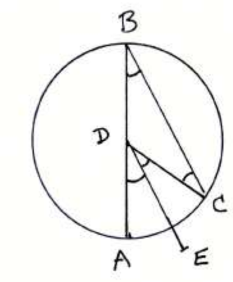
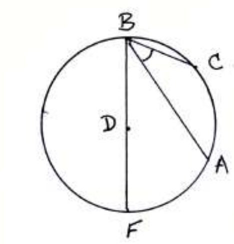
Axial Magnification

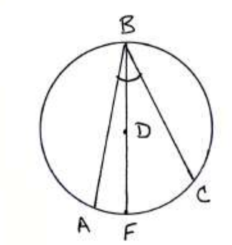
Gregg Baldwin, OD
2022

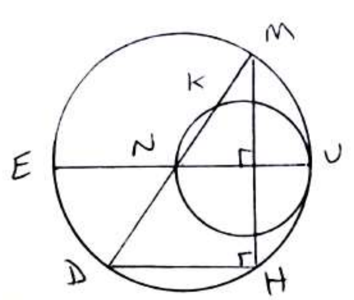


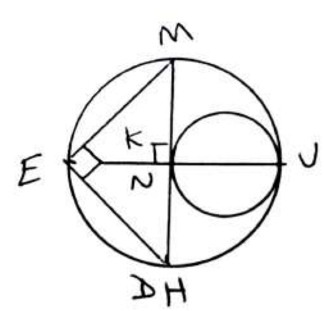
$\angle ADC = 2\angle ABC$



$\angle ADC = \angle FDC - \angle FDA$, or:
$\angle ADC = \angle FDC + \angle FDA$
$\angle ADC = 2(\angle FBC +/- \angle FBA)$
$\angle ADC = 2\angle ABC$
Therefore $\angle ABC$ can be defined as:
$\sim AC/BF$, or $1/2(\sim AC/DF)$.

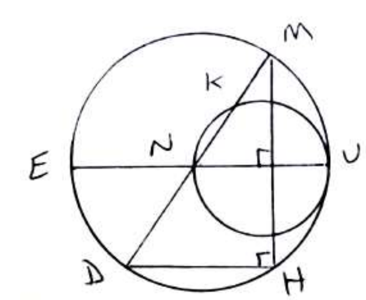


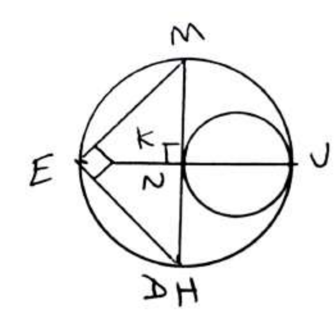




$2\sim KU/UN = 2\angle MNU = \angle MNH$
As $K \Rightarrow N$ and $D \Rightarrow H$, $2\sim KU/UN \Rightarrow \pi$
$\angle ABC + \angle BCA + \angle CAB = \sim CA/BF + \sim AB/BF + \sim BC/BF = \pi$

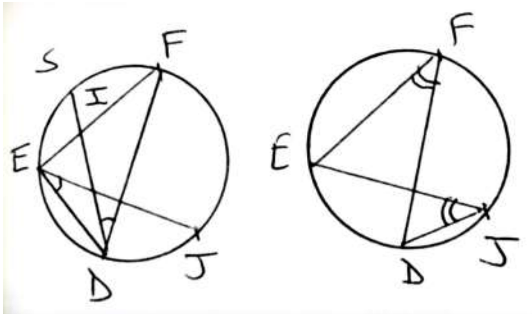
If two angles of two triangles are the same, their third angles are the same. They are consequently the same shape, (are $\cong$), with equal side ratios.

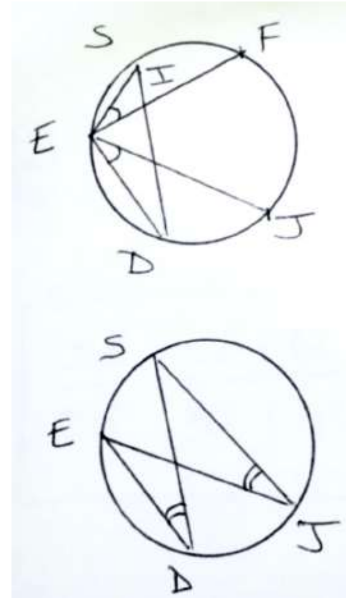




$\sim UK/UN = \sim MH/MD = 2\sim UM/UE = 2\sim UM/2UN$

$\sim UK = \sim UM$





SD || FJ

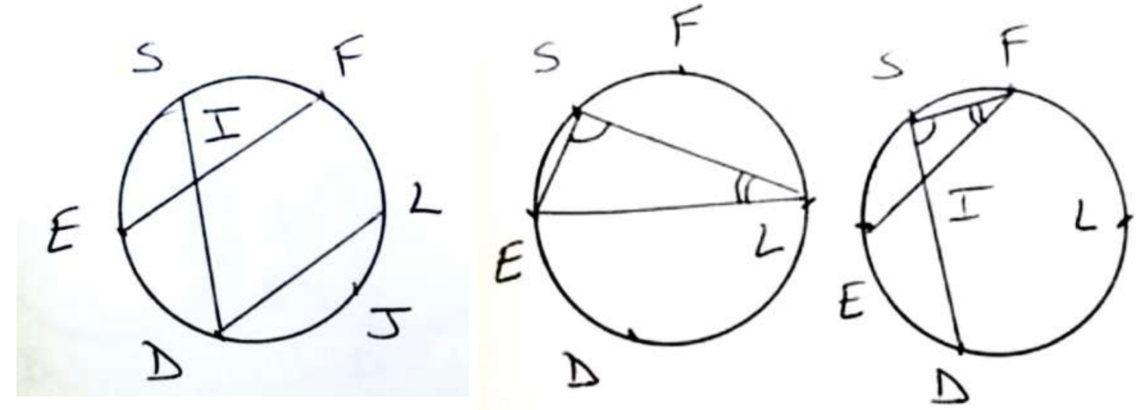
ΔEJD ≅ ΔDFI; FD/FI = JE/JD

ΔEJS ≅ ΔEDI; EI/ED = ES/EJ

[(FD)(EI)]/[(FI)(ED)]

= [(JE)(ES)]/[(JD)(EJ)] = SE/SF

IE/IF = [(SE)(DE)]/[(SF)(DF)]



LD || FE ; DE/DF = LF/LE; IE/IF = (SE)(LF)/(SF)(LE)

FE/FI = {(SE)(LF) + (SF)(LE)}/(SF)(LE)

LD || FE; ~EL = ~FD

ΔLSE ≅ ΔFSI; LS/FS = LE/FI; LS = {(FS)(LE)}/FI

Ptolemy's Theorem:

(FE)(LS) = (SE)(LF) + (SF)(LE)

NS/NC = NC/NB

NK/NC = CN/CK

ΔNSC = ΔKWB =

ΔKNP

NC = KP

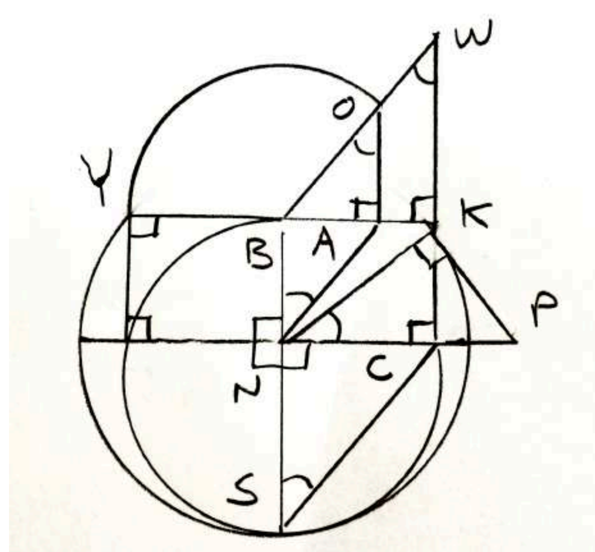
ΔCKP = ΔBNA = ΔAOB

NA = KP

NC = NA = OB

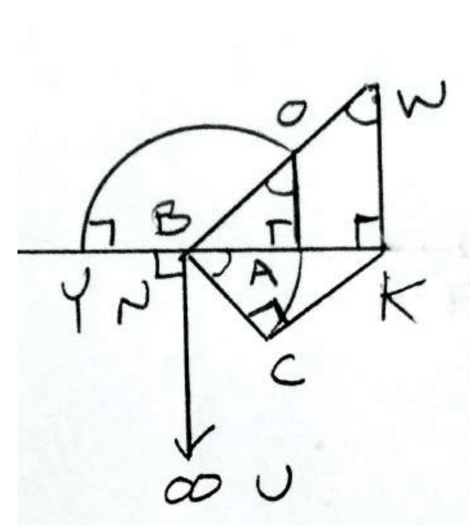
NC = KB = YB

WK = NS **=** YN



Keeping only:

NA = NC, and ΔCNK ≅ ΔAOB ≅ ΔKWB:



As N ⇒ B, WK ⇒ YN

because:

WK/OA ⇒ NK/NA

= NK/NC

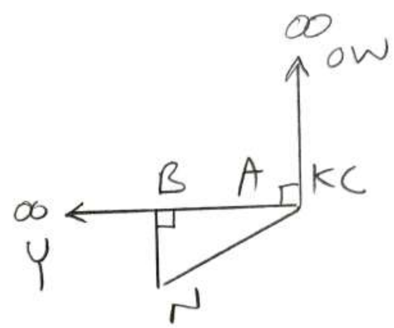
= OB/OA

= WB/WK

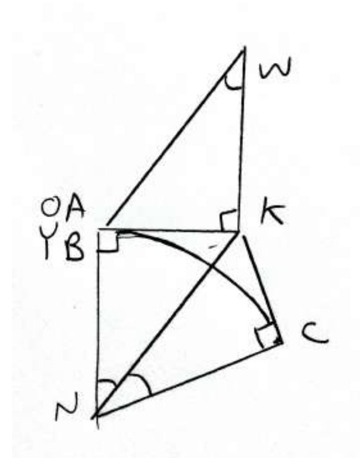
so that:

WK ⇒ OB ⇒ YN

Keeping only:
NA = NC, and $\Delta CNK \cong \Delta AOB \cong \Delta KWB$:

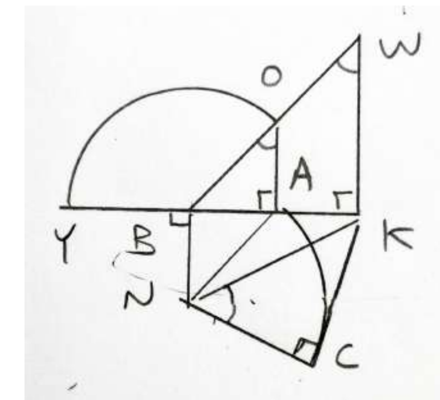


As A ⇒ K, WK ⇒ YN

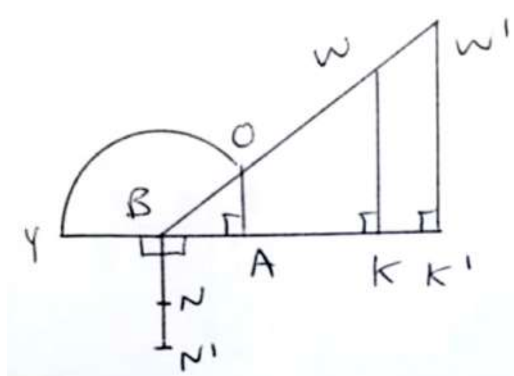


As A ⇒ B, WK ⇒ YN

Therefore,
whenever A lies on KB of right triangle ΔKBN;
if NA = NC, and
$\Delta CNK \cong \Delta AOB \cong \Delta KWB$,
then WK = YN



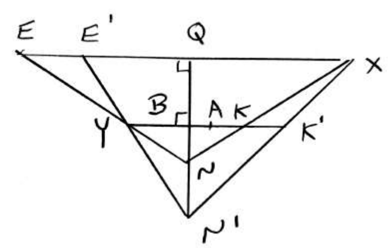
which also can be shown directly, using the equations:
$(CK/CN)^2 = (AB/AO)^2 = (KB/KW)^2 = (CK^2 + AB^2)/(CN^2 + AO^2)$
since: $KB^2 = KN^2 - BN^2 = KN^2 - (NC^2 - AB^2) = CK^2 + AB^2$
then: $WK^2 = CN^2 + AO^2$, which equals:
$AN^2 + AO^2 = BA^2 + BN^2 + BO^2 - BA^2 = YN^2$



OB/OA = NK/NA = N'K'/N'A

KW = YN
K'W' = YN'

KB/YN = K'B/YN'



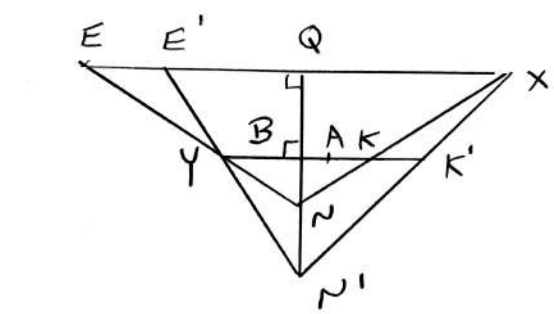
QX/EN = KB/YN
= K'B/YN' = QX/E'N'

EN = E'N'

Only one N'K'X exists for NKX since only one E'N' exists equal to EN.

When EN is changed to become the smallest segment through Y,
bound by the right angle EQN:

E' lies at E, and
N' lies at N.



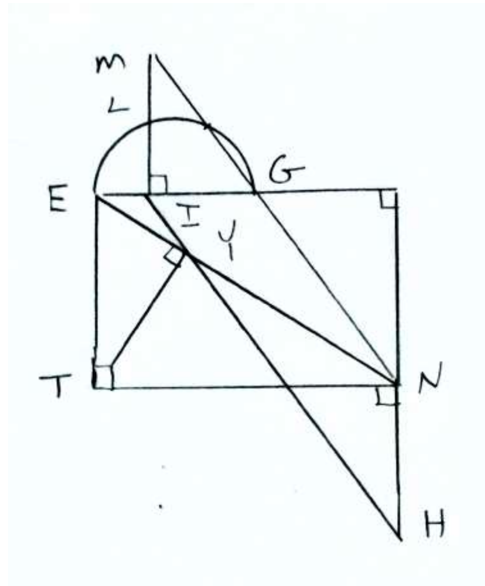
Also, QX varies with EN because:
QX/EN = KB/YN
= KB/KW, which is a constant.

To specify EN as the shortest hypotenuse through Y:

NE || GL
TY || EL
HI || NM
HI = NM > NL

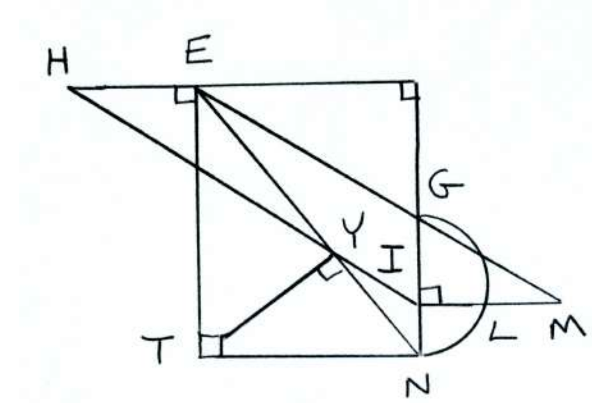
NL is the hypotenuse of right triangle NEL, so:

NL > NE
HI > NE



But also:

NE || GL
TY || NL
HI || EM
HI = EM > EL

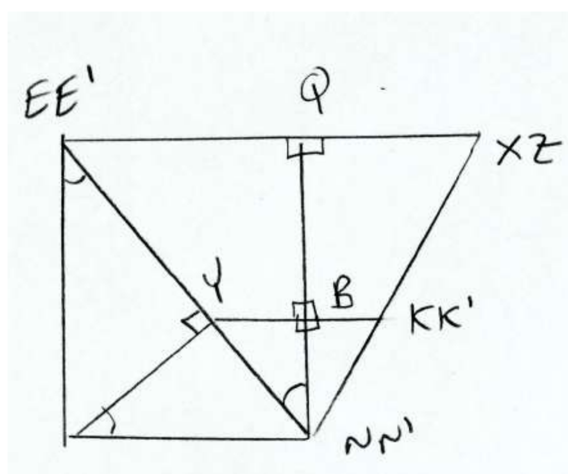


EL is the hypotenuse of right triangle ENL, so:

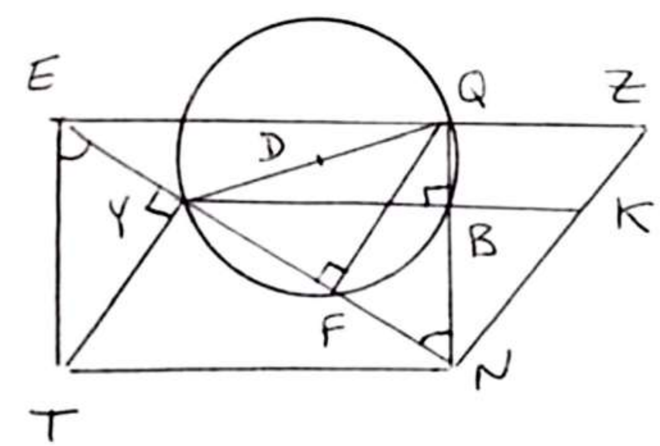
EL > EN
HI > EN

Let X = Z when EN is the shortest segment through Y included in right angle EQN.

