Tangential Refraction at a Plane Surface

Gregg Baldwin, OD
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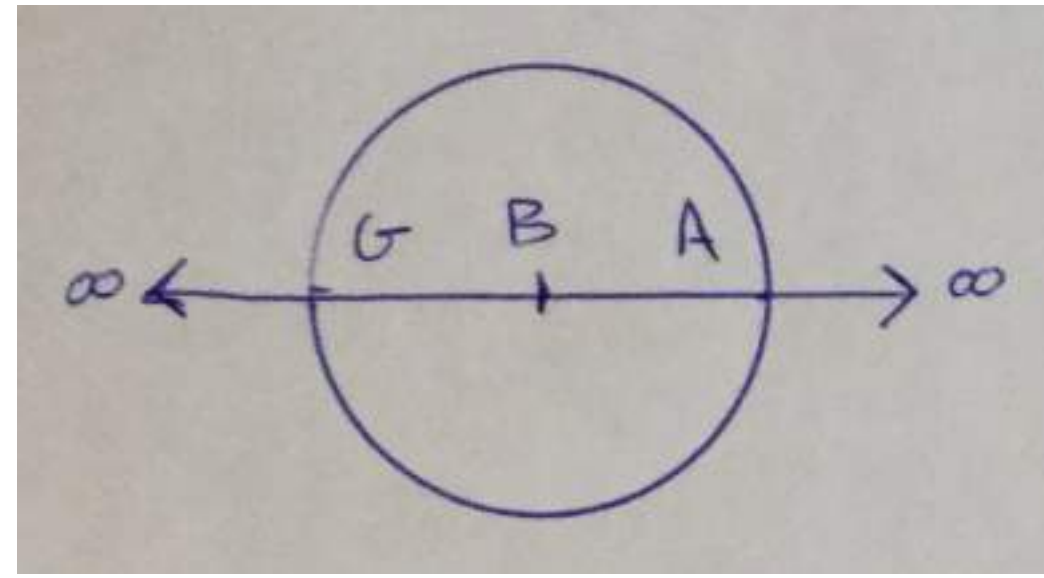
Introduction

Continuity of Conic Sections /Plane Geometry

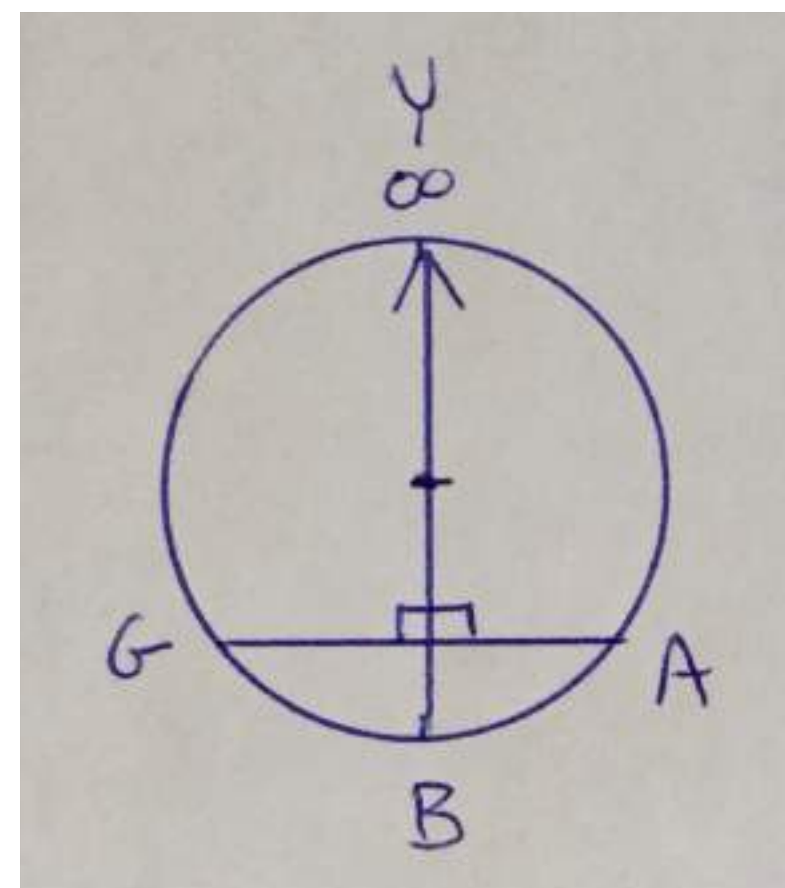
Although solid geometry can be used to visualize the continuity of conic sections, this can be also visualized with plane geometry.

Circle

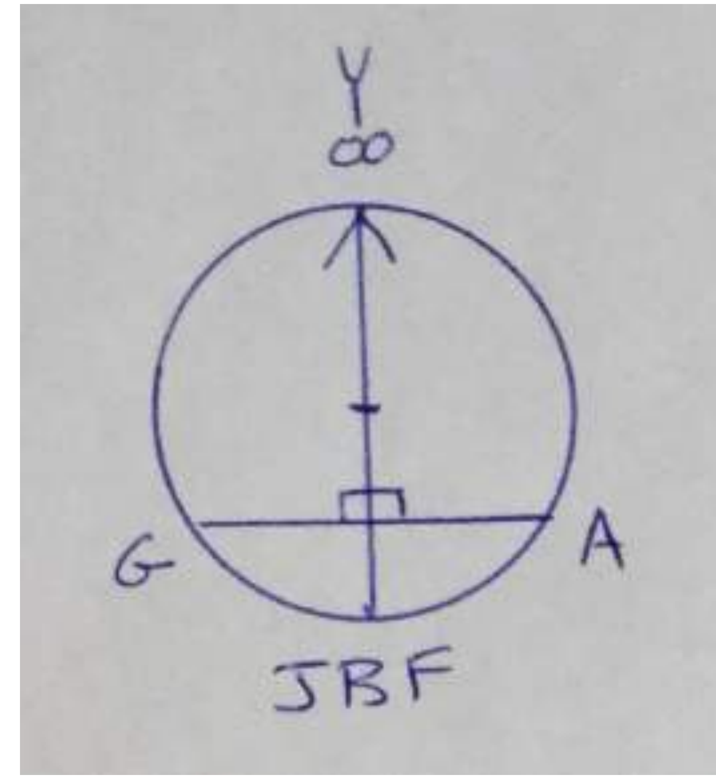
If we consider a circle with center B and diameter GBA with an “axis” infinitely long through GBA:



We can represent GBA along a circle of infinite diameter with BY, and draw BG = BA. This infinitely large reference circle is equally divided along ray BY, with Y at infinity.

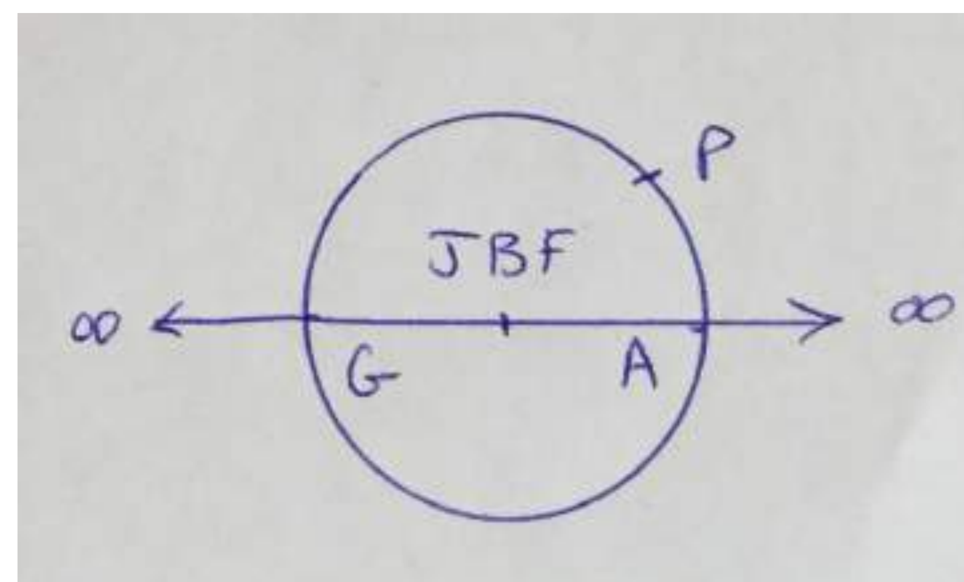


If we call points J & F, (both of which in this case lie at B), the "focal points" of the finite circle, we can consider the shape of the finite circle with diameter GBA to equal its "eccentricity" = e = BF/BA = 0.



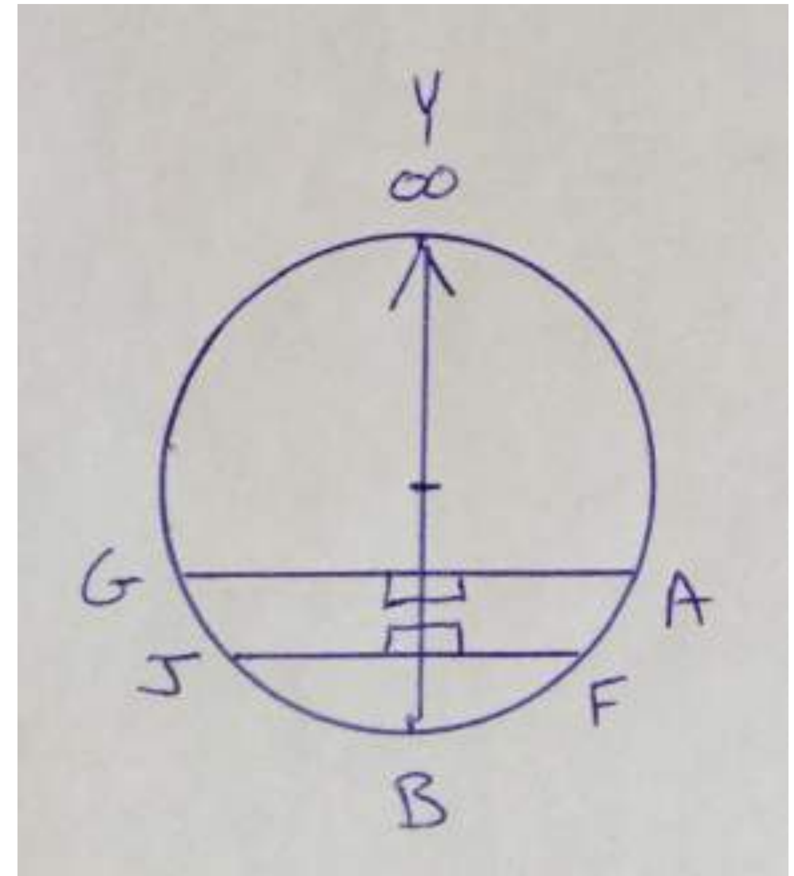
We will have drawn a defined circle where AJ + AF = AG along its diameter GJBFA, if it is also true that:

