Images Seen Through Water

Gregg Baldwin OD
2022

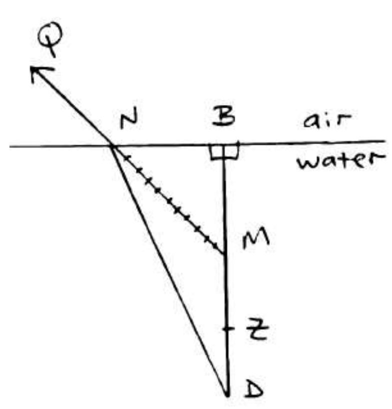
References:

Isaac Barrows Optical Lectures, 1667
Translated by H.C. Fay
Edited by A.G. Bennett
Publisher:
The Worshipful Company of Spectacle Makers
London, England; 1987
ISBN # 0-951-2217-0-1

Plane and Solid Geometry
G. A. Wentworth; 1899 revised edition

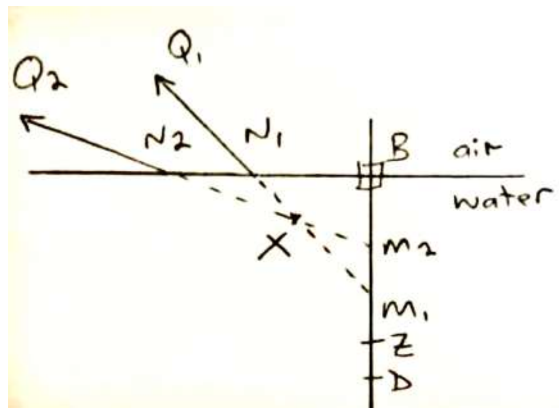
Object in water; image seen from air

If underwater object D is a perpendicular distance DB from the plane of the water surface in all radial directions, the image of object D along that perpendicular, when seen from directly above in air, is at Z, and BD/BZ = 4/3.



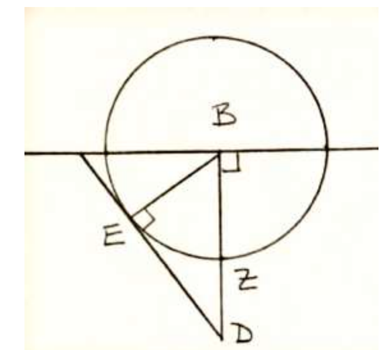
Isaac Barrow showed that the image of object D, when seen from Q *obliquely* along image ray MNQ, also lies above the object, but towards the observer relative to DB.

Isaac Barrow described a way to find all oblique image rays MNQ through a designated point X, without knowing their points of refraction (N) along the surface of the water, or their intersections (M) with the perpendicular DB.

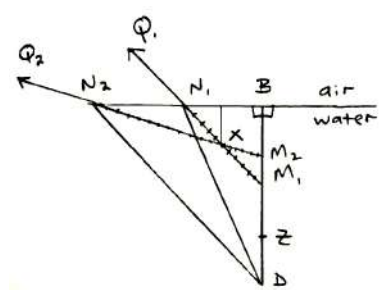


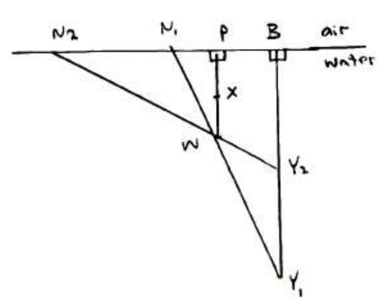
He first drew a reference right triangle created by drawing BE = BZ as shown, which created the following constant ratios for air/water refraction:

BD/BZ = BD/BE = 4/3
DB/DE = 4/√(16-9) = 1.5
ED/EB = [√(16-9)]/3 = 0.87



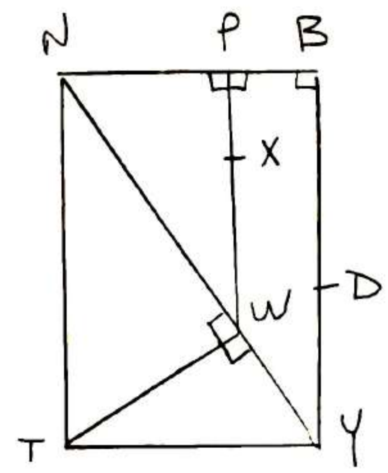
He showed that, given DB and the designated point X, if we draw the reference line segment PXW as shown, so that:
PW/PX = DB/DE = 1.5
then all image rays through X, (MXNQ) are found using:
DB/YN = ED/EB = 0.87
by drawing all possible reference lines of length YN = DB/0.87 through W, in order to locate the required positions of N.





This means that for any given DB, there can be a maximum of two image rays through the designated point X, since only two reference line segments within the right angle ∠(Y)B(N), and equaling his calculated constant YN, can fit through point W.

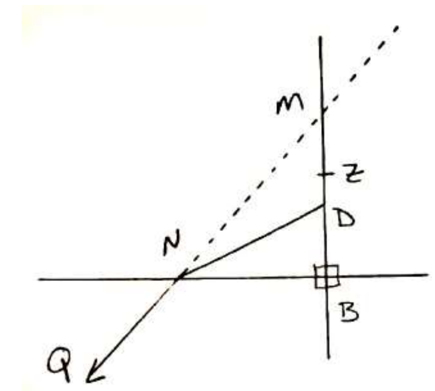
Isaac Barrow showed that YN can be drawn as the shortest segment through W bounded by the right angle ∠(Y)B(N) when right triangles ΔYBN, ΔNWT, and ΔTWY are all drawn as similar.