In order to find Z given ΔYBN, we must find E = E’ using:

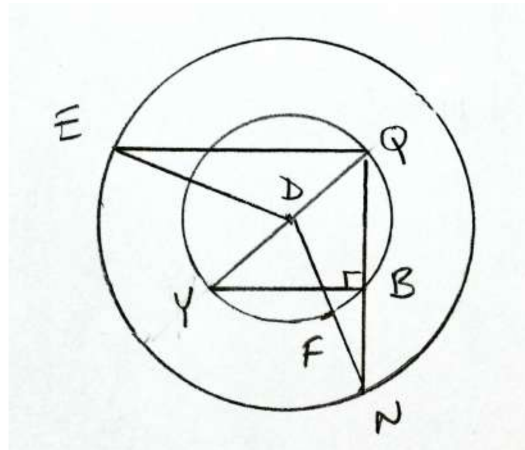


ΔYBN ≅ ΔNYT ≅ ΔNTE



In order to find Z given ΔYBQ, we must find EN = E’N’ by making ΔTYE a right triangle.

Draw a concentric circle around ⊙YBQ using its center at D, (the midpoint of hypotenuse YQ), containing an arc ~EN, so that YF lies on its chord EN. The arc intercepted by ∠DEN then equals that intercepted by ∠DNE.



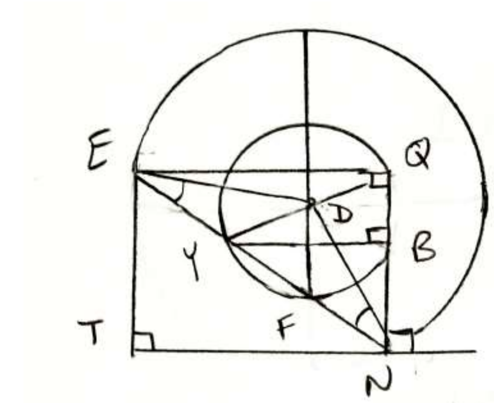
∠DEY = ∠DNF

DY = DF ; DE = DN

ΔEDY = ΔNDF

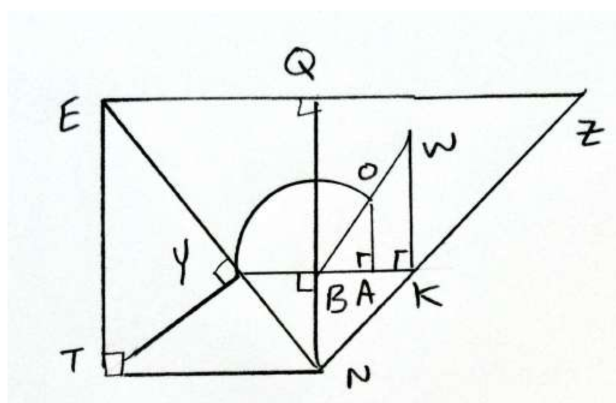
EY = NF

Since ΔQFN is a right triangle, so is ΔTYE.



WK = YN

Given   ΔBAO:



use ΔBNY to find ΔBKW and ΔQBY,

use ΔQBY or ΔBKW to find ΔBNY.

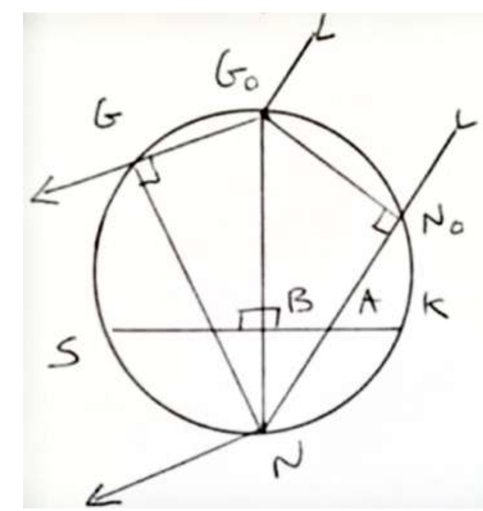
$\Delta N_oNK \cong \Delta KNA$

because:

~NS = ~NK

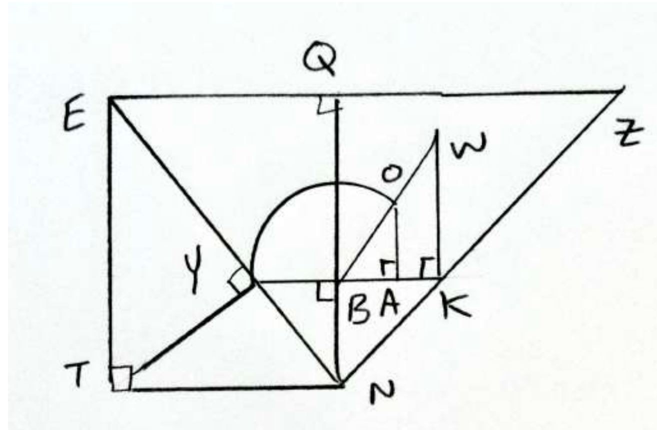
across diameter G_oN.

Wavefront G_oN_o refracts into wavefront GN along G_oN, since it travels G_oG in the same time it travels N_oN.



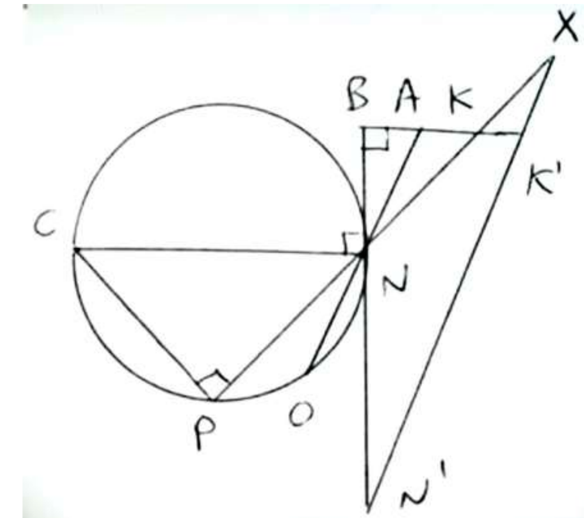
$\mathbb{R} = NN_o/GG_o = NN_o/NK = NK/NA$

Therefore, if
ℝ = OB/OA,
and WK = YN;
then,
ℝ = NK/NA



and Z is the clear image of object A refracted at N (= N’), along BN, because the two possible refracted rays through Z coincide at N.

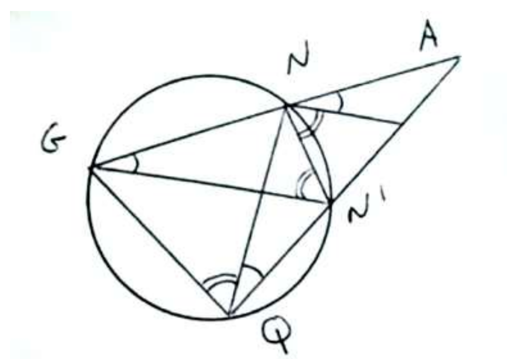
ΔKNA ≅ ΔOCP
ℝ = NK/NA
= N’K’/N’A
= CO/CP

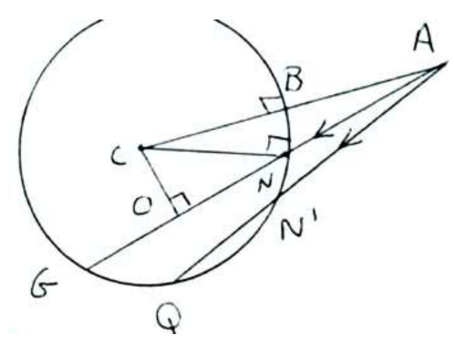


Real object A:

ΔANN’ ≅ ΔAQG
AG/AN’ = QG/NN’

(AG + AN’)/2AN’
= (QG + NN’)/2NN’

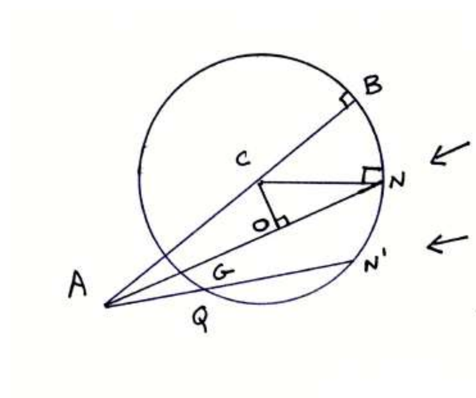




Virtual object A, which can not be projected on a screen due to refraction at BN:

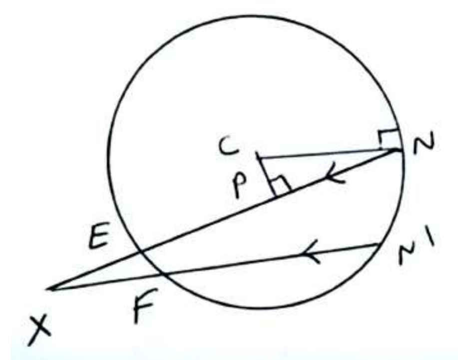
ΔANN’ ≅ ΔAQG
AG/AN’ = QG/NN’

(AG + AN’)/2AN’
= (QG + NN’)/2NN’



Real image at X,
(will be defined as clear
as N’ ⇒ N, and X ⇒ Z),

can be projected on a
screen:



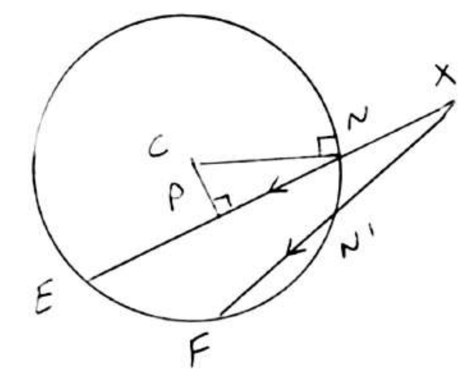
ΔXNN’ ≅ ΔXFE

XE/XN’ = EF/NN’

(XE + XN’)/2XN’
= (EF + NN’)/2NN’

Virtual image at X,
(will be defined as clear
as N’ ⇒ N, and X ⇒ Z),

can not be projected on
a screen:



ΔXNN’ ≅ ΔXFE

XE/XN’ = EF/NN’

(XE + XN’)/2XN’
= (EF + NN’)/2NN’

(AG + AN’)/2AN’ = (QG + NN’)/2NN’
(XE + XN’)/2XN’ = (EF + NN’)/2NN’

(QG + NN’)/(EF + NN’)
= [(AG + AN’)/2AN’][2XN’/(XE + XN’)]

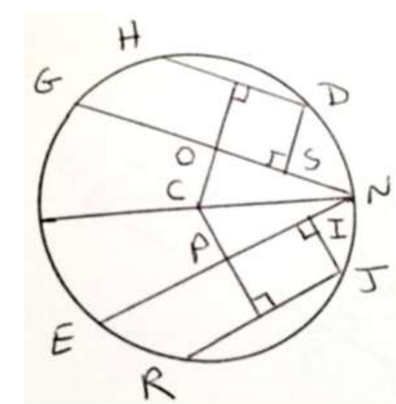
As N’ ⇒ N, X ⇒ Z, and:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ (QG + NN’)/(EF + NN’)

⇒ (AO/AN)(ZN/ZP)

Also, when HD = QN’
and RJ = FN’

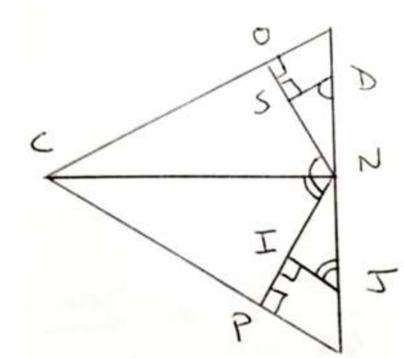
(~QG + ~NN’)/(~EF + ~NN’)
= 2(~ND)/2(~NJ) = ~ND/~NJ



As N’ ⇒ N, X ⇒ Z, and:

~DJ ⇒ line segment DJ, so:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ ND/NJ



DS/JI = CO/CP
JI/JN = NP/NC
DN/DS = NC/NO
ND/NJ = (NP/NO)(CO/CP)

As N’ ⇒ N, X ⇒ Z, and:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ (NP/NO)(CO/CP)

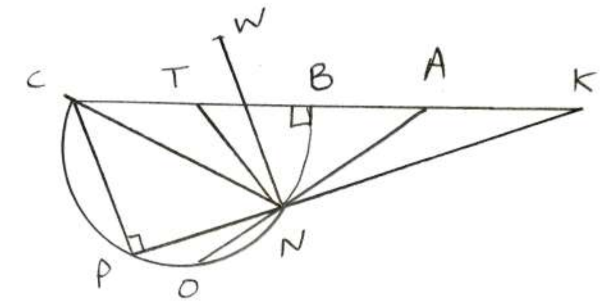
and therefore:
(AO/AN)(ZN/ZP) ⇒ (NP/NO)(CO/CP)

Thus ℝ = CO/CP, and Z, (along both NP and CW), is the clear image of A refracted along ~BN, when:

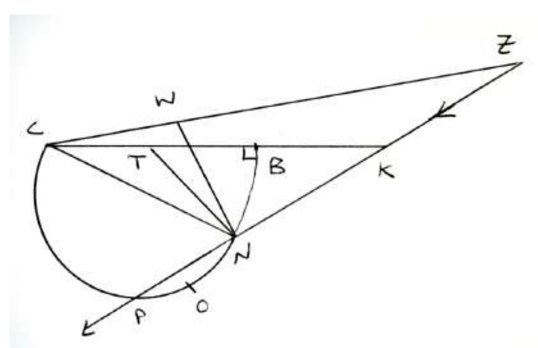
NT||CO, so:
AO/AN = CO/NT and:

NW||CP, so:
ZN/ZP = NW/CP
and:

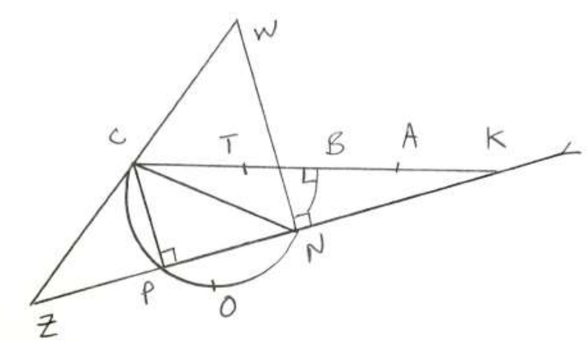
NW/NT = NP/NO
(ΔWNT ≅ ΔPNO)

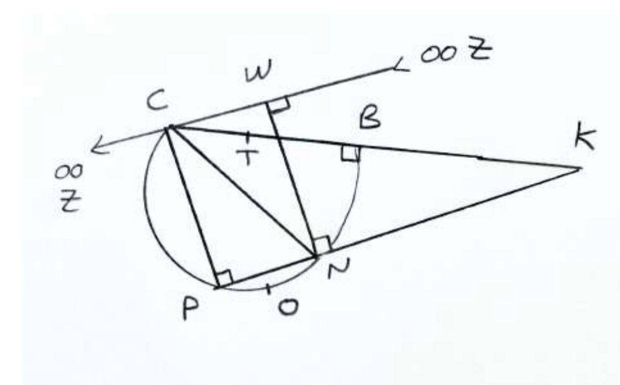


The off-axis rays from any on-axis object A, (real or virtual), can not form a virtual on-axis image at Z because NW must be less than CP for Z to be virtual; but NW must also be greater than NT.

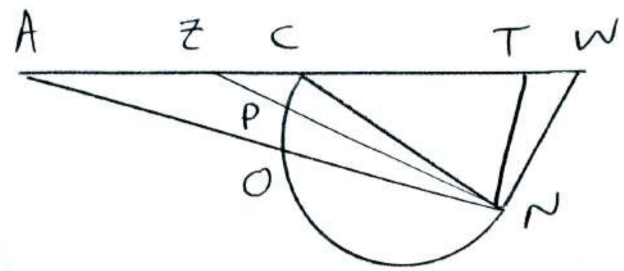


The off-axis rays from any real on-axis object A can not form a real on-axis image at Z because NW must be greater than (or equal to) CP for Z to be real; but NW must also be greater than NT.





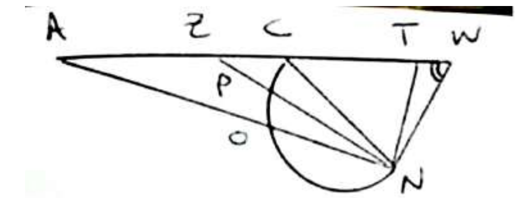
The off-axis rays from a virtual on-axis object A *can* form a real on-axis image at Z, if NW is greater than CP, and WT lies along the axis.

