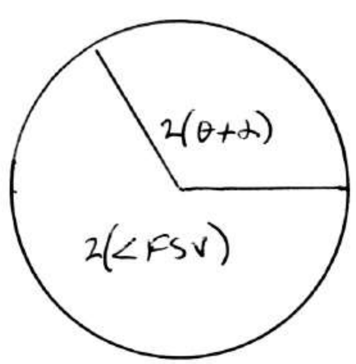

An angle on a circle equals its inscribed arc, divided by the arc's diameter. Since the sum of all angles measured on a circle's circumference add to π, when measured from a circle's center they add to 2π.

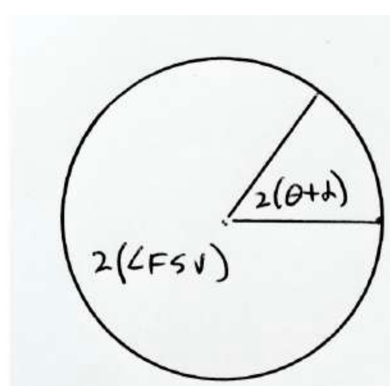





Therefore:

$$2(\angle FSV) + 2\,(\theta + \alpha) = 2\pi$$



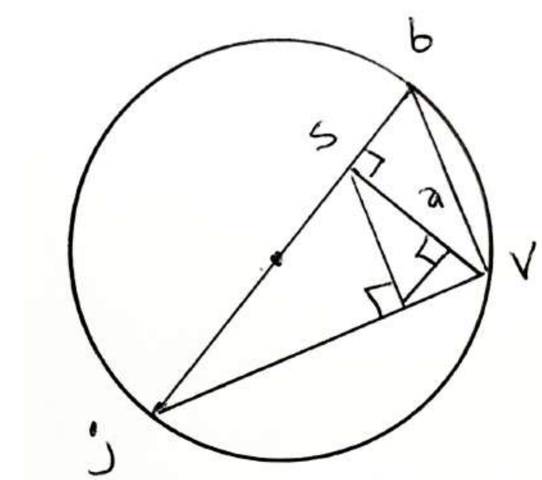






When:

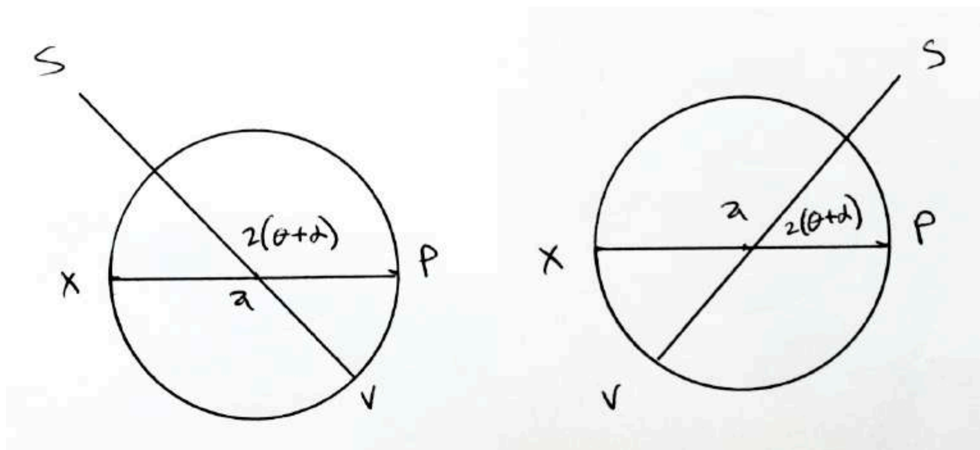
$$\frac{SO^2}{SV^2} = \frac{Sj^2}{SV^2} = \frac{aS}{aV}$$

as drawn:



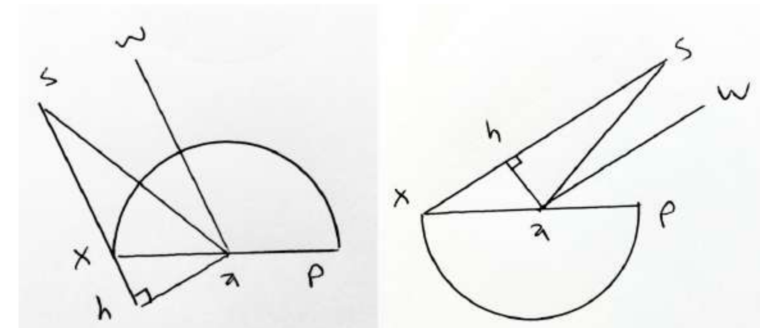
If we draw diameter XaP so:

aX = aV, and ∠SaP = 2 (θ + α)






$$\frac{SO^2}{SV^2} = \frac{aS}{aX} = \frac{ah/aX}{ah/aS} = \frac{\sin 2\theta}{\sin 2\alpha}$$