Keeping P constant, as we vary length YN = DB/0.87 through W to find its minimum, the position of D must vary, while PW/PX (= DB/DE) = 1.5 can remain unchanged. Therefore, when the object is in water, Isaac Barrow's analysis can find the image ray XMNQ for a designated clear image X, and an undesignated object D.

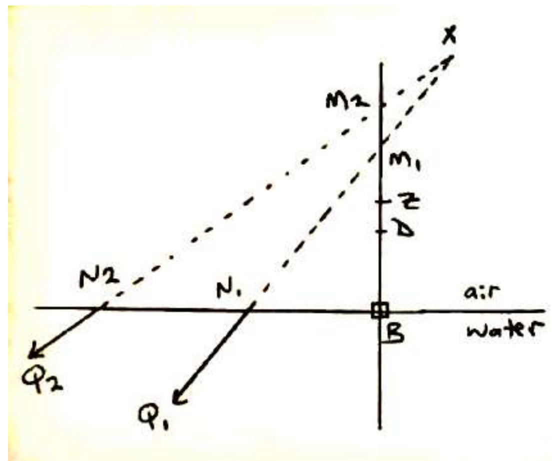
Object in air; image seen from water

If object D is in air, and at a perpendicular distance DB from the surface of water in all radial directions, the image of the object along that perpendicular when seen from underwater is at Z, and BZ/BD = 4/3.



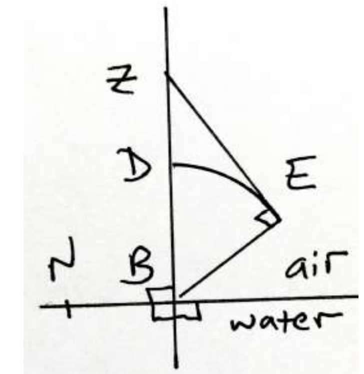
Isaac Barrow showed that the image of object D, when seen from Q *obliquely* along image ray MNQ, also lies above the object, but away from the observer relative to DB.

Isaac Barrow described a way to find all oblique image rays MNQ through a point X, without knowing their points of refraction (N) along the surface of the water, or their intersections (M) with the perpendicular DB.

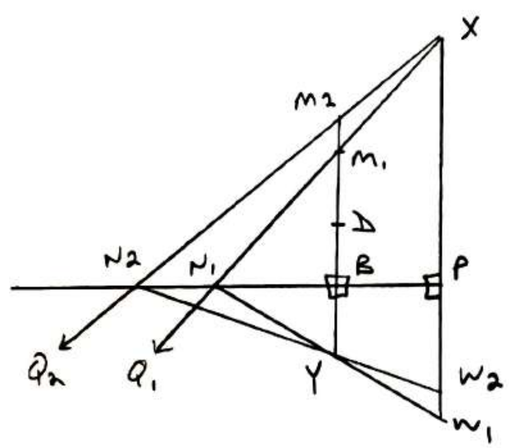


He first drew a reference right triangle created by drawing BE = BD as shown, which created the following constant ratios for air/water refraction:

$BZ/BD = BZ/BE = 4/3$
$ZB/ZE = 4/\sqrt{(16-9)} = 1.5$
$EZ/EB = \sqrt{(16-9)}/3 = 0.87$

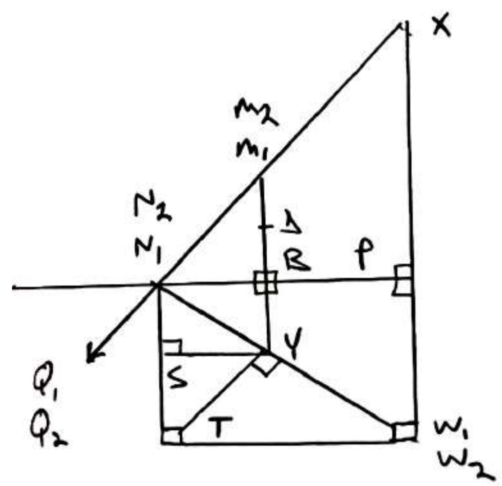


He showed that, given DB and the designated point X, if we draw
$BY/BD = ZB/ZE = 1.5$
then all image rays through X, (XMNQ) are found using:
$XP/WN = MB/YN$
$= EZ/EB = 0.87$
by drawing all possible reference lines of length $WN = XP/0.87$ through Y, in order to locate the required positions of N.



This means that for any given DB, there can be a maximum of two image rays through the designated point X, since only two reference line segments within the right angle ∠(W)P(N), and equaling his calculated constant WN, can fit through point Y.

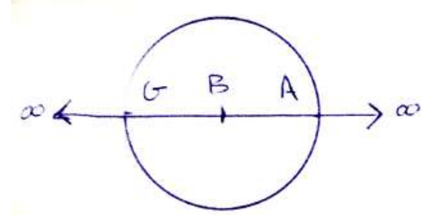
Isaac Barrow showed that WN can be drawn as the shortest segment through Y bounded by the right angle ∠(W)P(N) when right triangles ΔWPN, ΔNYT, and ΔWYT are all drawn as similar.



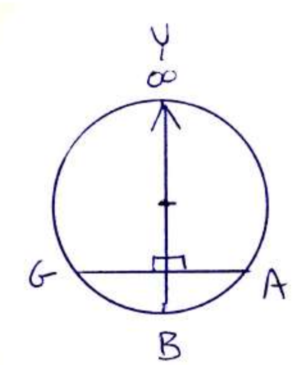
Keeping P constant, as we vary length WN = XP/0.87 through Y to find its minimum, the position of X must vary, while BY = DB(1.5) can remain unchanged. Therefore, when the object is in air, Isaac Barrow's analysis can find the image ray XMNQ for a designated object D, and an undesignated clear image X.