PJ + PF = AG

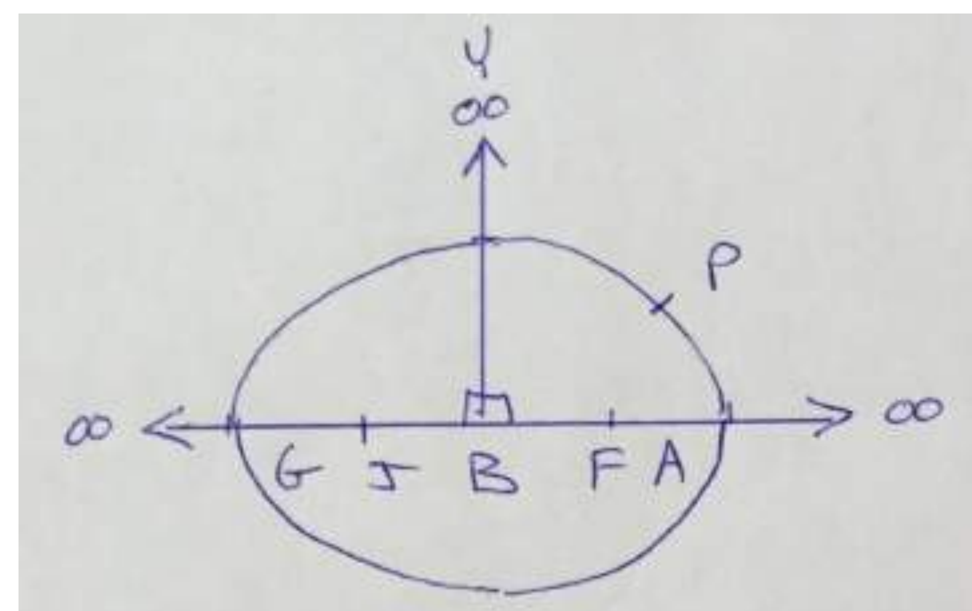


Ellipse

Draw: $0 < BF = BJ < \infty$
so that: $0 < e = BF/BA < 1$

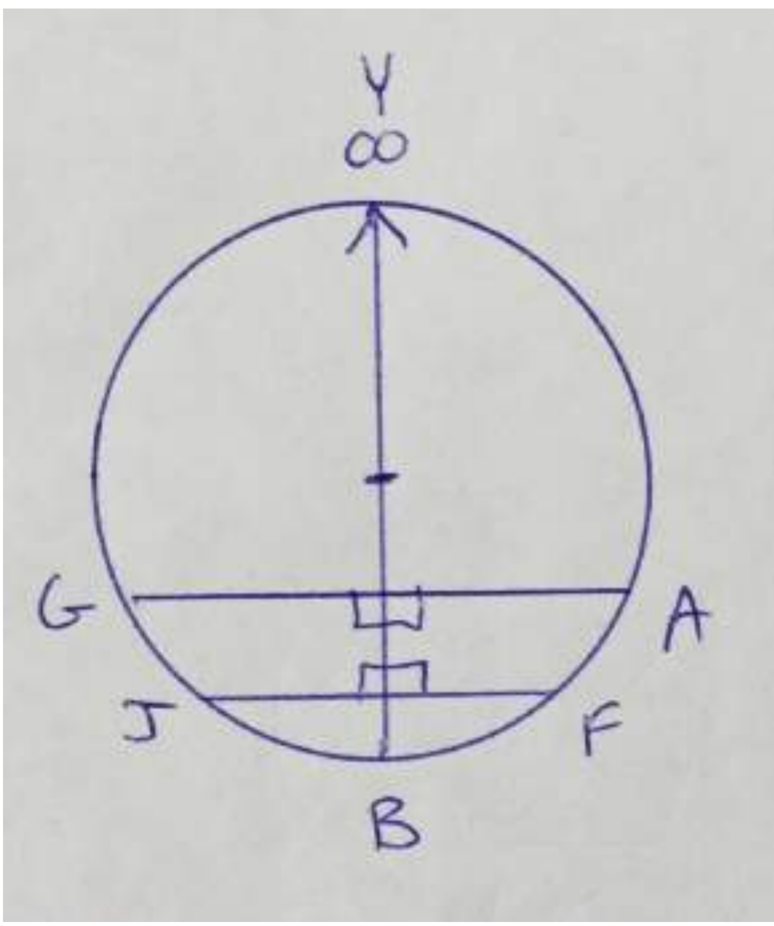


We will have drawn a defined ellipse where AJ + AF = AG along its "major axis" GJBFA, if it is also true that PJ + PF = AG.

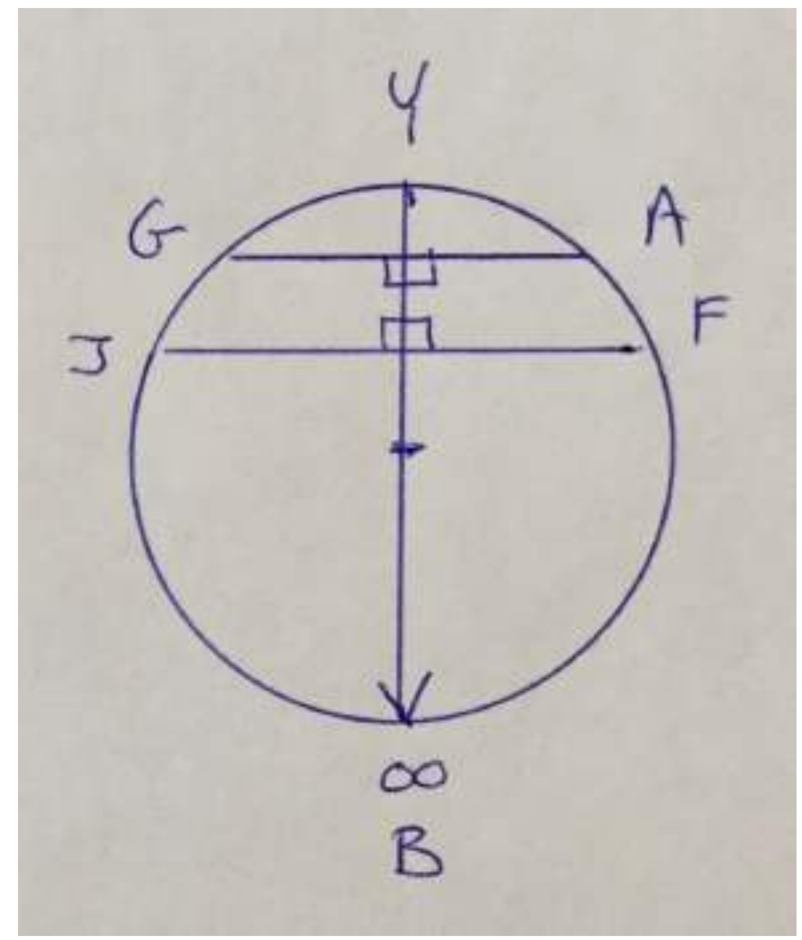


Hyperbola

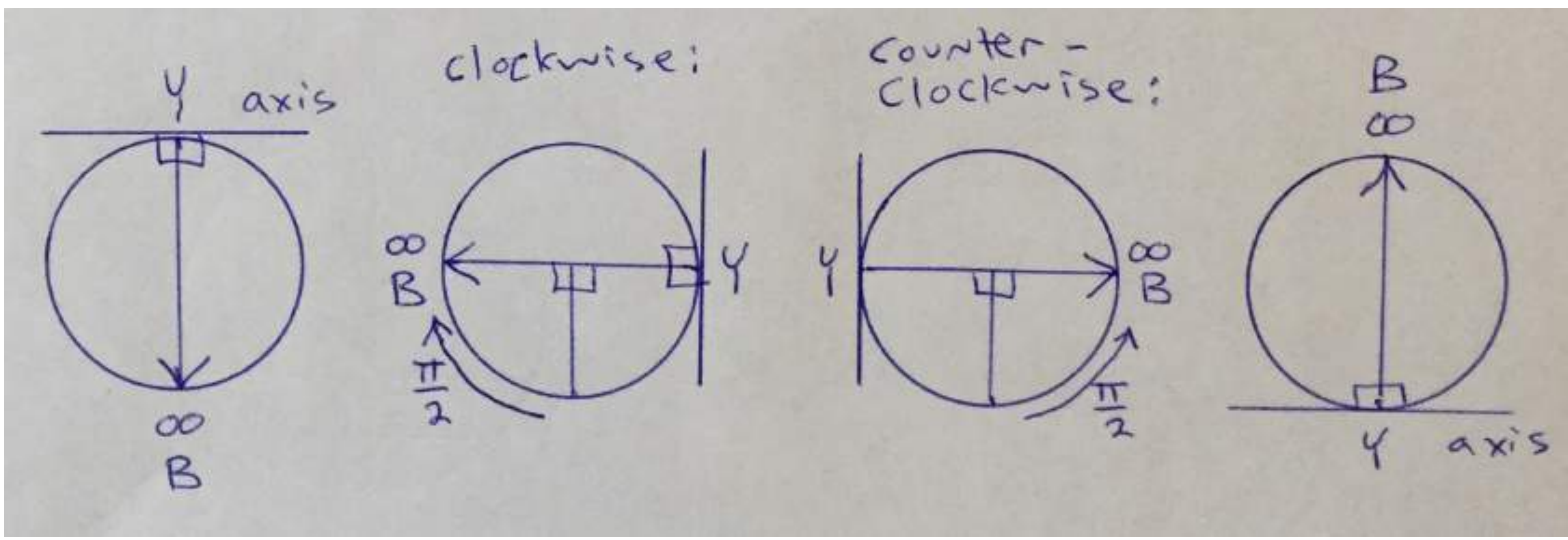
As B $\Rightarrow \infty$



becomes:

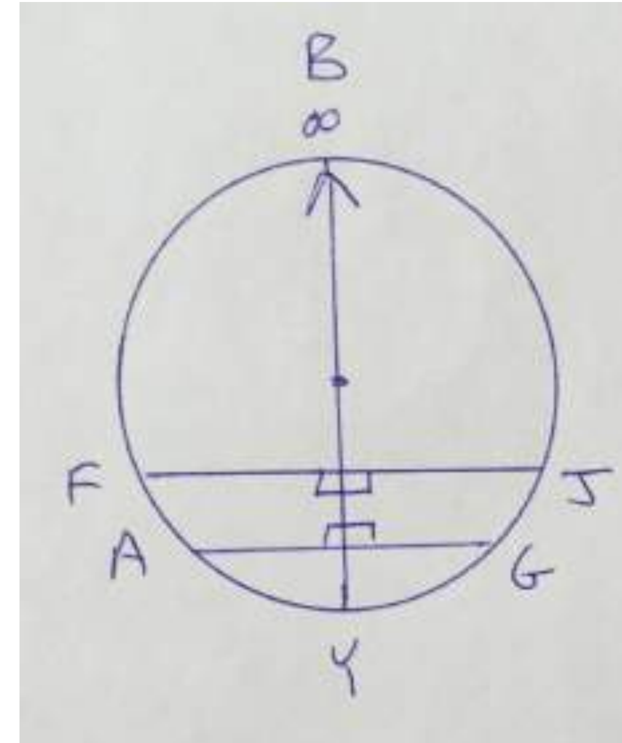


However, the infinitely large reference circle must then rotate by π radians in either direction to remain a circle equally divided by an infinitely long upward ray with its base on the axis; and it is this common perspective that allows for an appreciation of the continuity of these curves without using solid geometry.