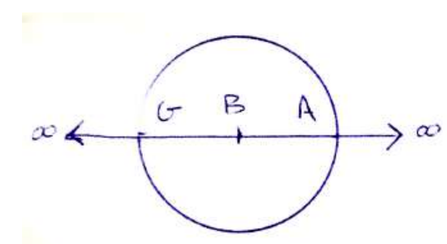

When aw || sX, we have divided the doubled angle 2 (θ + α) = ∠SaP
into 2θ = ∠WaP, and 2α = ∠WaS.



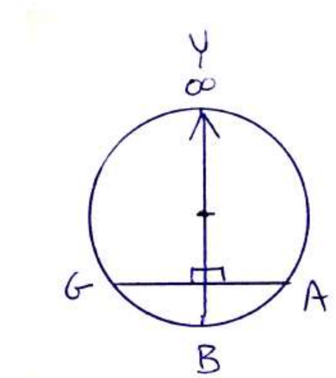
# 2). refraction along a line



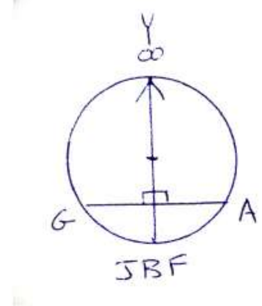
If we consider a circle with center B and diameter GBA with an "axis" infinitely long through GBA:




We can represent GBA along a circle of infinite diameter BY, and draw BG = BA. This infinitely large reference circle is equally divided along ray BY, with Y at infinity.

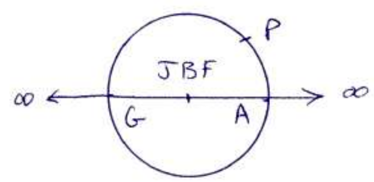


If we call points J & F, (both of which in this case lie at B), the “focal points” of the finite circle, we can consider the shape of the finite circle with diameter GBA to equal its “eccentricity” = e = BF/BA = 0.




We will have drawn a circle where AJ + AF = AG along its diameter GJBFA, if it is also true that:

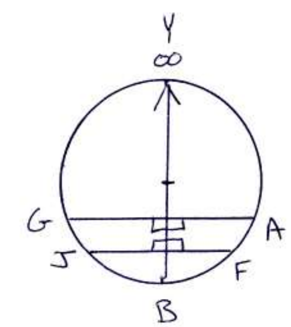
PJ + PF = AG

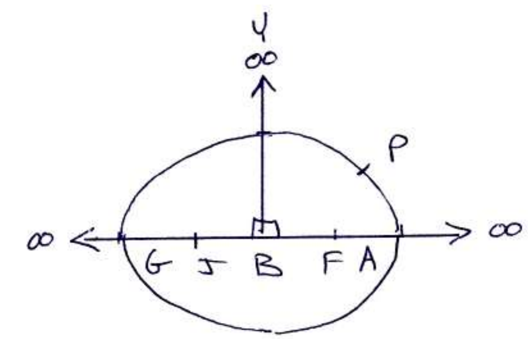





If we draw: 0 < e = BF/BA < 1

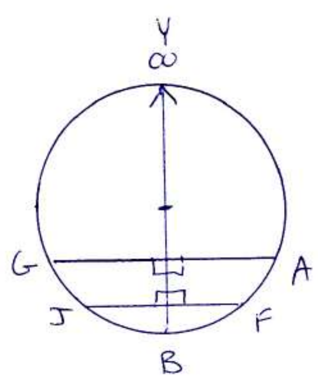
we will have drawn a **ellipse** where AJ + AF = AG along its “major axis” GJBFA, if it is also true that PJ + PF = AG.