Using conic sections

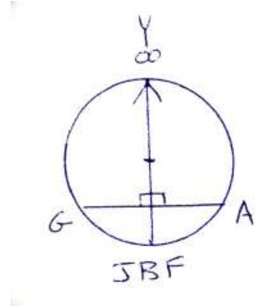
If we consider a circle with center B and diameter GBA with an "axis" infinitely long through GBA:



we can represent GBA along a circle of infinite diameter BY, and draw BG = BA.

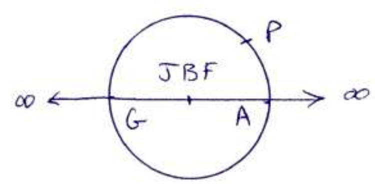


If we call points J & F, (both of which in this case lie at B), the “focal points” of the finite circle, we can consider the shape of the finite circle with diameter GBA to equal its “eccentricity” = e = BF/BA = 0.



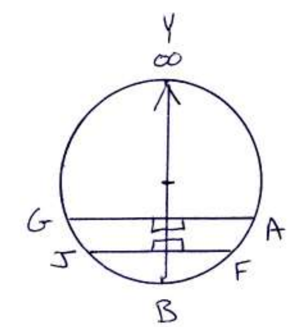
We will have drawn a finite circle where AJ + AF = AG along its diameter and “axis” GJBFA, if it is also true that:

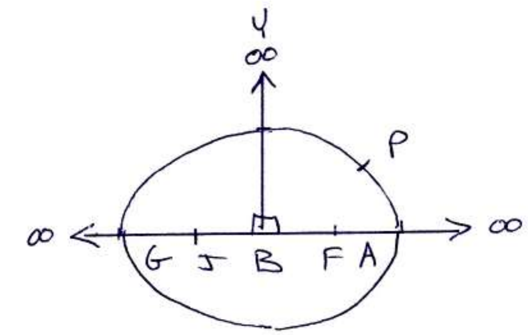
PJ + PF = AG



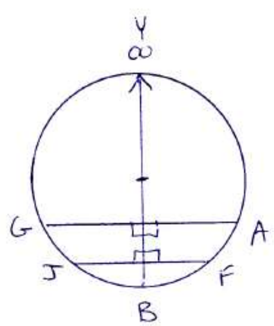
If we draw: 0 < e = BF/BA < 1

we will have drawn a finite **ellipse** where AJ + AF = AG along its “major axis” GJBFA, if it is also true that PJ + PF = AG.

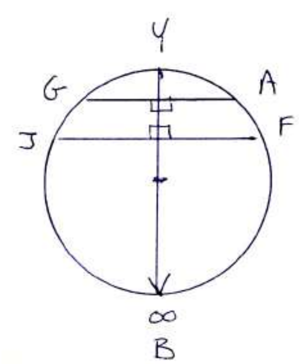




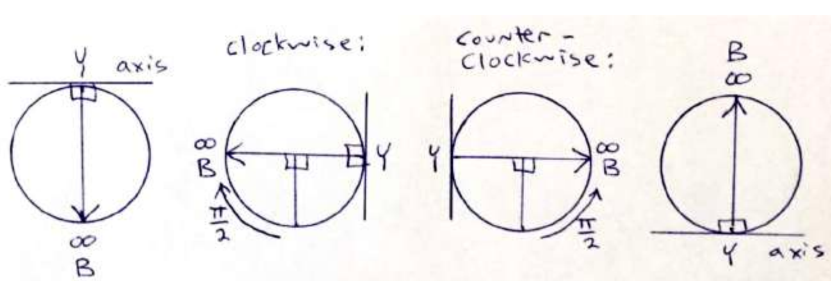
As:



becomes:

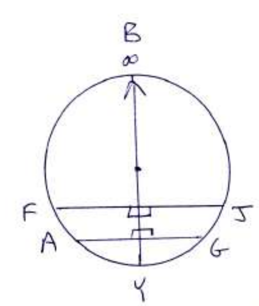


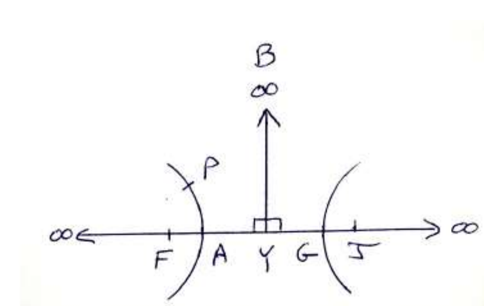
and rotates:



If we draw: 0 < e = YF/YA > 1

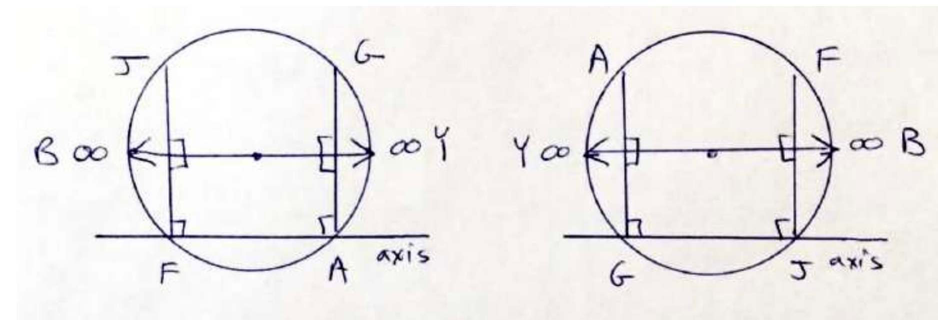
we will have drawn a **hyperbola** where AJ - AF = AG along its “transverse axis” FAYGJ, if it is also true that PJ - PF = AG.