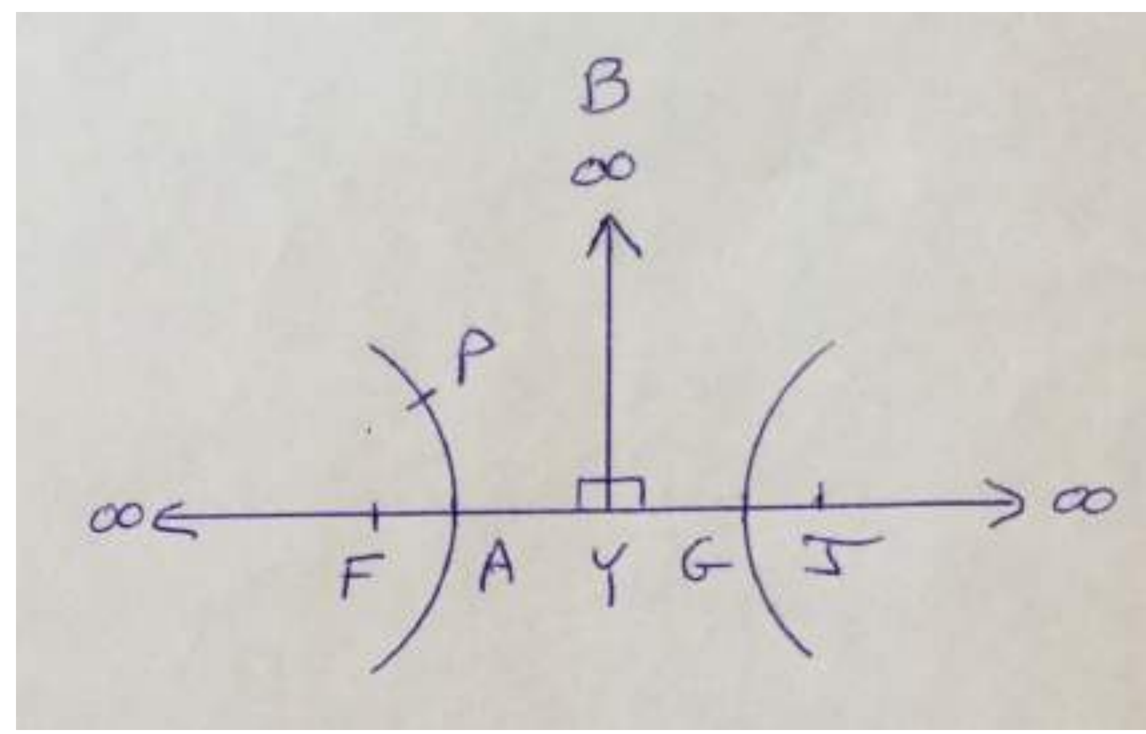


Draw: $0 < YF = YJ < \infty$

so that: $0 < e = YF/YA > 1$



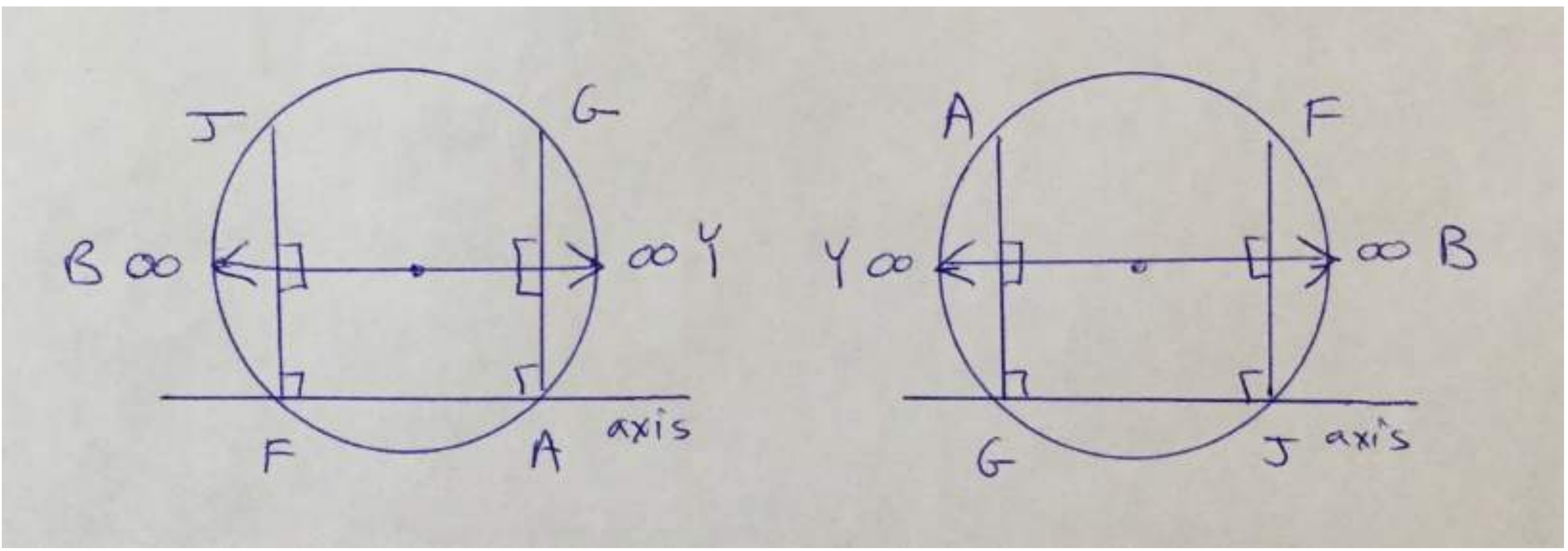
We will have drawn a defined hyperbola where AJ - AF = AG along its "transverse axis" FAYGJ, if it is also true that PJ - PF = AG.



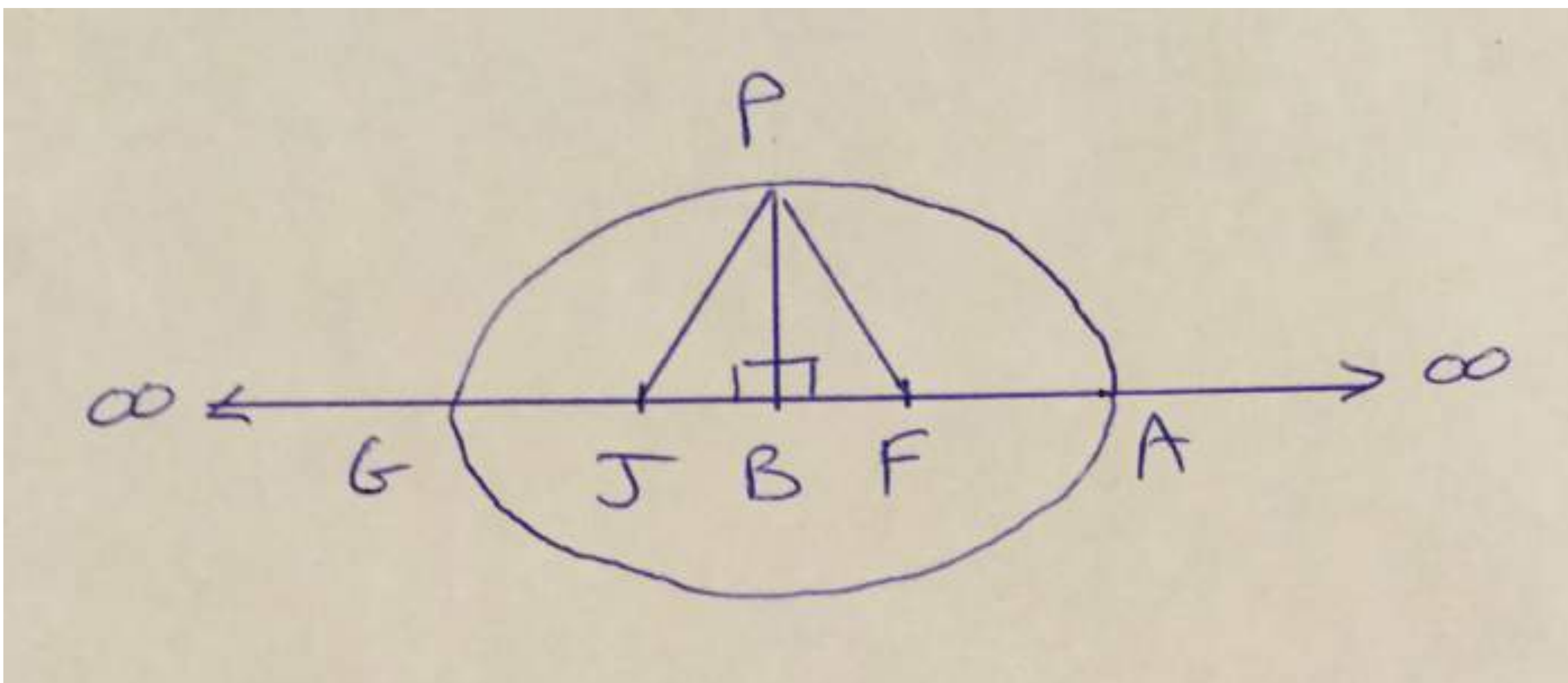
Parabola

When the infinitely large reference circle only rotates by π/2 radians in either direction, it no longer remains a circle equally divided by an infinitely long upward ray with its base on the axis. In fact, reference points B and Y are both infinitely far from the resulting curves they reference, and can be no longer used to specify the curves, or their eccentricity. However, due to the halfway rotation of the reference circle, we can assume either of the resulting curves would have an eccentricity halfway between that of an ellipse ($e < 1$), and that of an hyperbola ($e > 1$).