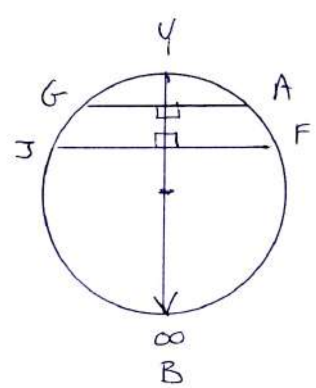






As:



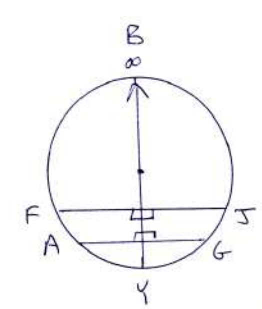

becomes:

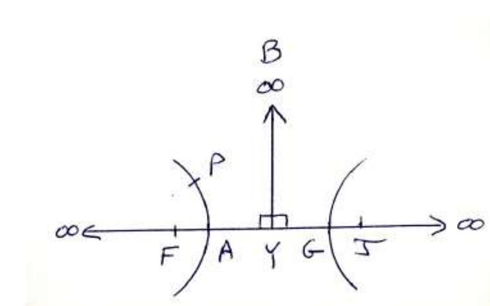




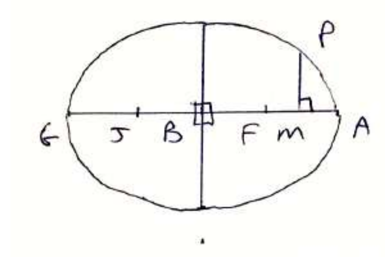

If we draw: 0 < e = YF/YA > 1

we will have drawn a **hyperbola** where AJ - AF = AG along its “transverse axis” FAYGJ, if it is also true that PJ - PF = AG.



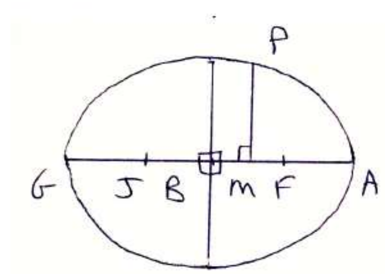



# Ellipse




2(BF)  =  MJ - MF

2(BM)  =  MJ + MF




2(BF)  =  MJ + MF

2(BM)  =  MJ - MF



$PJ^2 - FP^2 = (MJ^2 + MP^2) - (MF^2 + MP^2)$

(PJ + FP) (PJ - FP)  =  (MJ + MF) (MJ - MF)

AG (PJ - FP)  =  2(BM) 2(BF)

PJ - FP  =  [2(BM) 2(BF)]/2(BA)

eccentricity = e = BF/BA

PJ - FP  =  2(BM)e



Since:

FP + PJ  =  AG  =  2(BA)