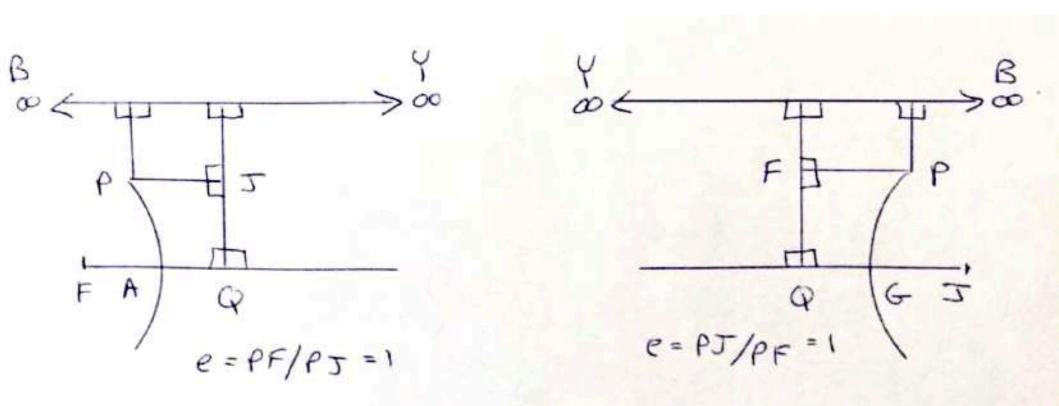


When the infinitely large reference circle only rotates by π/2 radians in either direction, it no longer remains a circle equally divided by an infinitely long upward ray with its base on an axis, because reference points B and Y are ***both*** infinitely far. However, due to the halfway rotation of the reference circle, we can presume these curves resulting from clockwise and counter-clockwise rotation have an eccentricity halfway between that of an ellipse ($e < 1$), and that of an hyperbola ($e > 1$).

These resulting curves are defined as a parabolas ($e = 1$), and like the circle ($e = 0$), they represent a special case with a singular shape, or eccentricity.



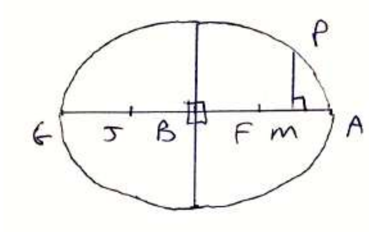
If we draw AF = GJ we will have drawn parabolas along their respective “axes” AF or GJ, if it is also true that PF = PJ. Since parabolas represent the eccentricity as an ellipse transforms into an hyperbola, (or visa versa), QA = QG.



Ellipse

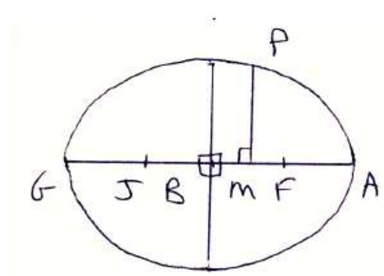
$2(BF) = MJ - MF$

$2(BM) = MJ + MF$



$2(BF) = MJ + MF$

$2(BM) = MJ - MF$



$PJ^2 - FP^2 = (MJ^2 + MP^2) - (MF^2 + MP^2)$

$(PJ + FP)(PJ - FP) = (MJ + MF)(MJ - MF)$

$AG(PJ - FP) = 2(BM)\,2(BF)$

$PJ - FP = [2(BM)\,2(BF)]/2(BA)$

eccentricity = e = BF/BA

$PJ - FP = 2(BM)e$

Since:

$FP + PJ = AG = 2(BA)$

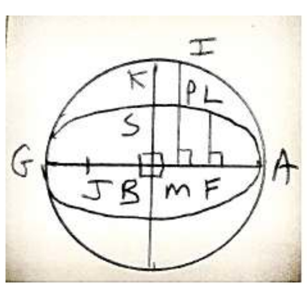
$(FP + PJ) + (PJ - FP) = 2(PJ) = 2(BA) + 2(BM)e$

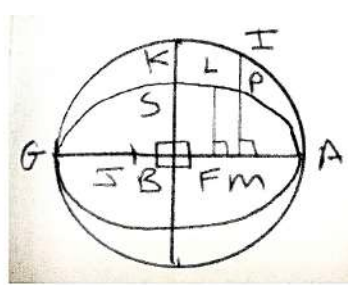
$(FP + PJ) - (PJ - FP) = 2(FP) = 2(BA) - 2(BM)e$