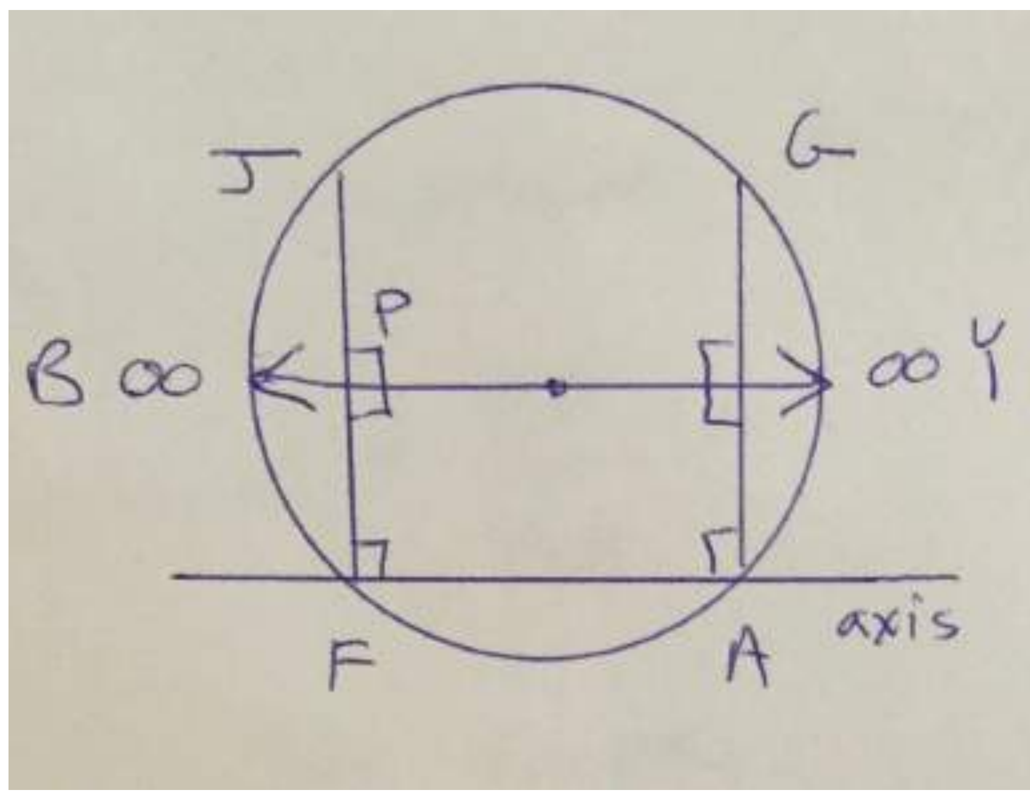
These resulting curves are defined as a parabolas (e = 1), and like the circle (e = 0), they represents a special case with a singular shape, or eccentricity.



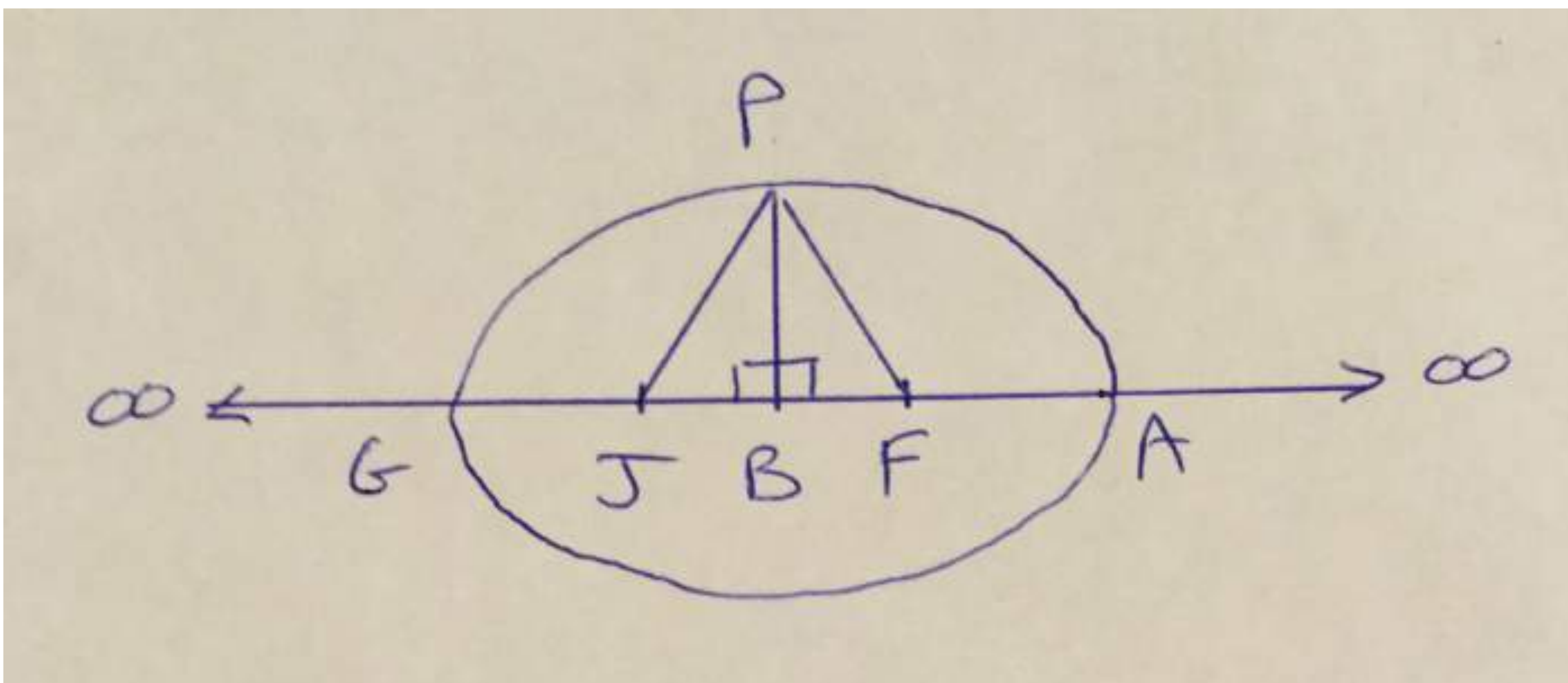
The path of an ellipse opens towards G from A only so long as PF and PJ become more equal. The ellipse remains open towards G as a parabola only as PF = PJ $\Rightarrow \infty$.



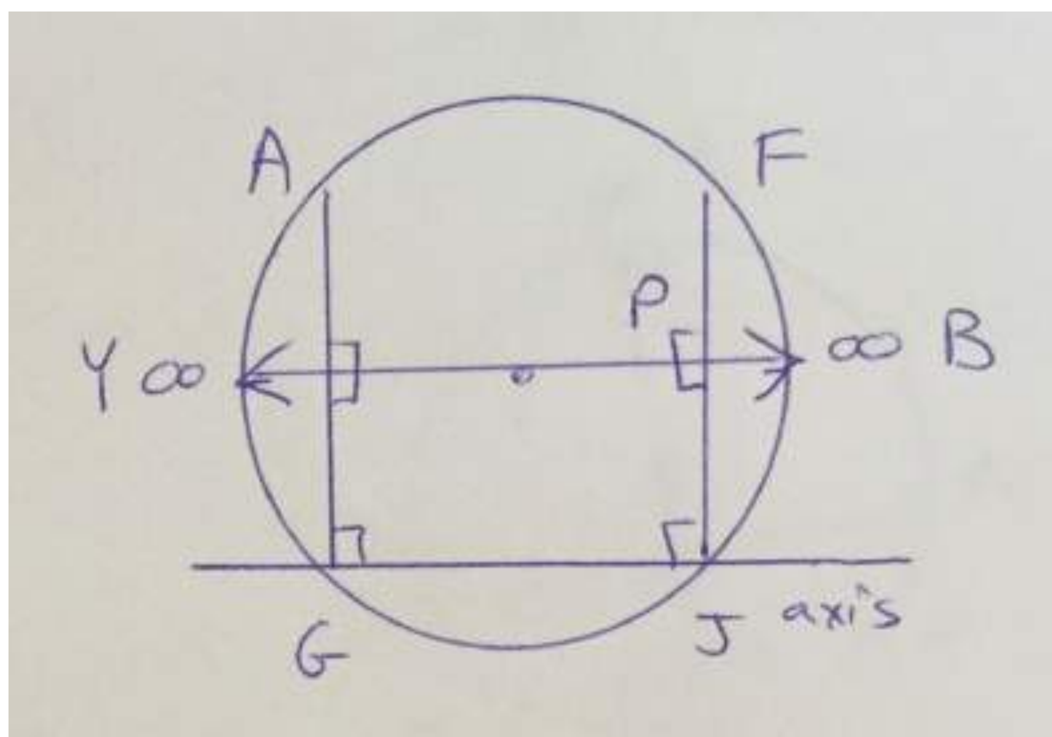
When P then moves from ∞ back toward A, this parabolic shape can be maintained using this knowledge by locating P on the following diagram halfway between F and J.



The path of an ellipse opens towards A from G only so long as PF and PJ become more equal. The ellipse remains open towards A as a parabola only as $PF = PJ \Rightarrow \infty$.

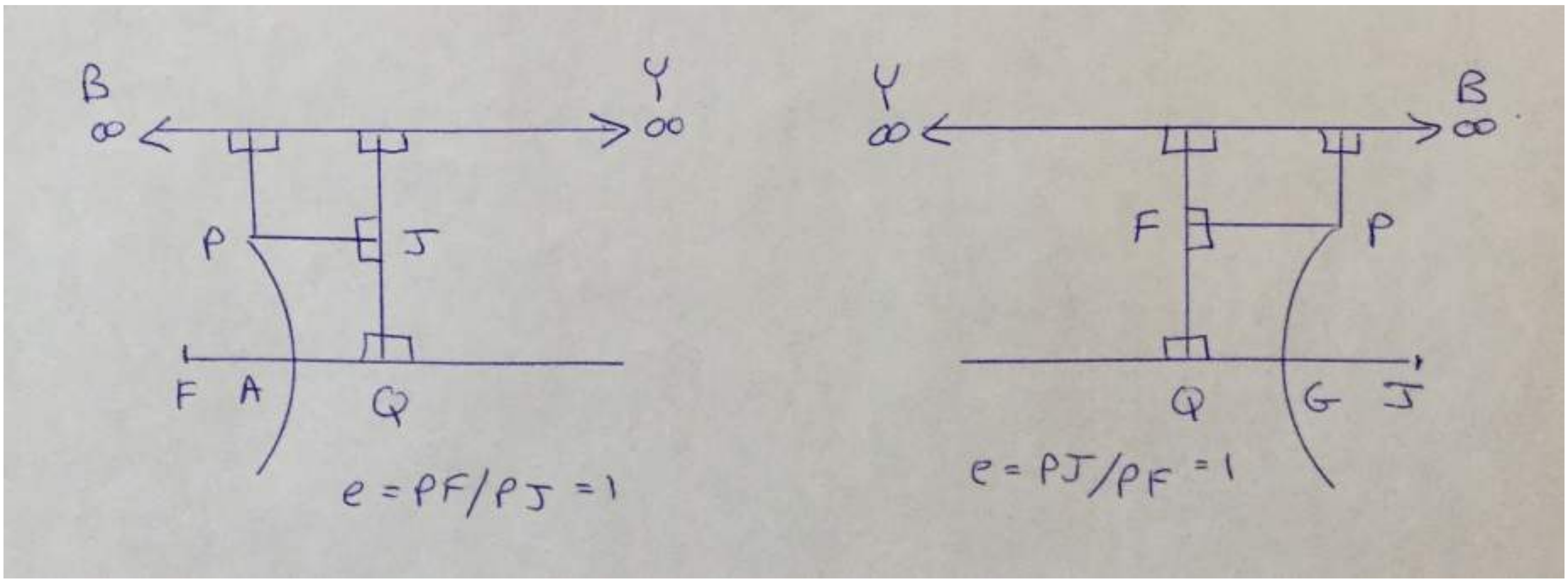


When P then moves from ∞ back toward G, this parabolic shape can be maintained using this knowledge by locating P on the following diagram halfway between F and J.