(FP + PJ) + (PJ - FP)  =  2(PJ)  =  2(BA) + 2(BM)e

(FP + PJ) - (PJ - FP)  =  2(FP)  =  2(BA) - 2(BM)e

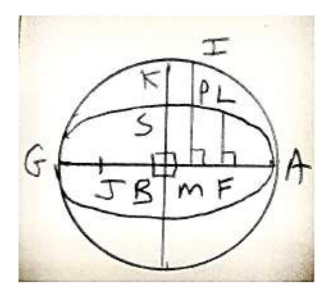
**PJ = BA + (BM)e**

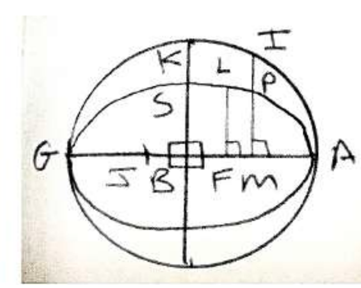
**PF = BA - (BM)e**



FM  =  BF - BM

FM  =  BM - BF




$FM^2 = BF^2 + BM^2 - 2(BF)BM$

e = BF/BA = FB/FS

$BA^2 = BF^2 + BS^2$

$PF^2 = [BA - (BM)e]^2$

$PF^2 = BA^2 + (BM)^2e^2 - 2(BM)BF$

$PM^2 = PF^2 - FM^2$

$PM^2 = [BA^2 + (BM)^2e^2 - 2(BM)BF]$

$- [BF^2 + BM^2 - 2(BF)BM]$

$PM^2 = BS^2 + BM^2(e^2 - 1)$

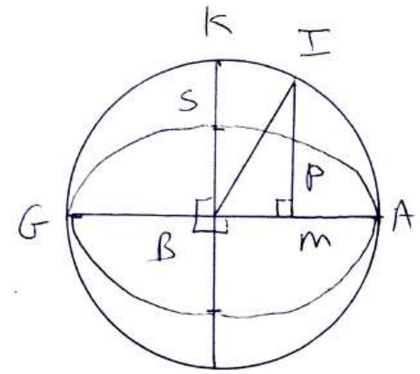
$PM^2 = BS^2 - BM^2(1-e^2)$

$(PM)^2BA^2 = (BS)^2BA^2 - BM^2[BA^2 - BF^2]$

$(PM)^2BA^2 = BS^2[BA^2 - BM^2]$

$(MP/MI)^2 = (BS/BA)^2$

$MP/MI = BS/BK$





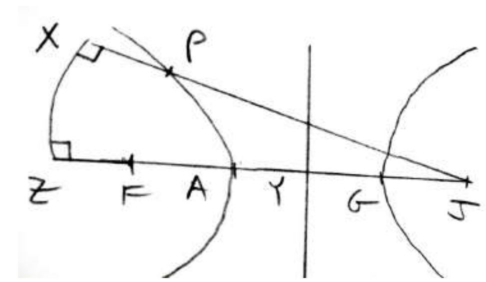

# Hyperbola

Draw hyperbola arm AP:

Make: $ZJ - AG = XP + FP$

So: $XJ - XP = FP + AG$

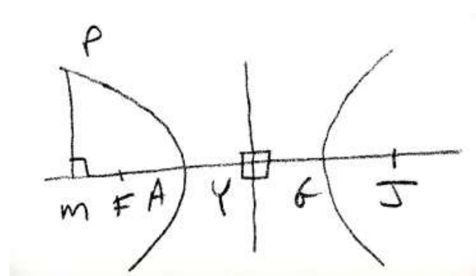
and $PJ - FP = AG$





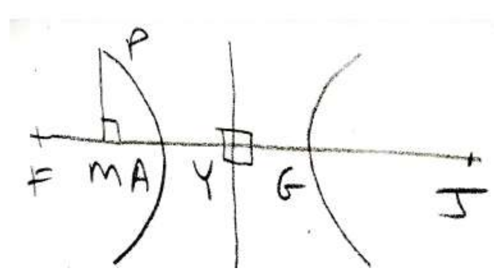

$MJ - MF = 2(YF)$

$MJ + MF = 2(YM)$




$MJ - MF = 2(YM)$

$MJ + MF = 2(YF)$






$PJ^2 - FP^2 = (MP^2 + MJ^2) - (MP^2 + MF^2)$

$(PJ + FP)(PJ - FP) = (MJ + MF)(MJ - MF)$

$(PJ + FP)AG = 2(YM)\,2(YF)$

$PJ + PF = [2(YM)\,2(YF)]/2(YA)$

eccentricity = $e = YF/YA$

$PJ + PF = 2(YM)e$

Since: $PJ - PF = AG = 2(YA)$

$(PJ + PF) + (PJ - PF) = 2(PJ) = 2(YM)e + 2(YA)$

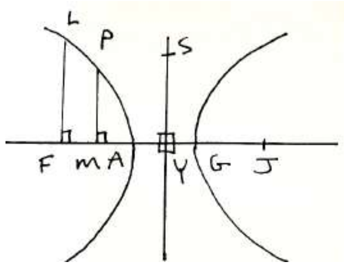
$(PJ + PF) - (PJ - PF) = 2(PF) = 2(YM)e - 2(YA)$