PJ = BA + (BM)e

PF = BA - (BM)e

$FM = BF - BM$ $\qquad$ $FM = BM - BF$





$FM^2 = BF^2 + BM^2 - 2(BF)BM$

$e = BF/BA = FB/FS$

$BA^2 = BF^2 + BS^2$

$PF^2 = [BA - (BM)e]^2$

$PF^2 = BA^2 + (BM)^2e^2 - 2(BM)BF$

$PM^2 = PF^2 - FM^2$

$PM^2 = [BA^2 + (BM)^2e^2 - 2(BM)BF]$

$\qquad - [BF^2 + BM^2 - 2(BF)BM]$

$PM^2 = BS^2 + BM^2(e^2 - 1)$

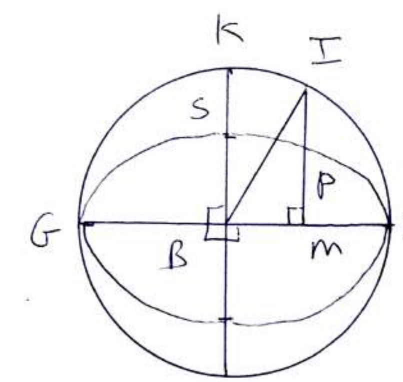
$PM^2 = BS^2 - BM^2(1-e^2)$

$(PM)^2BA^2 = (BS)^2BA^2 - BM^2[BA^2 - BF^2]$

$(PM)^2BA^2 = BS^2[BA^2 - BM^2]$

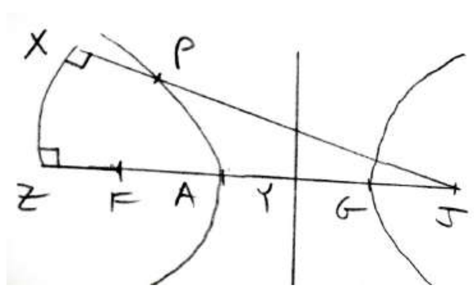
$(MP/MI)^2 = (BS/BA)^2$

$MP/MI = BS/BK$



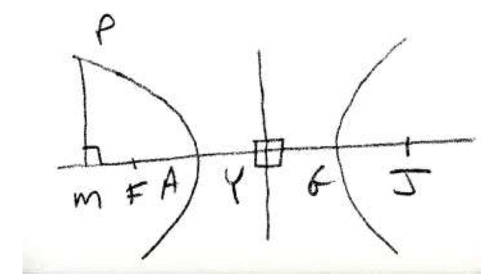
Hyperbola

Draw hyperbola arm AP:
Make: ZJ - AG = XP + FP
So: XJ - XP = FP + AG
and PJ - FP = AG



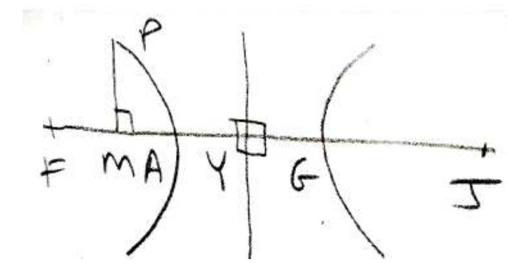
MJ - MF = 2(YF)

MJ + MF = 2(YM)



MJ - MF = 2(YM)

MJ + MF = 2(YF)



$PJ^2 - FP^2 = (MP^2 + MJ^2) - (MP^2 + MF^2)$

(PJ + FP) (PJ - FP) = (MJ + MF) (MJ - MF)

(PJ + FP)AG = 2(YM) 2(YF)

PJ + PF = [2(YM) 2(YF)]/2(YA)

eccentricity = e = YF/YA

PJ + PF = 2(YM)e

Since: PJ - PF = AG = 2(YA)

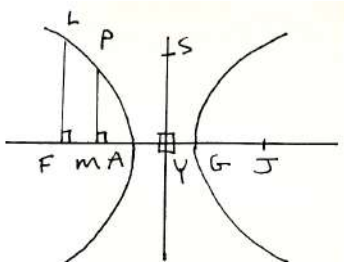
(PJ + PF) + (PJ - PF) = 2(PJ) = 2(YM)e + 2(YA)

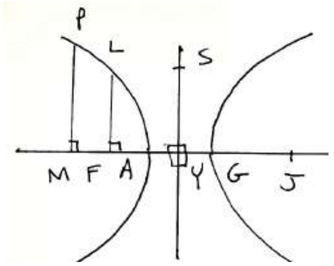
(PJ + PF) - (PJ - PF) = 2(PF) = 2(YM)e - 2(YA)