We have seen that as $PF = PJ$, and as $P \Rightarrow \infty$ from either A or G, an ellipse can transform into either of two parabolas opening toward each other with vertices at A or G, but this can not happen simultaneously. However, if we start with a hyperbola, and let $P \Rightarrow \infty$ from A, so that $PF \Rightarrow PJ$, we create two simultaneous parabolas facing away from each other.

QJ and QF can simply be chosen so that PF = PJ, and for these curves to be *simultaneously* true, Q must lie halfway between A and G. It is also obvious that FA must equal AQ, and that JG must equal GQ.



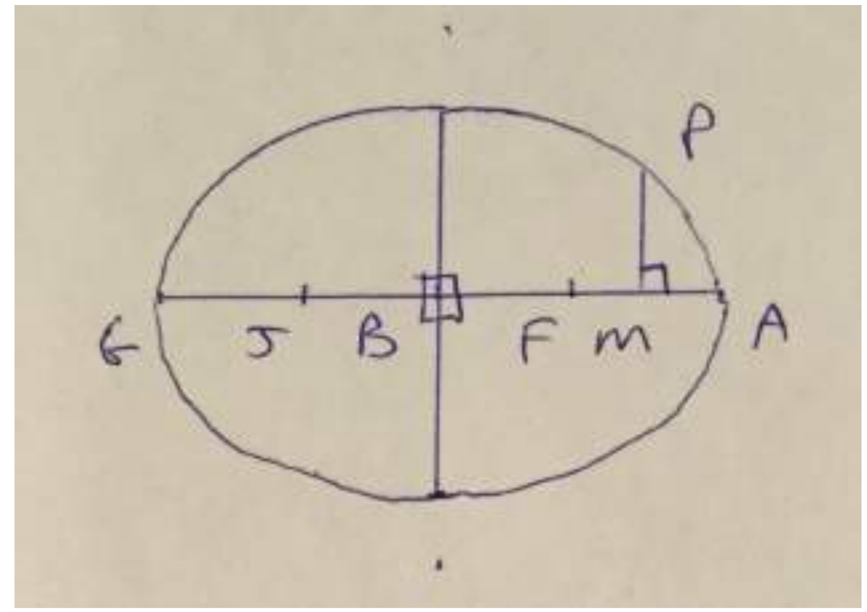
Characteristics of Conic Sections /Synthetic Geometry

The following discussions reference the 1899 revised edition of “Plane and Solid Geometry,” by G. A. Wentworth. (Later editions of this text do not discuss these curves). This text is particularly useful when studying Isaac Barrow’s 1667 Optical Lectures, because it also uses the language of *synthetic* geometry, rather than analytical geometry.

Ellipse

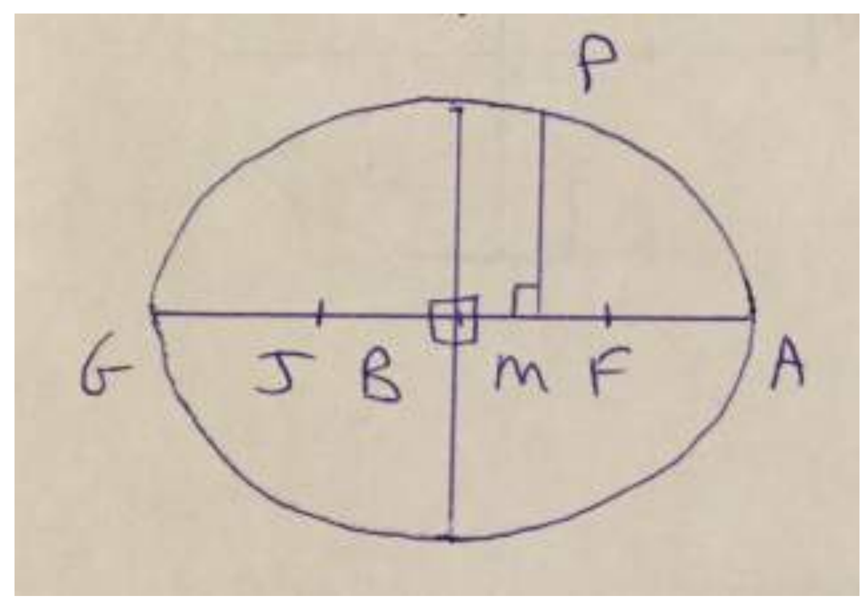
2(BF) = MJ - MF

2(BM) = MJ + MF



2(BF) = MJ + MF

2(BM) = MJ - MF



$$PJ^2 - FP^2 = (MJ^2 + MP^2) - (MF^2 + MP^2)$$

$$(PJ + FP)(PJ - FP) = (MJ + MF)(MJ - MF)$$

$$AG(PJ - FP) = 2(BM)\,2(BF)$$

$$PJ - FP = [2(BM)\,2(BF)]/2(BA)$$

$$\text{eccentricity} = e = BF/BA$$

$$PJ - FP = 2(BM)e$$

Since:

$$FP + PJ = AG = 2(BA)$$

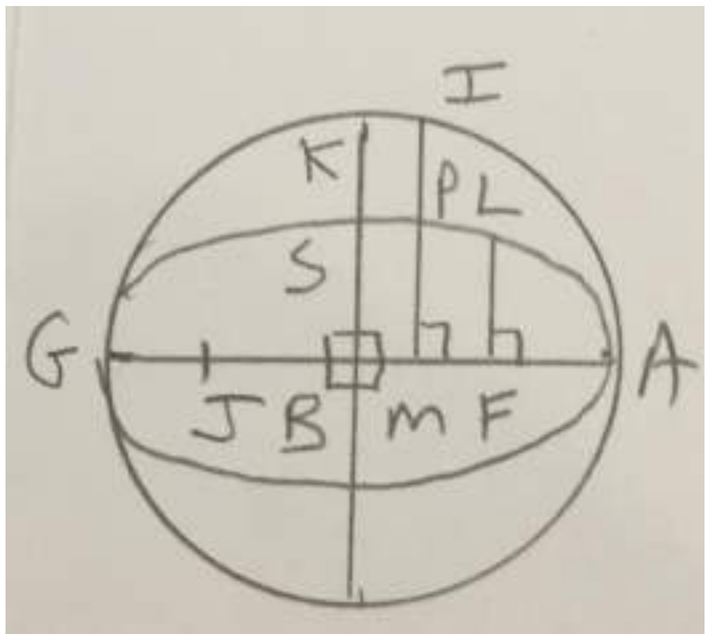
$$(FP + PJ) + (PJ - FP) = 2(PJ) = 2(BA) + 2(BM)e$$