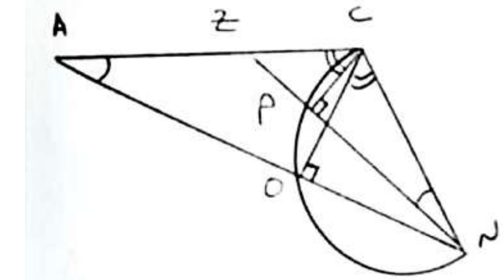


Since:
∠NWT = ∠NPO = ∠NCO
and NW || CP

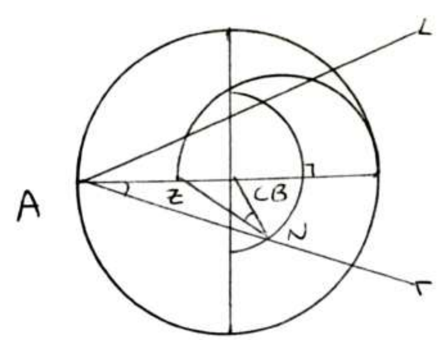
WT lies along the axis when:

ΔNCO ≅ ΔZCP





When off-axis rays from a virtual on-axis object A form a real on-axis image Z, this occurs at all points N because:

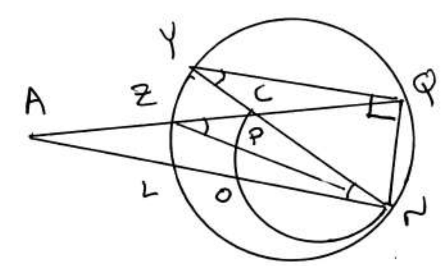


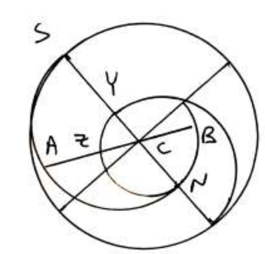
ΔACN ≅ ΔNCZ for all N,
(since they share proportional sides around a common angle).

This can also be demonstrated using similar right triangles:
ΔSAN ≅ CON, and ΔYZN ≅ ΔCPN,
so that: (AO/AN)(ZN/ZP) = (SC/SN)(YN/YC).

Since: CY/CN = CN/CS = (CY + CN)/(CN + CS) = NY/NS
(SC/SN) = (NC/NY), and:

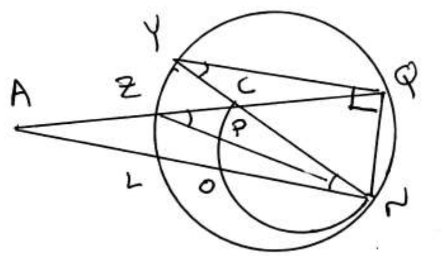
(AO/AN)(ZN/ZP) = CN/CY

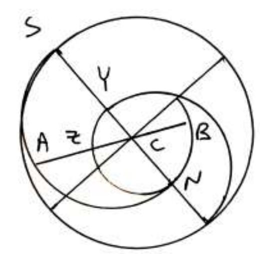




But it is also true that:
(CO/CP)(NP/NO) = CN/CY, because:

(CO/CP)(NP/NO) = (LY/LN)(PN/PC) =
= (QN/QY)(PN/PC) = (QN/QY)(ZN/ZY) =
QN (ZN)/QY(ZY) which, by the property of cyclic quadrilaterals discussed in slide #5, equals CN/CY





Keeping:

ℝ = (CO/CP) = (NO/NP)(AO/AN)(ZN/ZP)

constant, as N ⇒ B:

(BC/BC)(AC/AB)(ZB/ZC) ⇒ ℝ

Refraction through a circle’s center occurs when N lies at B, so that an object’s ray from A to N lies along ABC, and an image ray lies along BCZ. The locations of the object A and image Z along the optic axis BC are described by the equation:

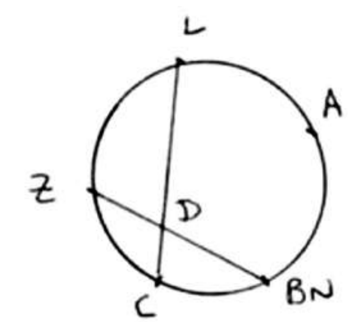
ℝ = CO/CP = (AC/AB)(ZB/ZC)

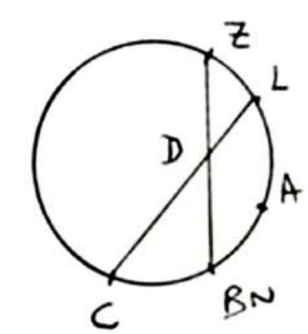
If we draw A and Z along the optic axis BC *as if* it were a circle, and draw CDL so that AL || ZB:

ΔACB ≅ ΔZCD, and:

(AC/AB)(ZB/ZC) =
(ZC/ZD)(ZB/ZC) =
(ZB/ZD)
so as the reference circle’s radius ⇒ ∞,

(ZB/ZD) ⇒ ℝ



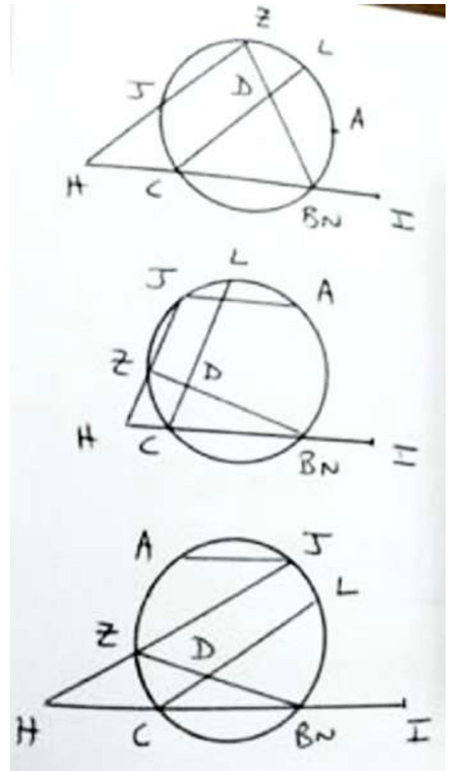


AL ll ZB
AZ = BL
~AZ = ~BL

HZ ll CL
ZC = LJ
~ZC = ~LJ

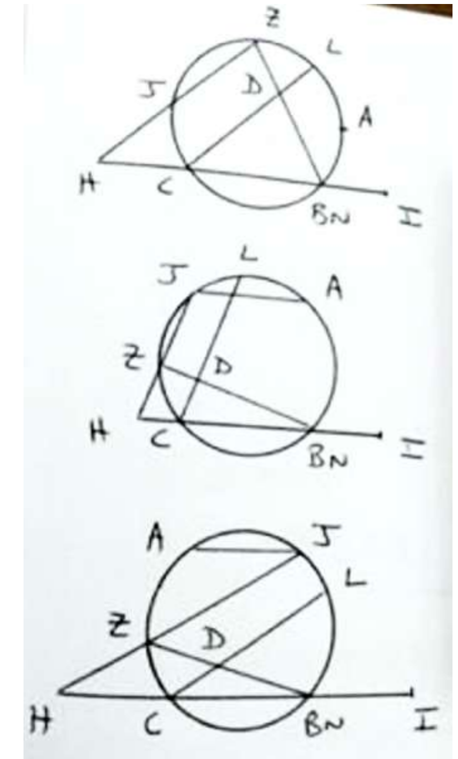
~AZ + ~ZC = ~AZC
~BL + ~LJ = ~BLJ

~AZC = ~BLJ
AJ ll CB

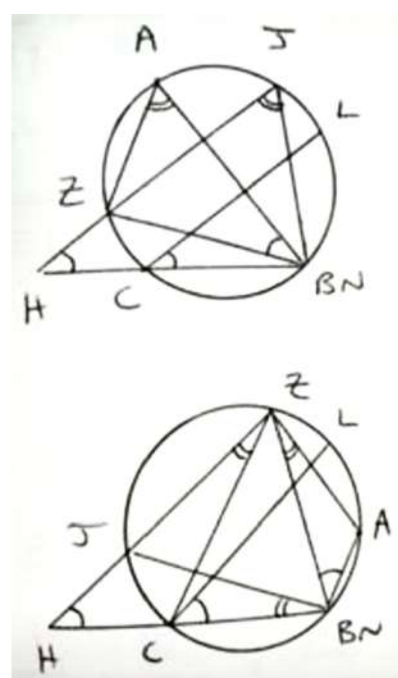


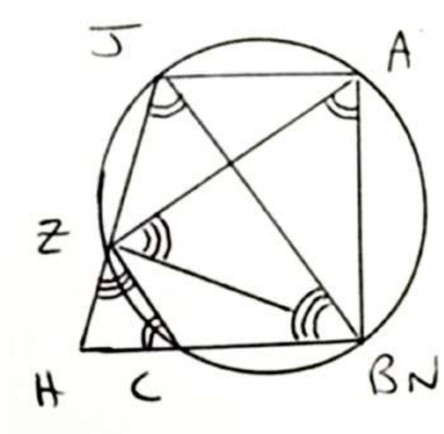
HZ ll CL
ZB/ZD = HB/HC
ΔHBZ ≅ ΔHJC
when ΔHJC = ΔIAB:
HC = IB, and:
IB/IA = HZ/HB