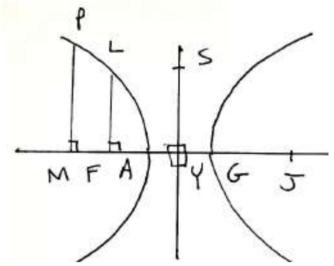
$\mathbf{PJ = (YM)e + YA}$

$\mathbf{PF = (YM)e - YA}$



$FM = YF - YM$ $FM = YM - YF$




$FM^2 = YF^2 + YM^2 - 2(YF)YM$

$e = YF/YA = AS/AY$

$YF^2 = YA^2 + YS^2$



$PF^2 = [(YM)e - YA]^2$

$PF^2 = YM^2e^2 + YA^2 - 2(YM)YF$

$PM^2 = PF^2 - FM^2$

$PM^2 = [YM^2e^2 + YA^2 - 2(YM)YF]$

$- [YF^2 + YM^2 - 2(YF)YM]$

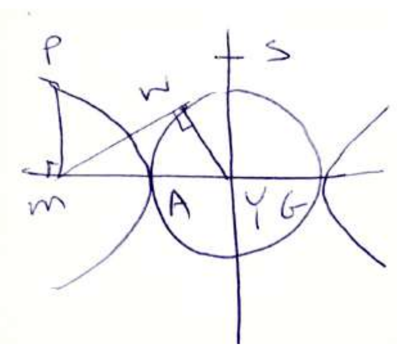
$PM^2 = YM^2(e^2 - 1) - YS^2$

$PM^2\ YA^2 = YM^2[YF^2 - YA^2] - YS^2\ YA^2$

$PM^2\ YA^2 = YS^2(YM^2 - YA^2)$

$(MP/MW)^2 = (YS/YA)^2$

$MP/MW = YS/YA$





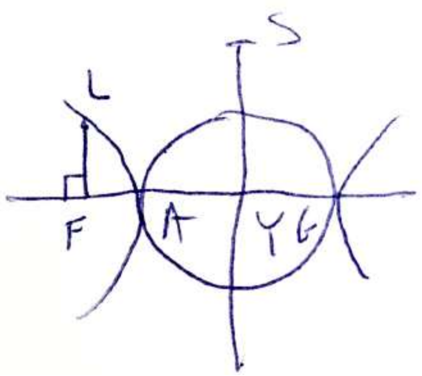

$MW^2 = (MA)MG$

$MP^2/(MA)MG = (YS/YA)^2 = FL^2/(FA)FG$

$(FA)FG = (YF - YA)(YF + YA)$

$(FA)FG = YF^2 - YA^2 = YS^2$

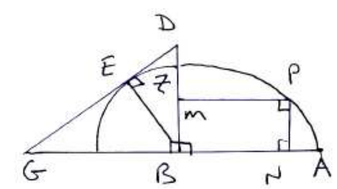
$FL/YS = YS/YA$



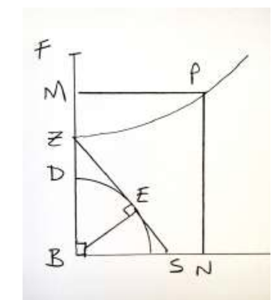
The following discussion will be presented in two columns for clarity. The left column represents the object in glass, and the right side column represents the object in air.



Given refraction along line GBNA, object D in glass, and image Z seen along BZD, a non-perpendicular image ray NM can be found using the reference semi-ellipse GZPA:



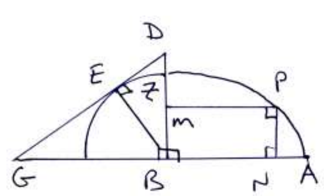

Given refraction along line BSN, object D in air, and image Z seen along BDZ, a non-perpendicular image ray NM can be found using the reference hyperbola arm ZP:

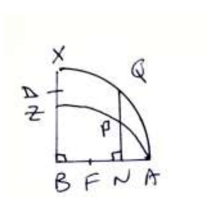


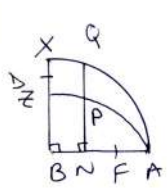

(with vertex designated as B instead of Y for consistency)



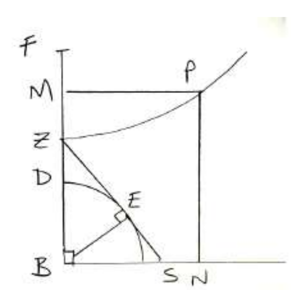
because:
$e = BF/BA = FB/FZ$
and: $NQ/NP = BX/BZ$