$$(FP + PJ) - (PJ - FP) = 2(FP) = 2(BA) - 2(BM)e$$

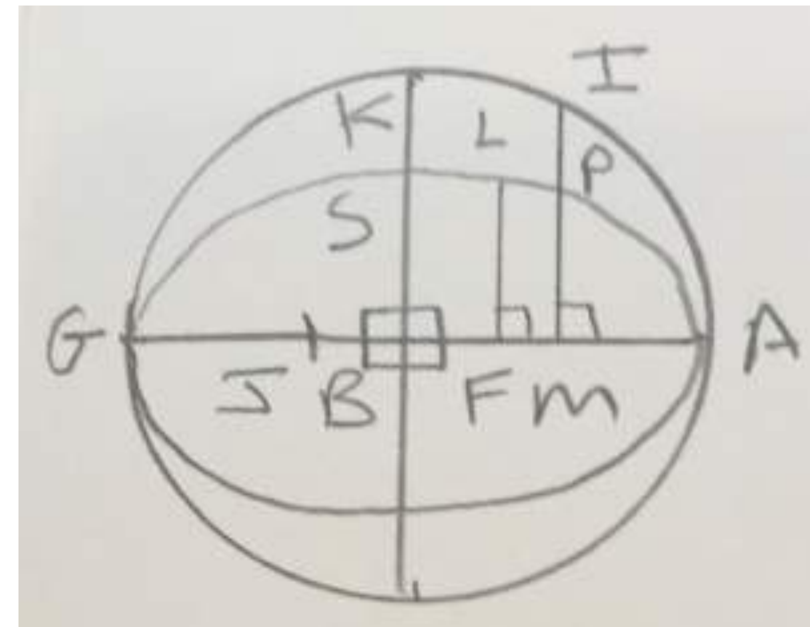
$$\mathbf{PJ = BA + (BM)e}$$

$$\mathbf{PF = BA - (BM)e}$$

$$FM = BF - BM$$



$$FM = BM - BF$$



$$FM^2 = BF^2 + BM^2 - 2(BF)BM$$

$$e = BF/BA = FB/FS$$

$$BA^2 = BF^2 + BS^2$$

$$PF^2 = [BA - (BM)e]^2$$

$$PF^2 = BA^2 + (BM)^2e^2 - 2(BM)BF$$

$$PM^2 = PF^2 - FM^2$$

$$PM^2 = [BA^2 + (BM)^2e^2 - 2(BM)BF]$$

$$- [BF^2 + BM^2 - 2(BF)BM]$$

$$PM^2 = BS^2 + BM^2(e^2 - 1)$$

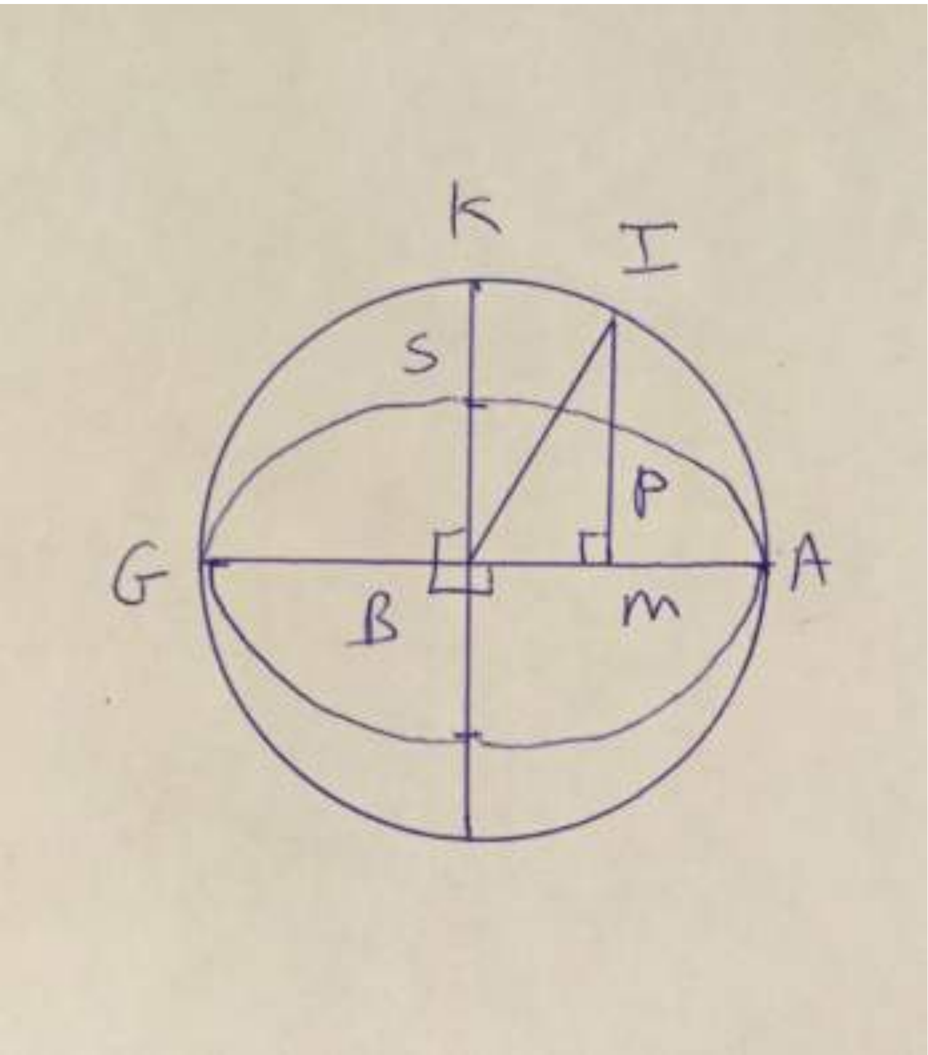
$$PM^2 = BS^2 - BM^2(1-e^2)$$

$$(PM)^2BA^2 = (BS)^2BA^2 - BM^2[BA^2 - BF^2]$$

$$(PM)^2BA^2 = BS^2[BA^2 - BM^2]$$

$$(MP/MI)^2 = (BS/BA)^2$$

$$MP/MI = BS/BK$$



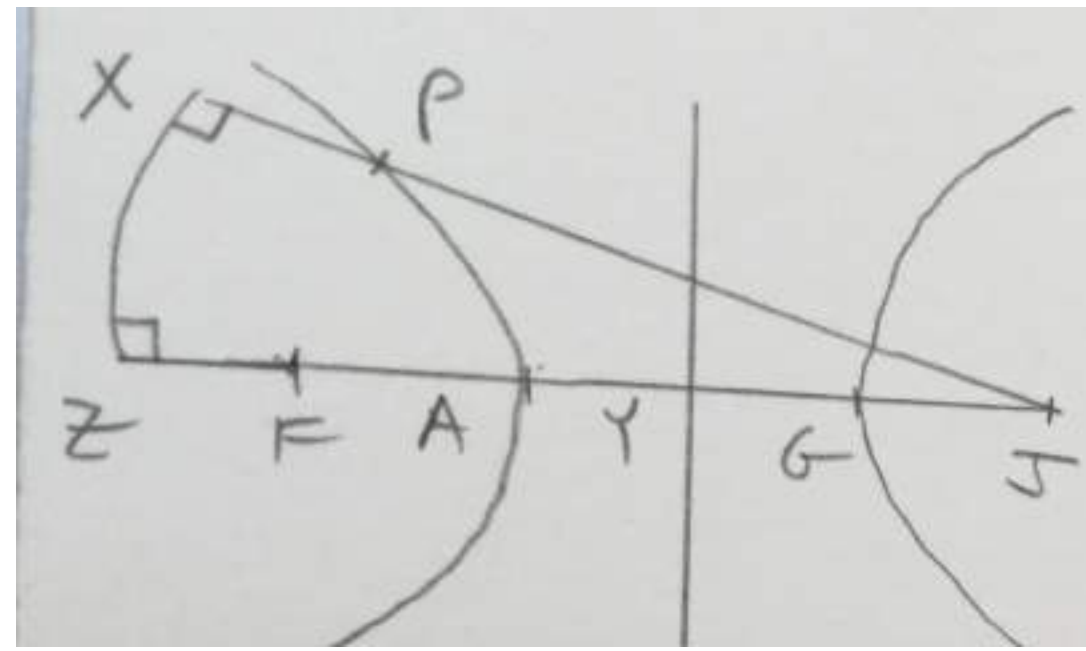
Hyperbola

Draw hyperbola arm AP:

Make: ZJ - AG = XP + FP

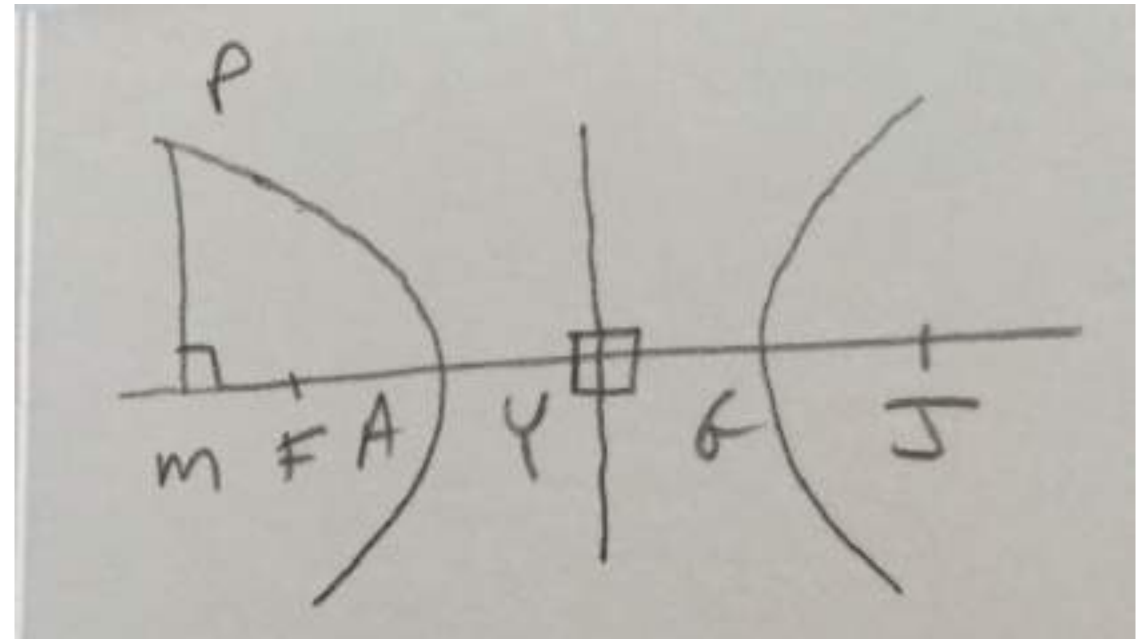
So: XJ - XP = FP + AG

and PJ - FP = AG



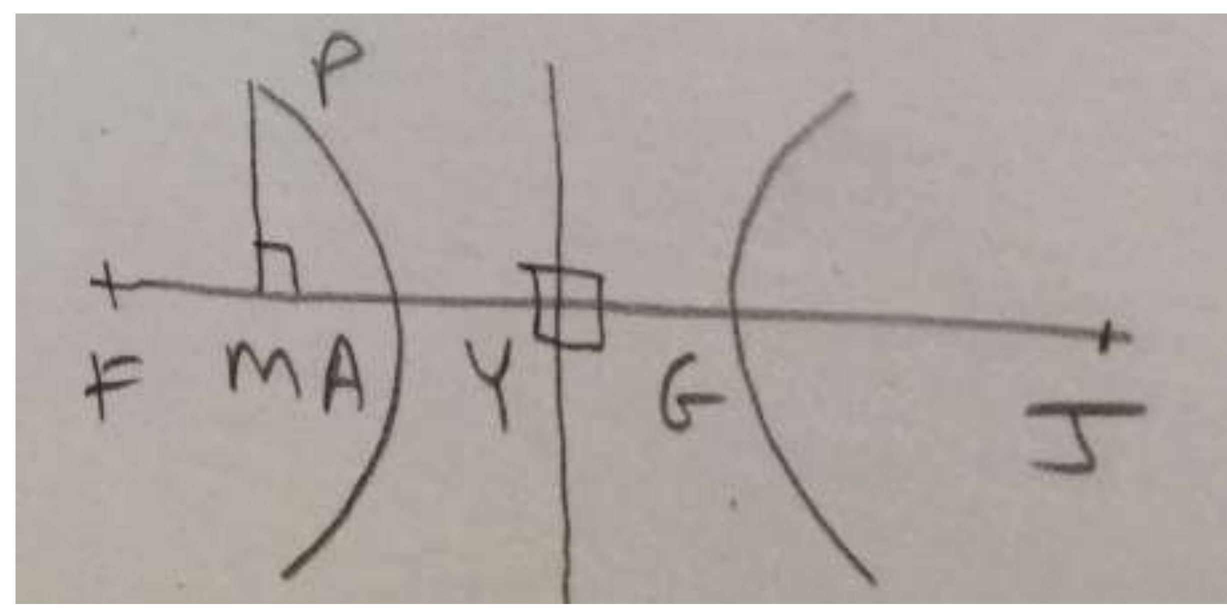
$$MJ - MF = 2(YF)$$

$$MJ + MF = 2(YM)$$



$$MJ - MF = 2(YM)$$

$$MJ + MF = 2(YF)$$



$$PJ^2 - FP^2 = (MP^2 + MJ^2) - (MP^2 + MF^2)$$

$$(PJ + FP)(PJ - FP) = (MJ + MF)(MJ - MF)$$

$$(PJ + FP)AG = 2(YM)\,2(YF)$$

$$PJ + PF = [2(YM)\,2(YF)]/2(YA)$$

$$\text{eccentricity} = e = YF/YA$$

$$PJ + PF = 2(YM)e$$

Since: $PJ - PF = AG = 2(YA)$

$$(PJ + PF) + (PJ - PF) = 2(PJ) = 2(YM)e + 2(YA)$$

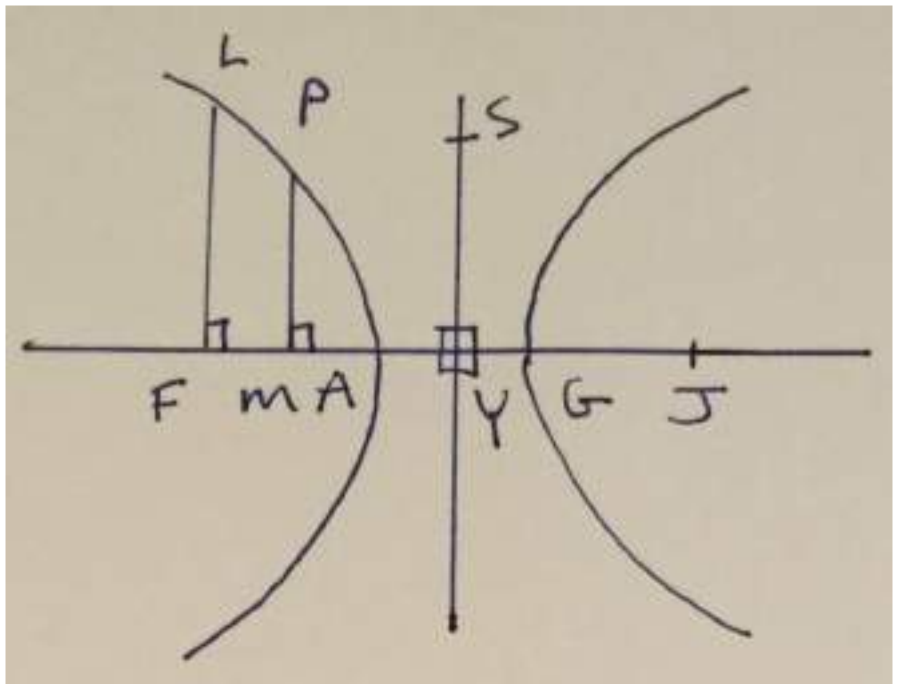
$$(PJ + PF) - (PJ - PF) = 2(PF) = 2(YM)e - 2(YA)$$