PJ = (YM)e + YA

PF = (YM)e - YA

$FM = YF - YM$ $FM = YM - YF$





$FM^2 = YF^2 + YM^2 - 2(YF)YM$

$e = YF/YA = AS/AY$

$YF^2 = YA^2 + YS^2$

$PF^2 = [(YM)e - YA]^2$

$PF^2 = YM^2e^2 + YA^2 - 2(YM)YF$

$PM^2 = PF^2 - FM^2$

$PM^2 = [YM^2e^2 + YA^2 - 2(YM)YF]$

$\qquad - [YF^2 + YM^2 - 2(YF)YM]$

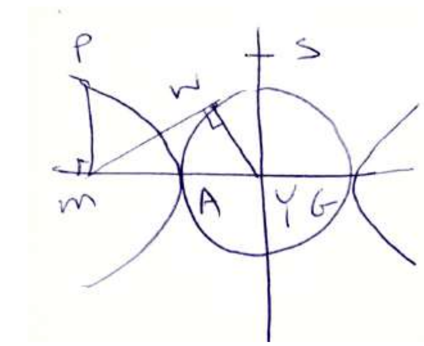
$PM^2 = YM^2(e^2 - 1) - YS^2$

$PM^2\ YA^2 = YM^2[YF^2 - YA^2] - YS^2\ YA^2$

$PM^2\ YA^2 = YS^2(YM^2 - YA^2)$

$(MP/MW)^2 = (YS/YA)^2$

$MP/MW = YS/YA$



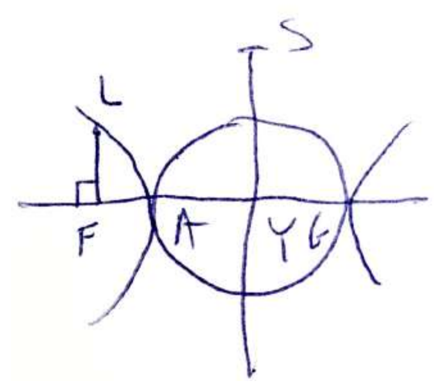
$MW^2 = (MA)MG$

$MP^2/(MA)MG = (YS/YA)^2 = FL^2/(FA)FG$

$(FA)FG = (YF - YA)\ (YF + YA)$

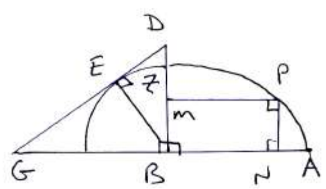
$(FA)FG = YF^2 - YA^2 = YS^2$

$FL/YS = YS/YA$

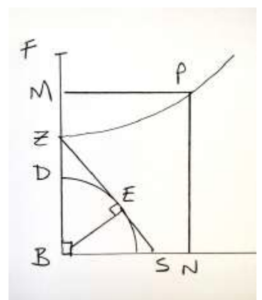


The following discussion will be presented in two columns for clarity. The left column represents the object in glass, and the right side column represents the object in air.

Given refraction along line GBNA, object D in glass, and image Z seen along BZD, a non-perpendicular image ray NM can be found using the reference semi-ellipse GZPA:

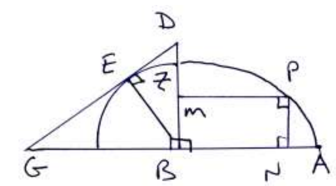


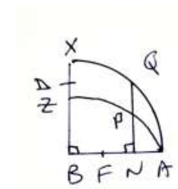
Given refraction along line BSN, object D in air, and image Z seen along BDZ, a non-perpendicular image ray NM can be found using the reference hyperbola arm ZP:

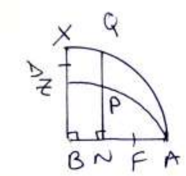


(with vertex designated as B instead of Y for consistency)

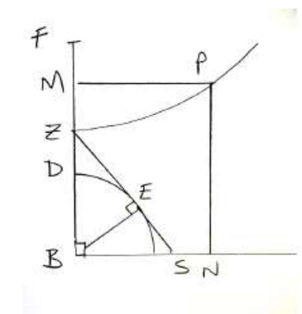
because:
$e = BF/BA = FB/FZ$
and: $NQ/NP = BX/BZ$

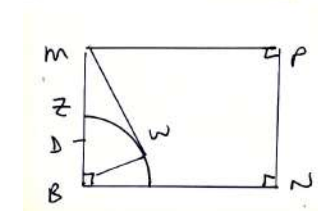






because:
$e = BF/BZ = ZS/ZB$
and: $MW/MP = BZ/BS$





$NQ/NP = BX/BZ$

$BZ^2/NP^2 = BA^2/(BA^2 - BN^2)$

$(BZ^2 - NP^2)/NP^2 = BN^2/(BA^2 - BN^2)$

$(BZ^2 - NP^2)/BN^2 = NP^2/(BA^2 - BN^2)$

$= NP^2/NQ^2 = BZ^2/BG^2 = BE^2/BG^2$
$= ED^2/BD^2 = (BD^2 - BZ^2)/BD^2$

$(BZ^2 - NP^2)/BN^2 = (BD^2 - BZ^2)/BD^2$

$MW/MP = BZ/BS$

$MW^2/MP^2 = (MB^2 - ZB^2)/BN^2$

$BZ^2/BS^2 = EZ^2/EB^2$
$= (ZB^2 - DB^2)/DB^2$

$(MB^2 - ZB^2)/BN^2$
$= (ZB^2 - DB^2)/DB^2$

$(NP^2 - BZ^2)/BN^2 = (BZ^2 - BD^2)/BD^2$

$(MN^2 - BZ^2)/BN^2 = BZ^2/BD^2$

$(MN^2 - BZ^2)/BZ^2 = BN^2/BD^2$

$MN^2/BZ^2 = (BN^2 + BD^2)/BD^2$

$MN^2/DN^2 = BZ^2/BD^2$

$MN/DN = BZ/BD$

$(MB^2 - ZB^2 + BN^2)/BN^2$
$= BZ^2/BD^2$

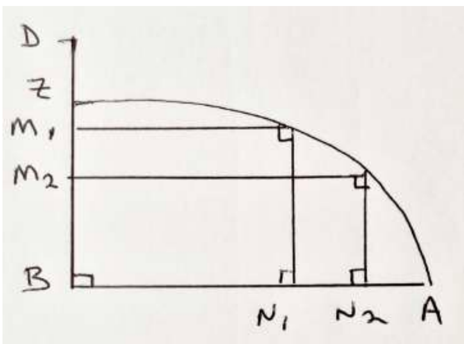
$(MN^2 - BZ^2)/BZ^2$
$= BN^2/BD^2$

$MN^2/ZB^2 = DN^2/DB^2$

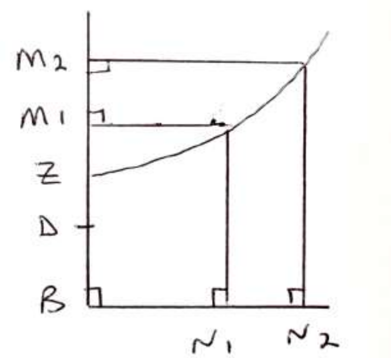
$MN^2/DN^2 = BZ^2/BD^2$

$MN/DN = BZ/BD$

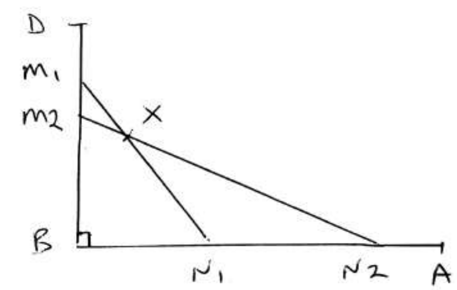
$\mathbb{R}$ = BD/BZ
$\mathbb{R}$ = N_1D/N_1M_1
$\mathbb{R}$ = N_2D/N_2M_2