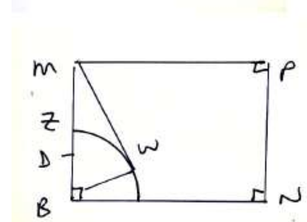



because:
$e = BF/BZ = ZS/ZB$
and: $MW/MP = BZ/BS$






$NQ/NP = BX/BZ$

$BZ^2/NP^2 = BA^2/(BA^2 - BN^2)$

$(BZ^2 - NP^2)/NP^2 = BN^2/(BA^2 - BN^2)$

$(BZ^2 - NP^2)/BN^2 = NP^2/(BA^2 - BN^2)$

$= NP^2/NQ^2 = BZ^2/BG^2 = BE^2/BG^2$
$= ED^2/BD^2 = (BD^2 - BZ^2)/BD^2$

$(BZ^2 - NP^2)/BN^2 = (BD^2 - BZ^2)/BD^2$

$MW/MP = BZ/BS$

$MW^2/MP^2 = (MB^2 - ZB^2)/BN^2$

$BZ^2/BS^2 = EZ^2/EB^2$
$= (ZB^2 - DB^2)/DB^2$

$(MB^2 - ZB^2)/BN^2$
$= (ZB^2 - DB^2)/DB^2$

$(NP^2 - BZ^2)/BN^2 = (BZ^2 - BD^2)/BD^2$

$(MN^2 - BZ^2)/BN^2 = BZ^2/BD^2$

$(MN^2 - BZ^2)/BZ^2 = BN^2/BD^2$

$MN^2/BZ^2 = (BN^2 + BD^2)/BD^2$

$MN^2/DN^2 = BZ^2/BD^2$

$MN/DN = BZ/BD$

$(MB^2 - ZB^2 + BN^2)/BN^2$
$= BZ^2/BD^2$

$(MN^2 - BZ^2)/BZ^2$
$= BN^2/BD^2$

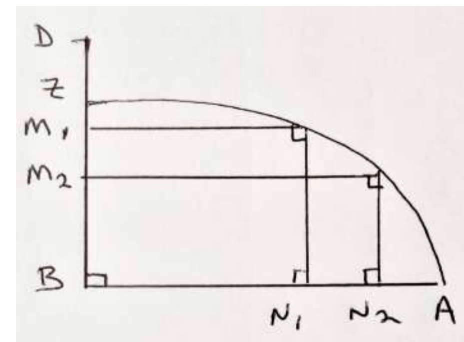
$MN^2/ZB^2 = DN^2/DB^2$

$MN^2/DN^2 = BZ^2/BD^2$

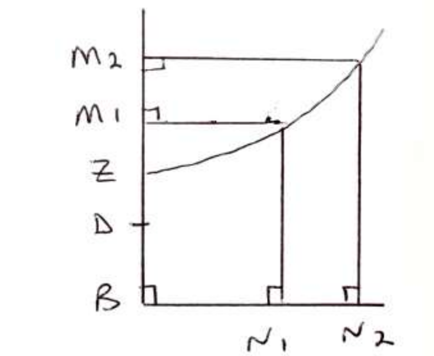
$MN/DN = BZ/BD$



$\mathbb{R} = BD/BZ$
$\mathbb{R} = N_1D/N_1M_1$
$\mathbb{R} = N_2D/N_2M_2$

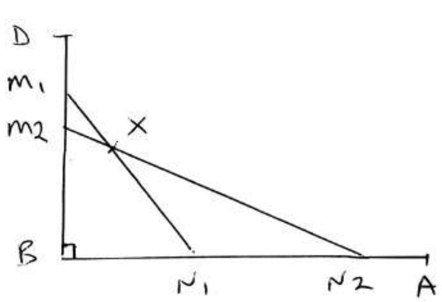



$\mathbb{R} = BZ/BD$
$\mathbb{R} = N_1M_1/N_1D$
$\mathbb{R} = N_2M_2/N_2D$

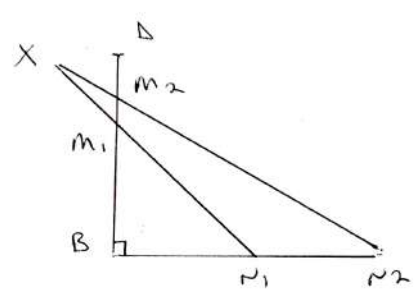




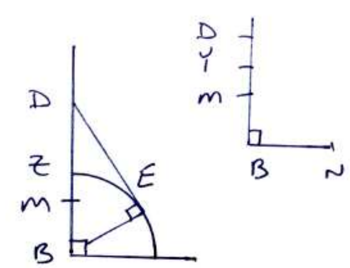

$BM_1 > BM_2$
and $N_1M_1$ crosses
$N_2M_2$ at X within the
right angle ∠DBA.




$BM_2 > BM_1$
and $N_1M_1$ crosses
$N_2M_2$ at X outside the
right angle ∠$DBN_2$.

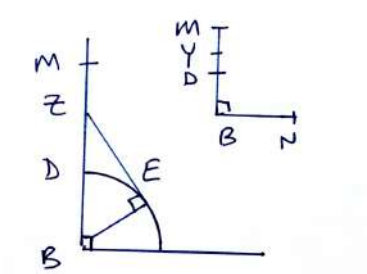





$\mathbb{R} = DB/BZ = ND/NM$

if: $BY/MB = DB/DE$
then: $DB/YN = ED/EB$
because:




$\mathbb{R} = BZ/DB = NM/ND$

if: $BY/DB = ZB/EZ$
then: $MB/YN = EZ/EB$
because:

$MB^2 = MN^2 - BN^2$

$MB^2 = MN^2 - YN^2 + BY^2$

$BY^2/(MN^2 - YN^2 + BY^2)$
$= DB^2/(DB^2 - BZ^2)$
$= DN^2/(DN^2 - MN^2)$

$BY^2/(YN^2 - MN^2) = DN^2/MN^2$

$BY^2 = YN^2 - BN^2$

$BY^2/DB^2 = BZ^2/(BZ^2 - EB^2)$

$BY^2/(BY^2 - DB^2) = BZ^2/DB^2$

$BZ^2/DB^2 = MN^2/DN^2$

$BY^2/MN^2 = (BY^2 - DB^2)/DN^2$

$(BY^2 + MN^2)/MN^2$
$= (BY^2 - DB^2 + DN^2)/DN^2$



$BY^2 = YN^2 - DN^2 + DB^2$

$(YN^2 - DN^2 + DB^2)/(YN^2 - MN^2)$
$= DN^2/MN^2$

$(YN^2 + DB^2)/YN^2 = DN^2/MN^2$

$DB^2/YN^2 = (DN^2 - MN^2)/MN^2$
$= (DB^2 - BZ^2)/DB^2 = ED^2/EB^2$

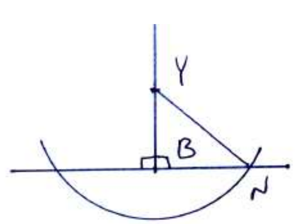
$(BY^2 + MN^2)/(BY^2 + BN^2)$
$= MN^2/DN^2$

$(MN^2 - BN^2)/NY^2$
$= (MN^2 - DN^2)/DN^2$

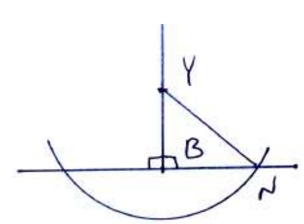
$MB^2/YN^2 = (BZ^2 - DB^2)/DB^2$
$= EZ^2/EB^2$



When given point M, after calculating BY with known BM, (as well as known DB/DE); we can use known DB, (as well as known ED/EB), to calculate YN and use that as a radius about Y to find N:

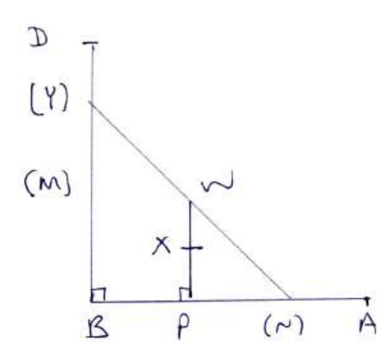



When given point M, after calculating BY with known DB, (as well as known ZB/ZE); we can use known MB, (as well as known EZ/EB), to calculate YN and use that as a radius about Y to find N:

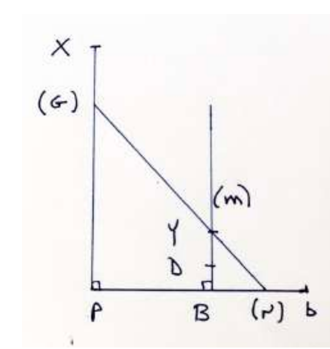





Since M must be known to find N, this gives no advantage over the previously described reference ellipse. However, it provides a way to find N on image ray MX(N) without knowing M.




Since M must be known to find N, this gives no advantage over the previously described reference hyperbola arm. However, it provides a way to find N on image ray XM(N) without knowing M.



To find an image ray through a given point X, first calculate PW with known PX and DB/DE using:
PW/PX = (BY/MB) = DB/DE

Since DB and ED/EB are also known, find the length of YWN using:
DB/YN = ED/EB

We can then find (N) by inserting the calculated length YWN within the right angle ∠DBA through W.

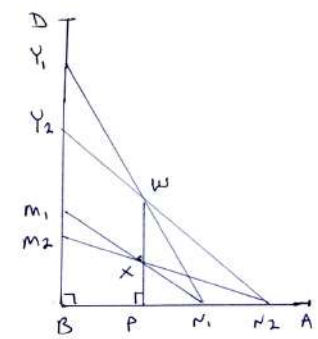
To find an image ray through a given point X, first calculate BY with known DB and ZB/ZE using:
BY/DB = ZB/EZ

Since PX and EZ/EB are also known, find the length of GYN using:
PX/GYN = (MB/YN) = EZ/EB

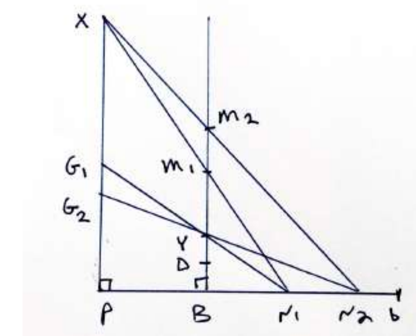
We can then find (N) by inserting the calculated length GYN within the right angle ∠XPb through Y.