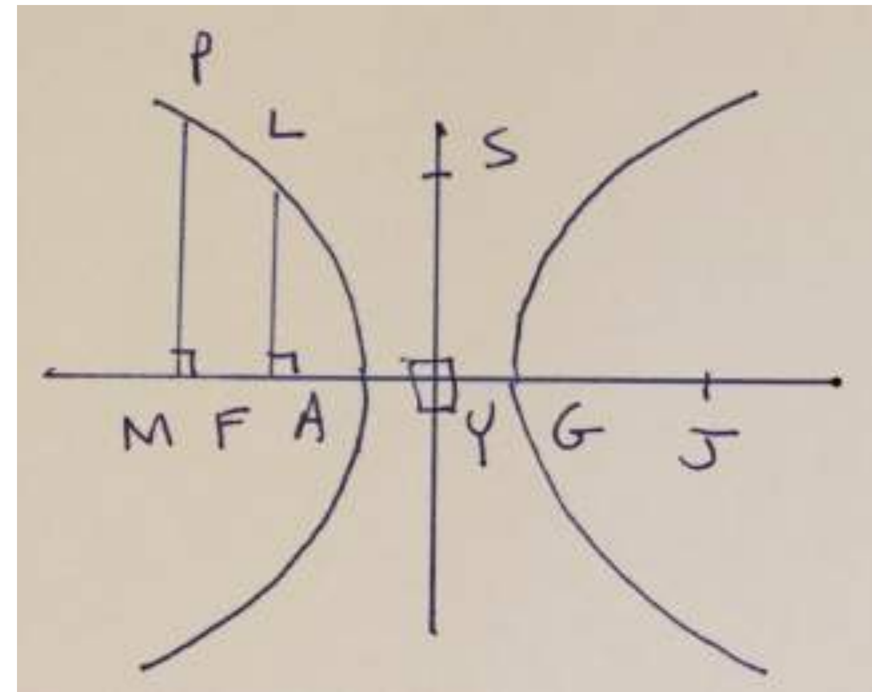
$PJ = (YM)e + YA$

$PF = (YM)e - YA$

$FM = YF - YM$

$FM = YM - YF$





$$FM^2 = YF^2 + YM^2 - 2(YF)YM$$

$$e = YF/YA = AS/AY$$

$$YF^2 = YA^2 + YS^2$$

$$PF^2 = [(YM)e - YA]^2$$

$$PF^2 = YM^2e^2 + YA^2 - 2(YM)YF$$

$$PM^2 = PF^2 - FM^2$$

$$PM^2 = [YM^2e^2 + YA^2 - 2(YM)YF]$$

$$- [YF^2 + YM^2 - 2(YF)YM]$$

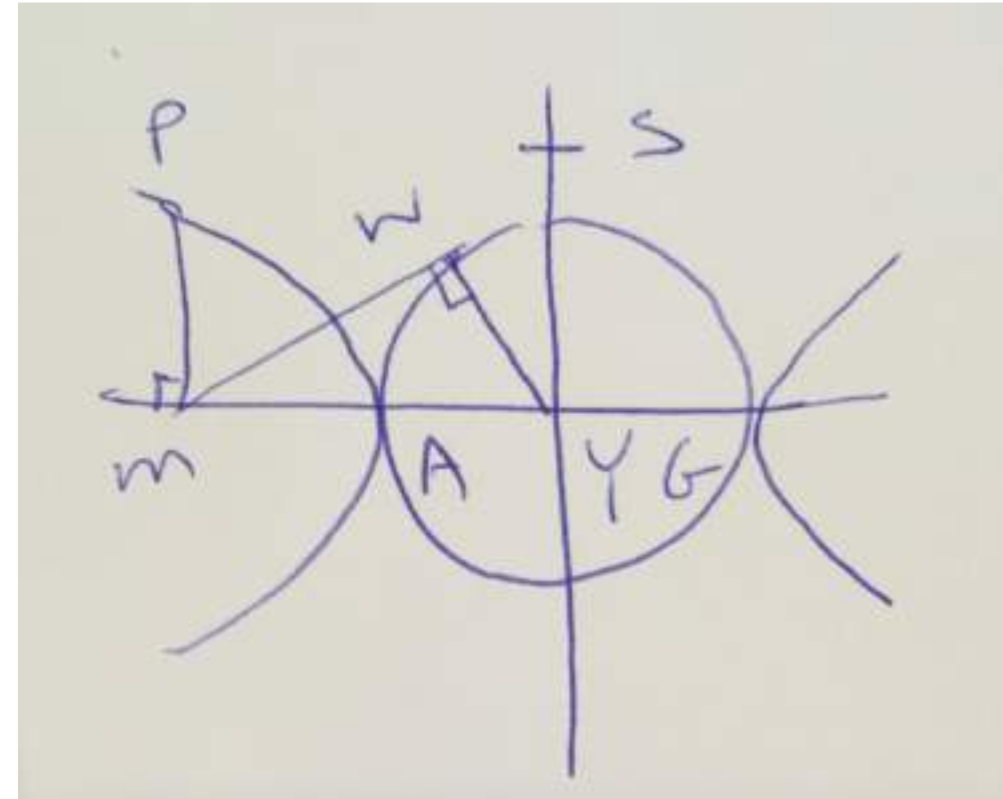
$$PM^2 = YM^2(e^2 - 1) - YS^2$$

$$PM^2\ YA^2 = YM^2[YF^2 - YA^2] - YS^2\ YA^2$$

$$PM^2\ YA^2 = YS^2(YM^2 - YA^2)$$

$$(MP/MW)^2 = (YS/YA)^2$$

$$MP/MW = YS/YA$$



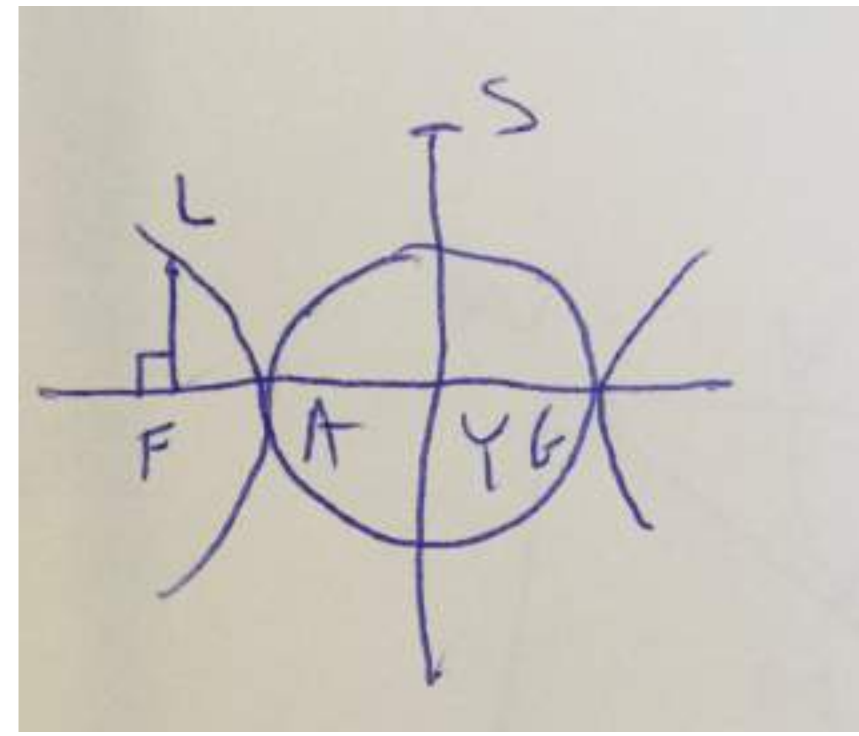
$MW^2 = (MA)MG$

$MP^2/(MA)MG = (YS/YA)^2 = FL^2/(FA)FG$

$(FA)FG = (YF - YA)(YF + YA)$

$(FA)FG = YF^2 - YA^2 = YS^2$

$FL/YS = YS/YA$



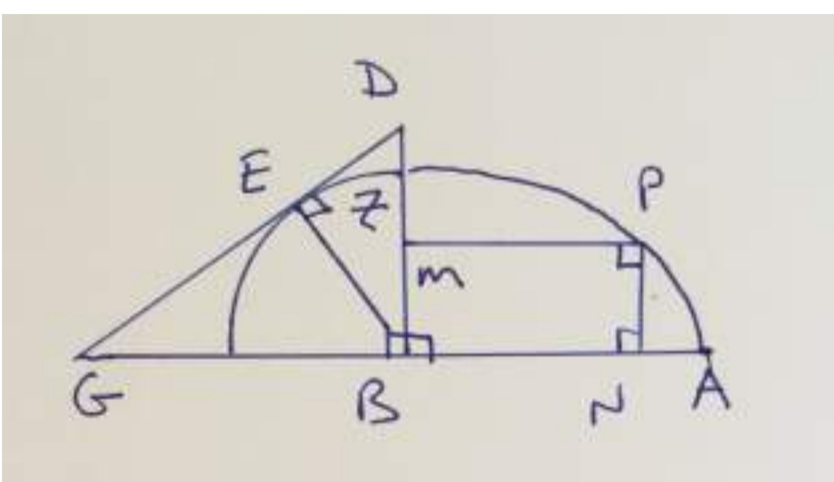
Tangential Refraction at a Plane Surface

The following discussion is based on material from *Isaac Barrow's Optical Lectures, 1667.* It references H. C. Fay's 1987 English translation published in 1987 by the "Worshipful Company of Spectacle Makers," and edited by A. G. Bennett and D. F. Edgar. It concerns itself solely with tangential refraction, or that within the plane that is a perpendicular cross section of the refracting plane surface. The discussion will be presented in two columns for clarity, one representing the condition with an object in glass, and image space in air; and the other with an object in air, and image space in glass.