This results in
Newton’s Equation:
as the reference circle
radius ⇒ ∞,
(AI)(ZH) = (BI)(BH)



ΔHCZ ≅ ΔHJB ≅ ΔBAZ
(HC/HZ) = (BA/BZ)
[1/(HZ)(BA)] = [1/(HC)(BZ)]





as the reference circle’s radius ⇒ ∞,

[1/(HZ)(BA)] = [1/(HC)(BZ)] ⇒ ℝ/(HB)(BZ)

and the resulting possible sums occur:

HZ = HB + BZ
HB = HZ + BZ
BZ = HZ + HB

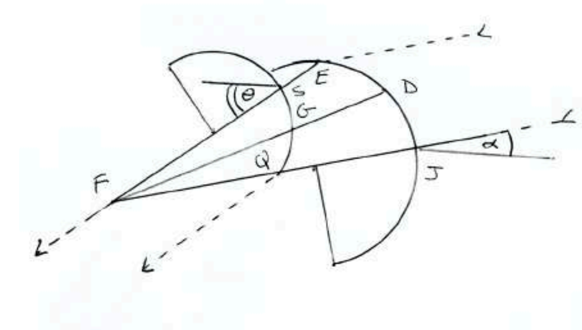
which, when multiplied by the above three factors, form the conjugate foci equations.

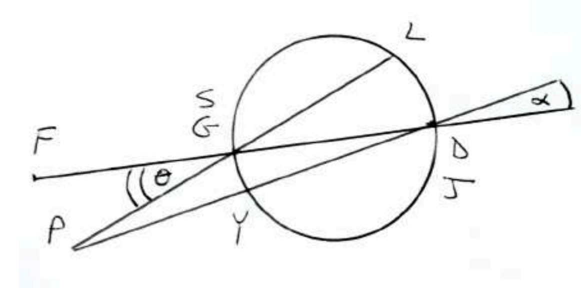
The conjugate foci equations allow for the effect of axial refraction at a circle to be expressed as the term:

(1/HC) = (ℝ/HB)

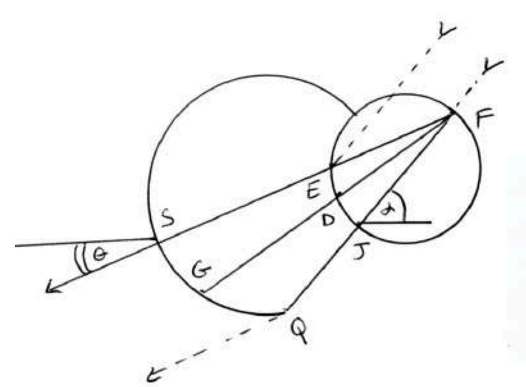
which is then additive with object vergence, defined as (1/BA); or image vergence, defined as (ℝ/BZ).

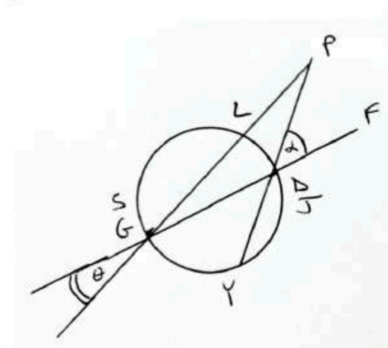
When distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis DGF as shown;
as ∠JFD = ∠SFG,
and both approach zero:





Or when distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis FDG, as shown;
as ∠JFD = ∠SFG,
and both approach zero:





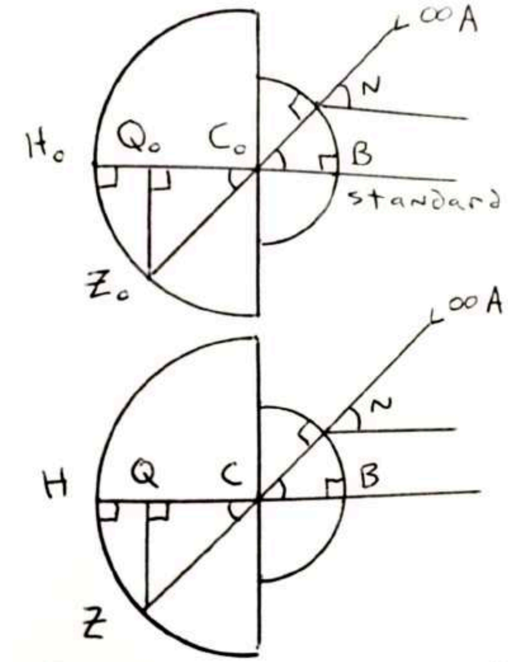
θ/α ⇒ (~LD/GD)/(~YG/GD) as P ⇒ F

θ/α ⇒ (FD/FG) as P ⇒ F

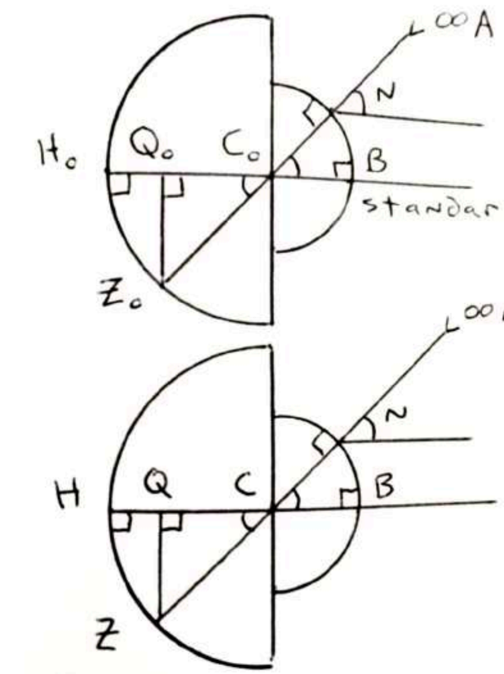
so that afocal axial angular magnification/minification equals:

FD/FG

The top diagram illustrates a standard single-surfaced eye with a distant object A, and resulting retinal image size H_oZ_o.



The bottom diagram illustrates any single-surfaced eye with a distant object A, and resulting retinal image size HZ.



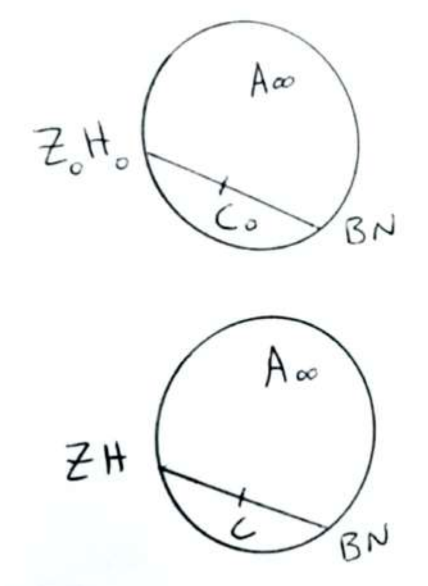
As N ⇒ B, the retinal image size

magnification, ZH/Z_oH_o, (relative to an arbitrary standard which factors out with subsequent comparisons), then approaches its axial value:

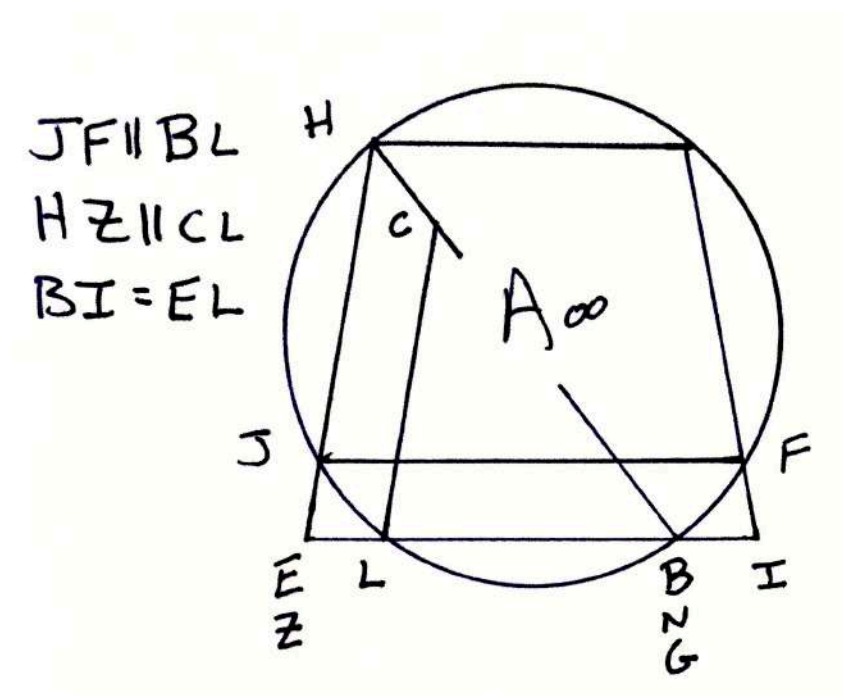
$$ZQ/Z_oQ_o = ZC/Z_oC_o = HC/H_oC_o$$

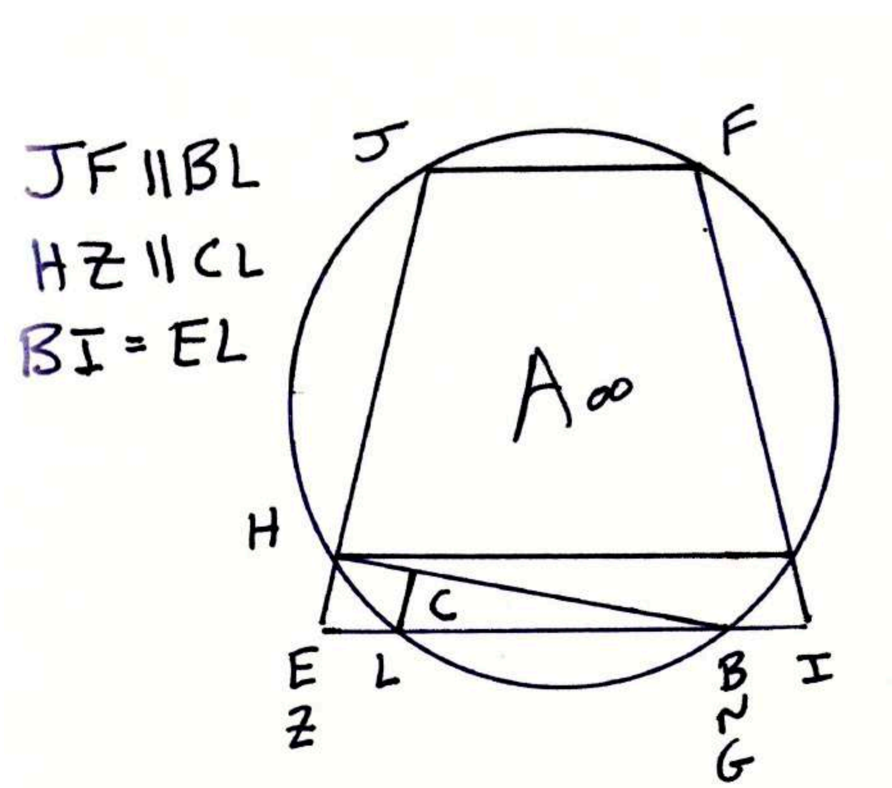
$$= (BH/\mathbb{R})/(BH_o/\mathbb{R}) = BH/BH_o$$

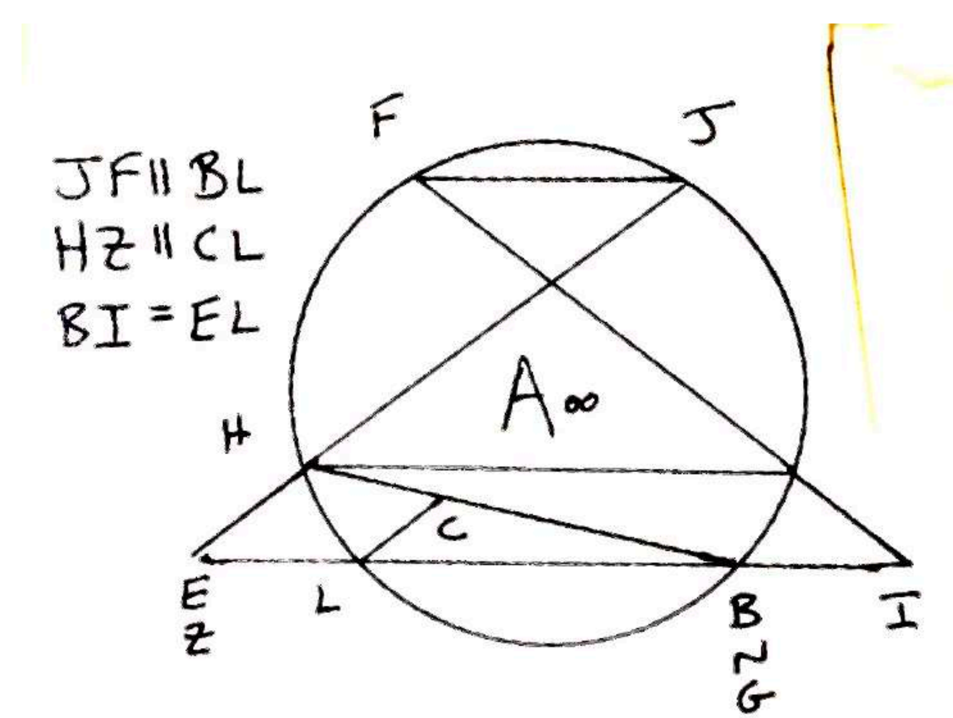
Once again representing the optic axis BCZ as a circle of infinite radius, the distant object A at ∞ is focused by the radius BC of the presumed single refracting surface towards the axial image Z, which lies at the retina H when there is no distance refractive error. (BH_o represents the standard axial length, and BC_o represents the standard single refracting curvature radius).



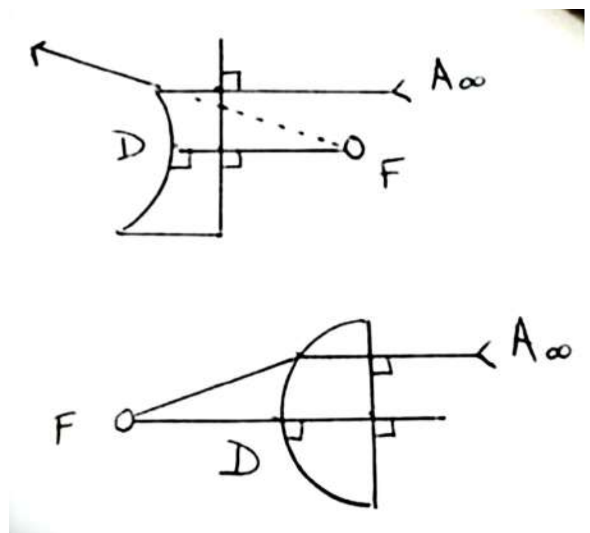
As pictured in the next three slides, additional refraction G (at B) will create an "ametropic" eye, with "distance refractive error," and a combination curvature effect with total radius BL instead of BC, moving image Z from the retina at H to its erroneous location at E. The "front focal point" of the "ametropic" eye is defined as point I.
A "distance correction" must focus the distant object towards F, so that JF || BL, in order to move image Z back to the retina H.







The distance correction at D:



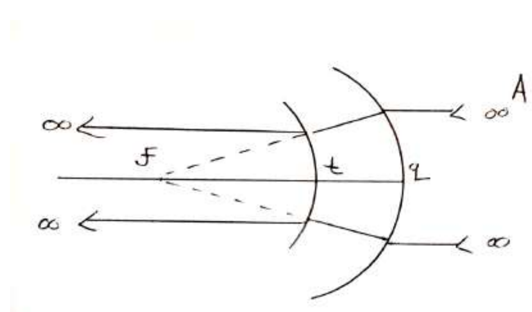
Since the distance correction D moves image Z from E to retina H, rays leaving the refractive error G (at B) after this correction is in place must be afocal. This results in afocal axial angular magnification equaling:

FD/FG (= FD/FB)

Therefore, the total axial magnification of distance correction equals:

$M = (BH/BH_o)(FD/FB)$

When the front surface of a spectacle lens that corrects distance refractive error is not flat, it is convex; and adds an additional “shape” factor, (fq/ft), to the afocal axial magnification of distance correction. (Point “t” lies at D, and FD/FB remains the “power” factor of the afocal axial magnification of distance correction).



“Axial Ametropia” occurs when E is at H_o, (and point I is therefore at I_o, the front focal point of the standard eye). The distance refractive error is then completely due to an axial length BZ, (or BH), that is not standard.

$\Delta H_oBH = \Delta EBH \cong \Delta EJL = \Delta I_oFB$

$(BH/BH_o) = (FB/FI_o)$

$M = (FB/FI_o)(FD/FB) = FD/FI_o$

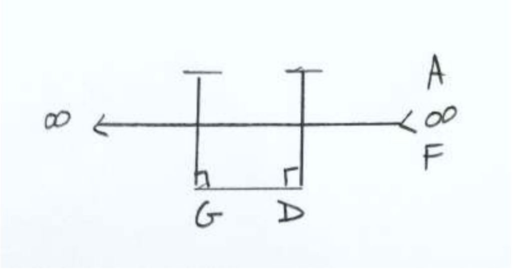
Therefore, in the case of axial ametropia, there is no total axial magnification of distance correction if the correction D lies at I_o.