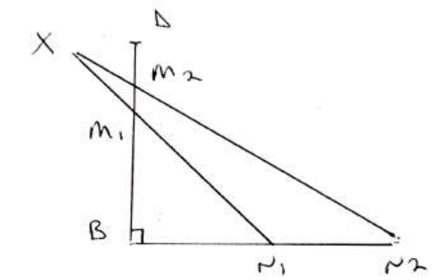
$\mathbb{R}$ = BZ/BD
$\mathbb{R}$ = N_1M_1/N_1D
$\mathbb{R}$ = N_2M_2/N_2D

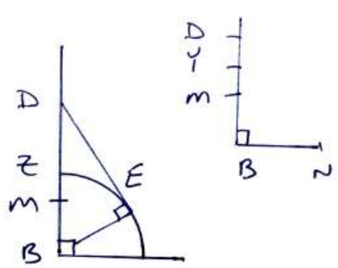


$BM_1 > BM_2$
and N_1M_1 crosses N_2M_2 at X within the right angle ∠DBA.



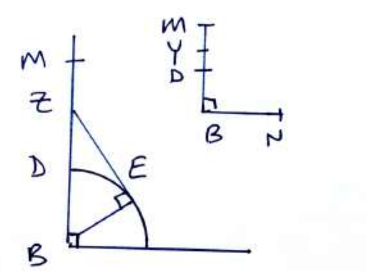
$BM_2 > BM_1$
and N_1M_1 crosses N_2M_2 at X outside the right angle ∠DBN_2.





$\mathbb{R}$ = DB/BZ = ND/NM

if: BY/MB= DB/DE
then: DB/YN = ED/EB
because:



$\mathbb{R}$ = BZ/DB= NM/ND

if: BY/DB= ZB/EZ
then: MB/YN= EZ/EB
because:

$MB^2 = MN^2 - BN^2$

$MB^2 = MN^2 - YN^2 + BY^2$

$BY^2/(MN^2 - YN^2 + BY^2)$
$= DB^2/(DB^2 - BZ^2)$
$= DN^2/(DN^2 - MN^2)$

$BY^2/(YN^2 - MN^2) = DN^2/MN^2$

$BY^2 = YN^2 - BN^2$

$BY^2/DB^2 = BZ^2/(BZ^2 - EB^2)$

$BY^2/(BY^2 - DB^2) = BZ^2/DB^2$

$BZ^2/DB^2 = MN^2/DN^2$

$BY^2/MN^2 = (BY^2 - DB^2)/DN^2$

$(BY^2 + MN^2)/MN^2$
$= (BY^2 - DB^2 + DN^2)/DN^2$

$BY^2 = YN^2 - DN^2 + DB^2$

$(YN^2 - DN^2 + DB^2)/(YN^2 - MN^2)$
$= DN^2/MN^2$

$(YN^2 + DB^2)/YN^2 = DN^2/MN^2$

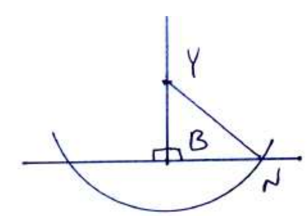
$DB^2/YN^2 = (DN^2 - MN^2)/MN^2$
$= (DB^2 - BZ^2)/DB^2 = ED^2/EB^2$

$(BY^2 + MN^2)/(BY^2 + BN^2)$
$= MN^2/DN^2$

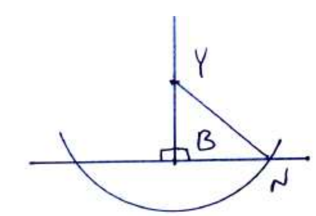
$(MN^2 - BN^2)/NY^2$
$= (MN^2 - DN^2)/DN^2$

$MB^2/YN^2 = (BZ^2 - DB^2)/DB^2$
$= EZ^2/EB^2$

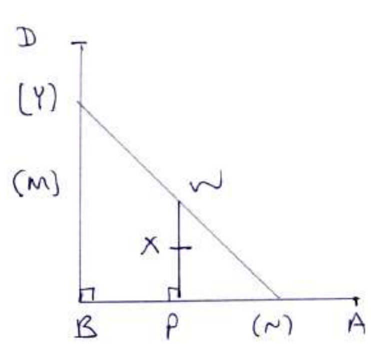
When given point M, after calculating BY with known BM, (as well as known DB/DE); we can use known DB, (as well as known ED/EB), to calculate YN and use that as a radius about Y to find N:



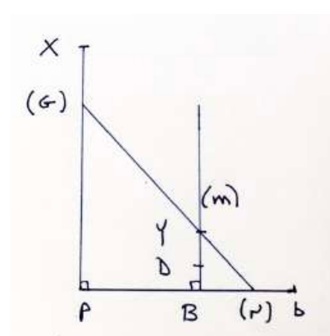
When given point M, after calculating BY with known DB, (as well as known ZB/ZE); we can use known MB, (as well as known EZ/EB), to calculate YN and use that as a radius about Y to find N:



Since M must be known to find N, this gives no advantage over the previously described reference ellipse. However, it provides a way to find N on image ray MX(N) without knowing M.



Since M must be known to find N, this gives no advantage over the previously described reference hyperbola arm. However, it provides a way to find N on image ray XM(N) without knowing M.