For any given calculated value of YN, a maximum of two line segments ($Y_1N_1 = Y_2N_2$) fit though W within the right angle ∠DBA.

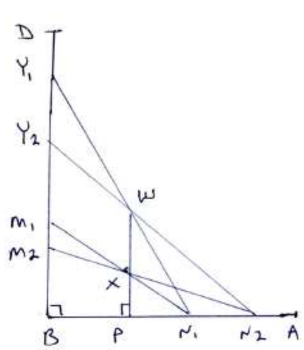



For any given calculated value of GN, a maximum of two line segments ($G_1N_1 = G_2N_2$) fit though Y within the right angle ∠XPb.

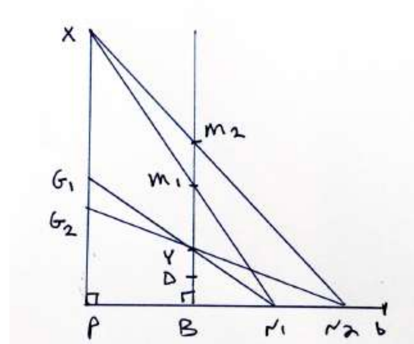





These two line segments are drawn to find both $N_1$ and $N_2$ for the image rays through X.




These two line segments are drawn to find both $N_1$ and $N_2$ for the image rays through X.

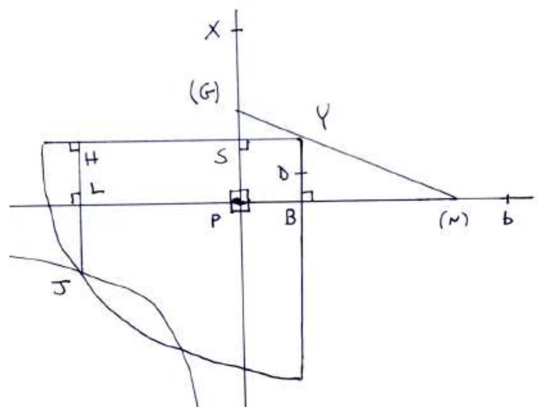





The clear image of X occurs when YN through its specified point W is its minimum possible length, so that $N_1$ lies at $N_2$. Since both BY/MB = PW/XP and DB/YN are constants, YN can be varied while keeping the image location XP constant, but not the object location DB.

The clear image of X occurs when GN through its specified point Y is its minimum possible length, so that $N_1$ lies at $N_2$. Since both MB/YN = XP/GN and BY/DB are constants, GN can be varied while keeping the object location DB constant, but not the image location XP.

Expanding on the right side column representing the object in air, (where GN can be varied while keeping the object location DB constant, but not the image location XP), consider Y to be on a reference hyperbola defined by: (LP)LJ = (BP)BY, and draw its opposite arm:

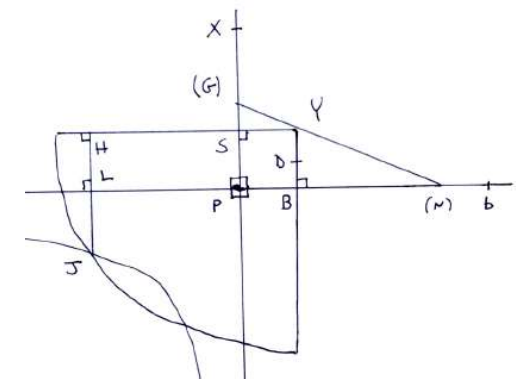





We know LP/BY = BP/LJ.
**If we construct BN = LP**, then BN/BY = BP/LJ.
But SY/SG = BN/BY = BP/LJ
and since SY = BP:

SG = LJ
SG + SP = LJ + HL
PG = HJ

and since by construction BN = LP:
PN = LB = HY
ΔNPG = ΔYHJ
GN = YJ






The reference radius length YJ intersects the reference hyperbola at a maximum of two possible points $J_1$ and $J_2$. Both $G_1YN_1$ and $G_2YN_2$ can be drawn by constructing BN = LP for each point J.

A clear image of object D occurs when $N_1$ and $N_2$ overlap, or when the reference radius length YJ = GN intersects the reference hyperbola at a single point J. The required GN for this condition gives the required location of N, as well as the location of the clear image at X, (remember that PX varies with GN).