"Refractive Ametropia" occurs when the retina H is at at H_o. The distance refractive error at G moving image Z to E is then completely due to a refracting radius BL that is not the standard BC_o.

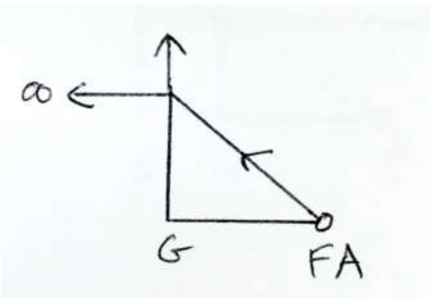
When the distance correction D lies at B:

$M = (BH/BH_o)(FD/FB) = 1$

There is no afocal axial angular magnification of distance correction with a distant object "A," and an emetropic eye whose refractive error at G (at B) is by definition zero, (with its focal point F at infinity).

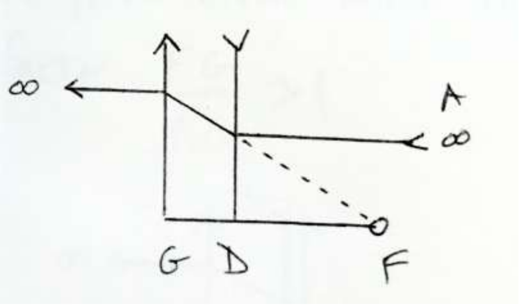


There is also no afocal axial angular magnification when object A is at the front focal point F of an uncorrected ametropic eye as shown, since this "myopic" system is not afocal, and involves only one refracting element G.



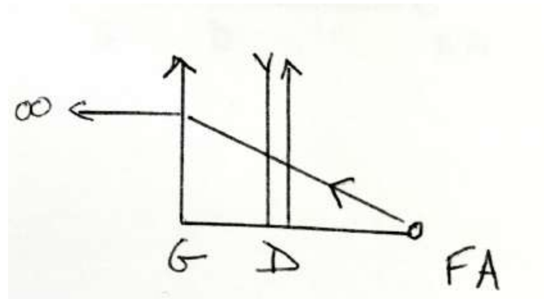
A distance myopic correction at D creates afocal axial angular minification:

$FD/FG < 1$



and this is relative to either the myopic eye with object A at its front focal point F, or the emetropic eye with object A at distance.

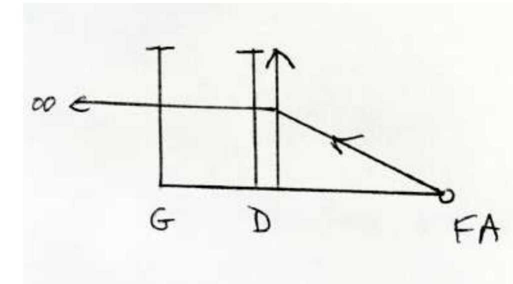
Removing the myopic distance correction at D with a converging lens at D removes this afocal axial angular magnification with the factor:



FG/FD > 1

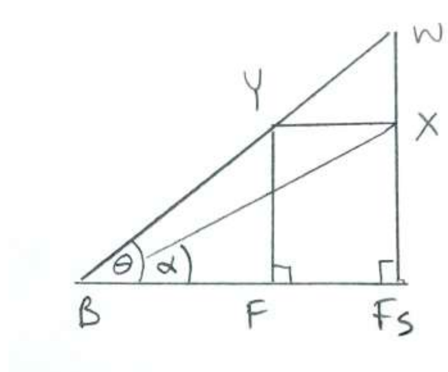
and this magnification of near correction is relative to the distance corrected myope.

If additional converging power is added to the converging lens so that the near focal point is in focus for an emetropic eye, which we then consider to be the reference eye, the magnification of near correction is still that which is removed with the factor:

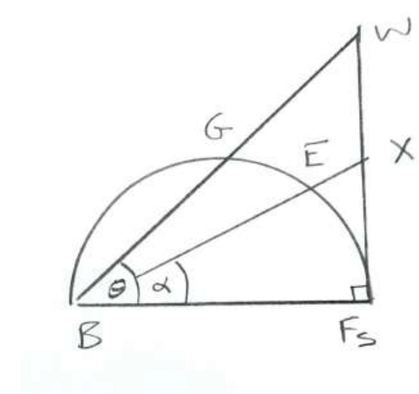


FG/FD > 1

When an object at a standard distance Fs is moved to F:



The object angular subtense magnification equals:



$\theta/\alpha = (\sim GFs/BFs)/(\sim EFs/BFs)$

as XFs ⇒ 0

the object angular subtense magnification approaches its axial value:

$\theta/\alpha \Rightarrow WFs/XFs = WFs/YF = BFs/BF$

which equals the axial
object angular subtense magnification.

The ratio describing axial object angular subtense magnification:

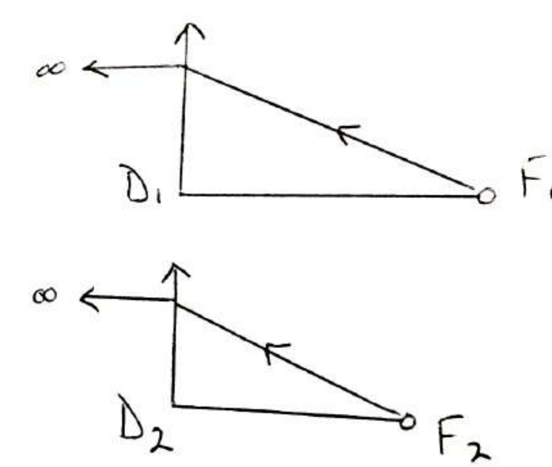
BFs/BF

when multiplied by the ratio describing near magnification due to a single converging lens producing parallel light for an emmetropic eye:

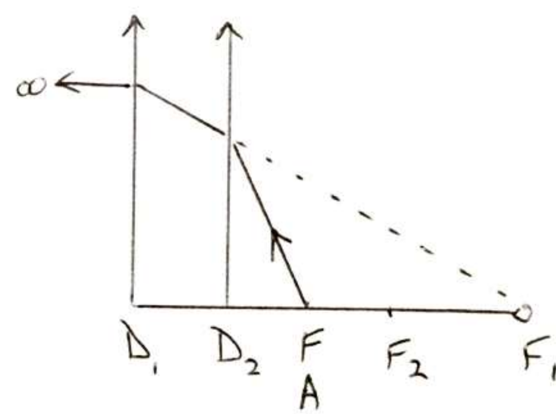
FB/FD

produces a ratio which factors out the object's actual distance to the eye, confirming that when a converging lens is used with its front focal point at the object, so that parallel light leaves the converging lens from the object, the image size is the same regardless of the object-to-eye distance.

When the converging lens at D is split into two converging lenses:



with the same combined focus F:



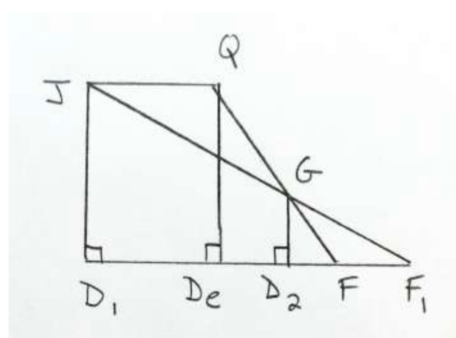
the ratio describing axial near magnification due to a single converging lens producing parallel light for an emmetropic eye:

FB/FD

must be expressed ***as if*** all convergence occurred at a single unknown axial point De:

FB/FDe

De can be located using triangles.



$D_2G/D_2F = DeQ/DeF$

$D_2G/D_2F_1 = D_1J/D_1F_1$

$D_2F(DeQ/DeF) = D_2F_1(D_1J/D_1F_1)$

$DeQ/DeF = (D_2F_1/D_2F)(D_1J/D_1F_1)$

$1/DeF = (D_2F_1/D_2F)(1/D_1F_1)$

$FB/FDe = (D_2F_1/D_2F)(FB/D_1F_1)$

Multiplying the axial object subtense magnification by the axial magnification of near correction (relative to the same eye without refractive error) produces:

$BFs/FDe = (D_2F_1/D_2F)(BFs/D_1F_1)$

The converging lens D_2 creates a virtual image F_1 of an object at F. When considering a stand magnifier with lens D_2, constant stand height D_2F, and reading spectacle add or ocular accommodation D_1, the stand magnifier's (constant) enlargement of the object at F equals:

$$E = D_2F_1/D_2F$$

The stand magnifier's axial magnification is its (constant) enlargement factor E, multiplied by what would be produced by D_1 alone, if the object A were at F_1.