To find an image ray through a given point X, first calculate PW with known PX and DB/DE using:
$PW/PX = (BY/MB) = DB/DE$

Since DB and ED/EB are also known, find the length of YWN using:
$DB/YN = ED/EB$

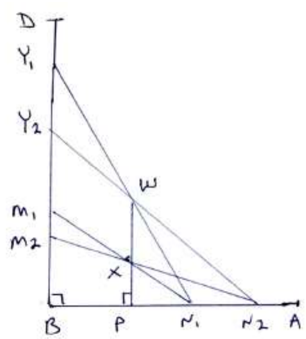
We can then find (N) by inserting the calculated length YWN within the right angle ∠DBA through W.

To find an image ray through a given point X, first calculate BY with known DB and ZB/ZE using:
$BY/DB = ZB/EZ$

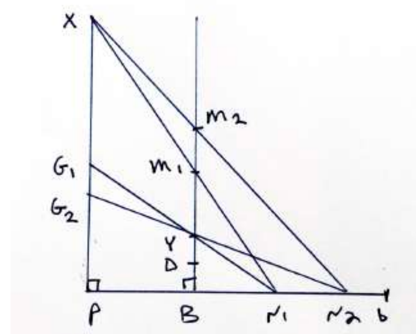
Since PX and EZ/EB are also known, find the length of GYN using:
$PX/GYN = (MB/YN) = EZ/EB$

We can then find (N) by inserting the calculated length GYN within the right angle ∠XPb through Y.

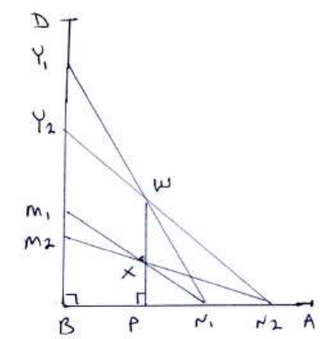
For any given calculated value of YN, a maximum of two line segments ($Y_1N_1 = Y_2N_2$) fit though W within the right angle ∠DBA.



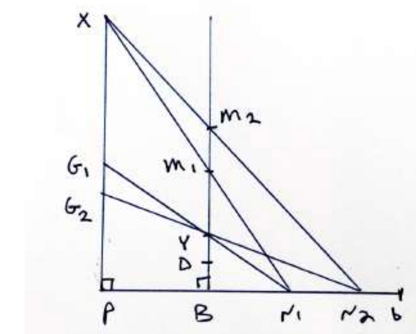
For any given calculated value of GN, a maximum of two line segments ($G_1N_1 = G_2N_2$) fit though Y within the right angle ∠XPb.



These two line segments are drawn to find both N_1 and N_2 for the image rays through X.



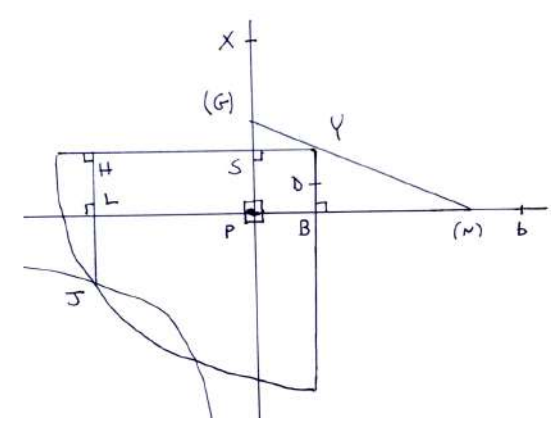
These two line segments are drawn to find both N_1 and N_2 for the image rays through X.



The clear image of X occurs when YN through its specified point W is its minimum possible length, so that N_1 lies at N_2. Since both BY/MB = PW/XP and DB/YN are constants, YN can be varied while keeping the image location XP constant, but not the object location DB.

The clear image of X occurs when GN through its specified point Y is its minimum possible length, so that N_1 lies at N_2. Since both MB/YN = XP/GN and BY/DB are constants, GN can be varied while keeping the object location DB constant, but not the image location XP.

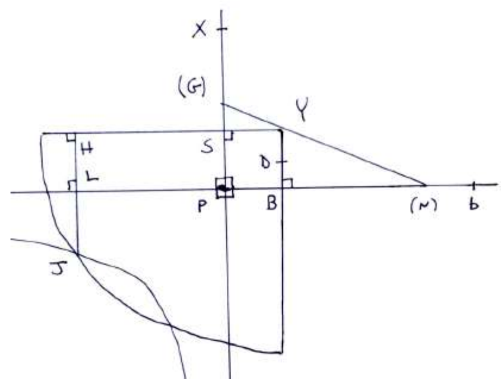
Expanding on the right side column representing the object in air, (where GN can be varied while keeping the object location DB constant, but not the image location XP), consider Y to be on a reference hyperbola defined by: (LP)LJ = (BP)BY, and draw its opposite arm:



We know LP/BY = BP/LJ.
If we construct BN = LP, then BN/BY = BP/LJ.
But SY/SG = BN/BY = BP/LJ
and since SY = BP:

SG = LJ
SG + SP = LJ + HL
PG = HJ

and since by
construction BN = LP:
PN = LB = HY
ΔNPG = ΔYHJ
GN = YJ



The reference radius length YJ intersects the reference hyperbola at a maximum of two possible points J_1 and J_2. Both G_1YN_1 and G_2YN_2 can be drawn by constructing BN = LP for each point J.

A clear image of object D occurs when N_1 and N_2 overlap, or when the reference radius length YJ = GN intersects the reference hyperbola at a single point J. The required GN for this condition gives the required location of N, as well as the location of the clear image at X, (remember that PX varies with GN).