Locating a tangential image ray through a point on the perpendicular containing the object

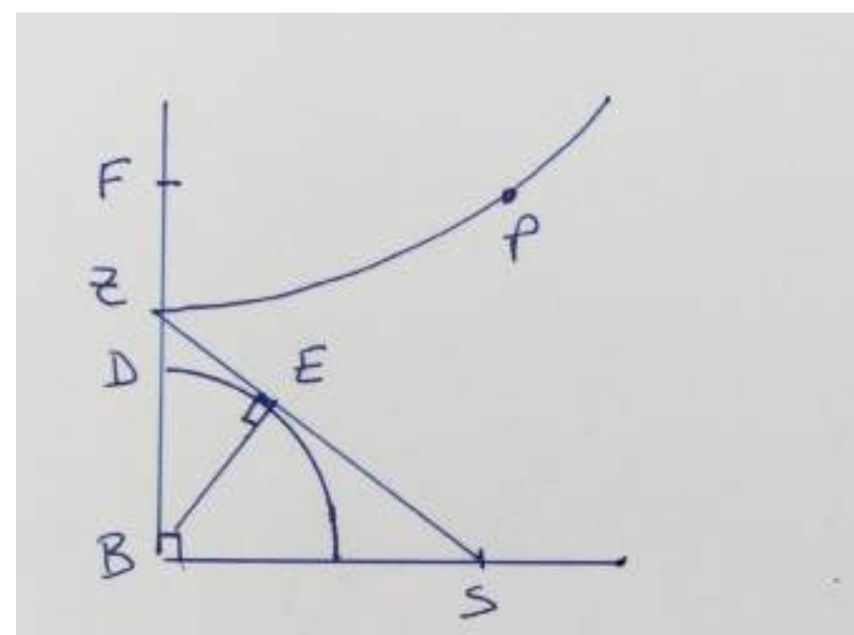
Object D in glass, image space in air

Given perpendicular object distance DB, perpendicular image distance ZB, and refracting plane cross section GA:

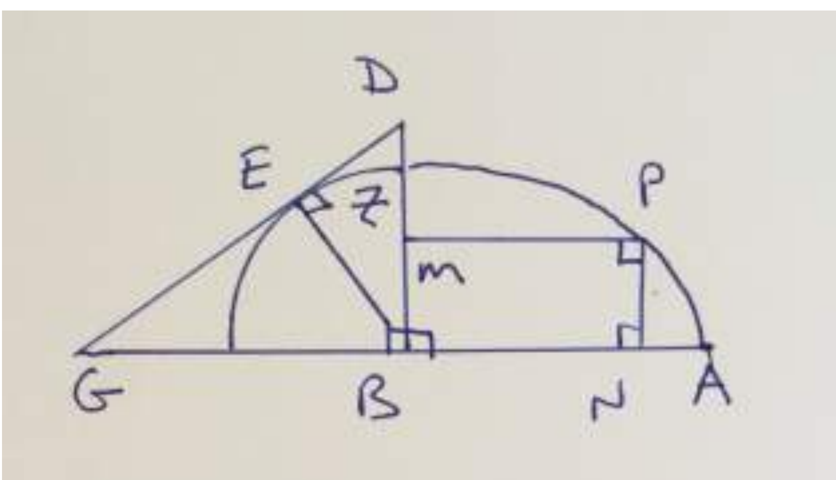


Object D in air, image space in glass

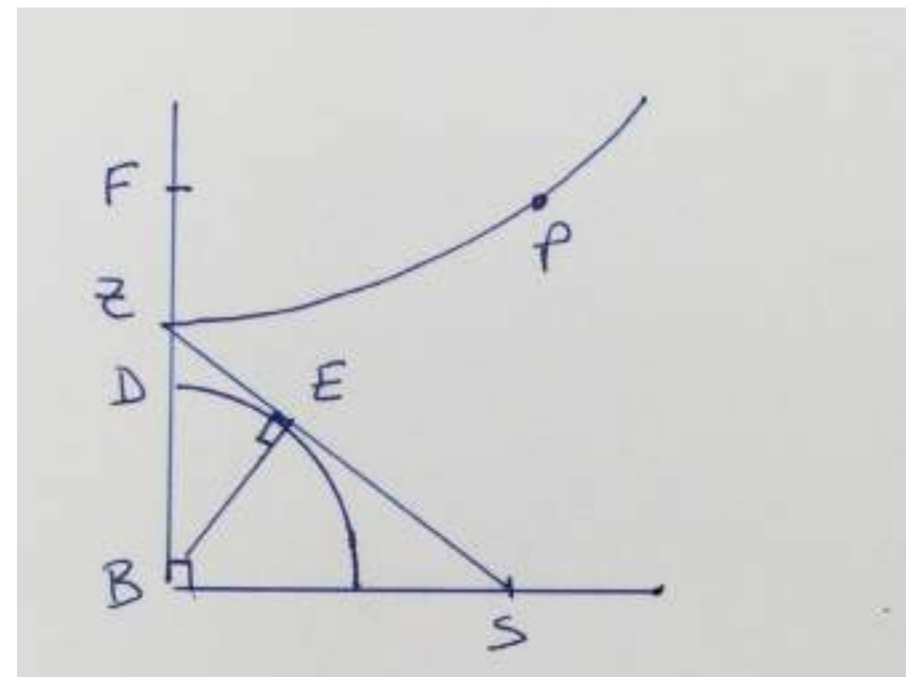
For ease of comparison, the perpendicular object distance has been designated DB, rather than DY, even though the reference curve is a hyperbola rather than an ellipse.



we can find any non-perpendicular refracted image ray (MN) using the reference semi-ellipse GZPA,

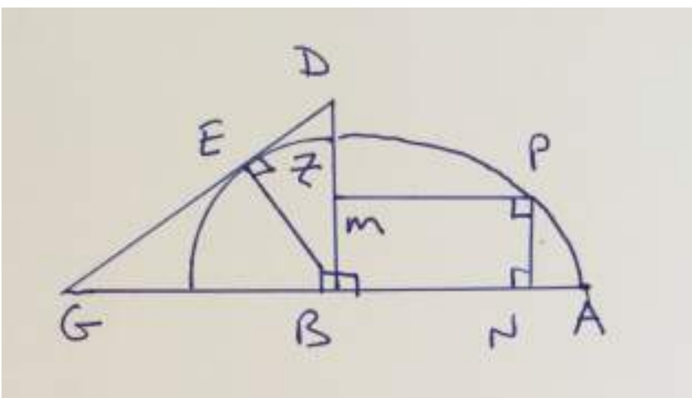


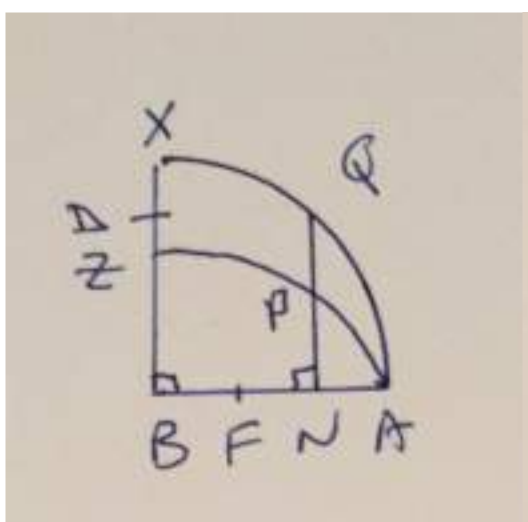
Given the perpendicular object distance DB, and perpendicular image distance ZB:

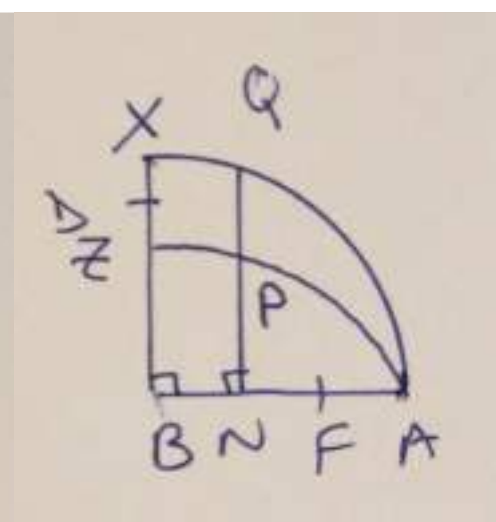


we can find any non-perpendicular refracted image ray using the reference hyperbola arm ZP, If:

e = BF/BA = FB/FZ

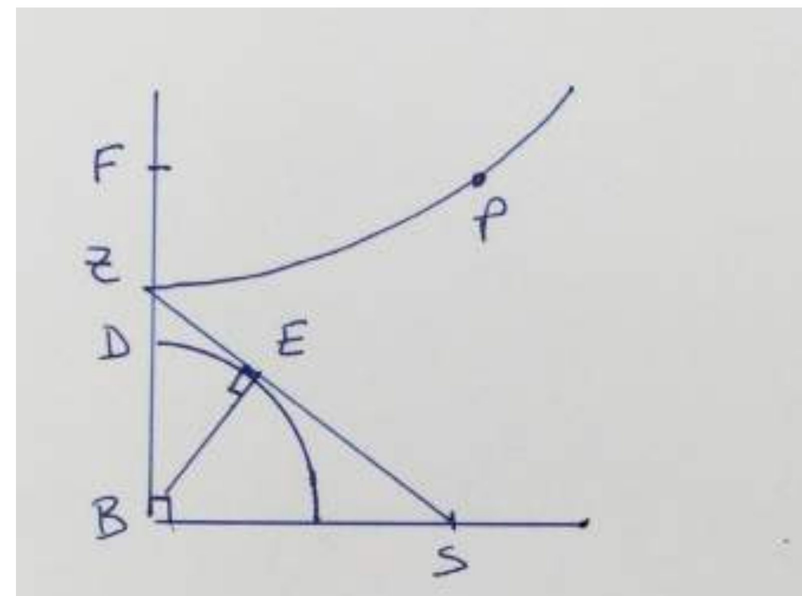






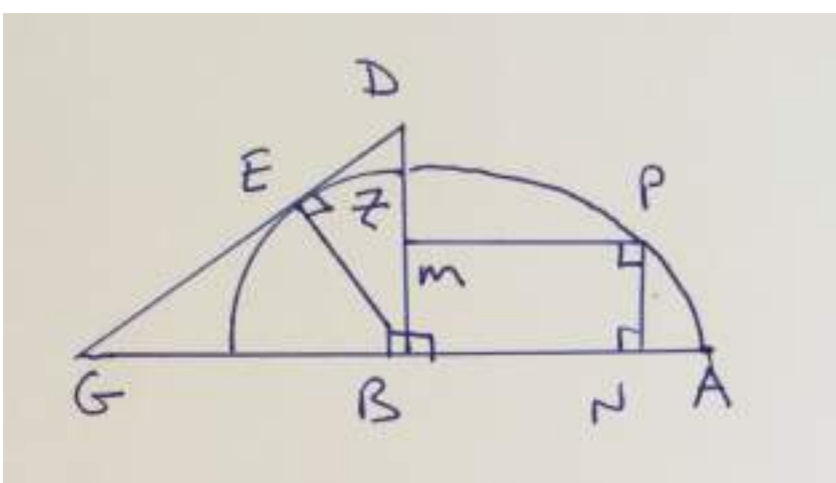
If:

e = BF/BZ = ZS/ZB
(e = ZB/ZE)



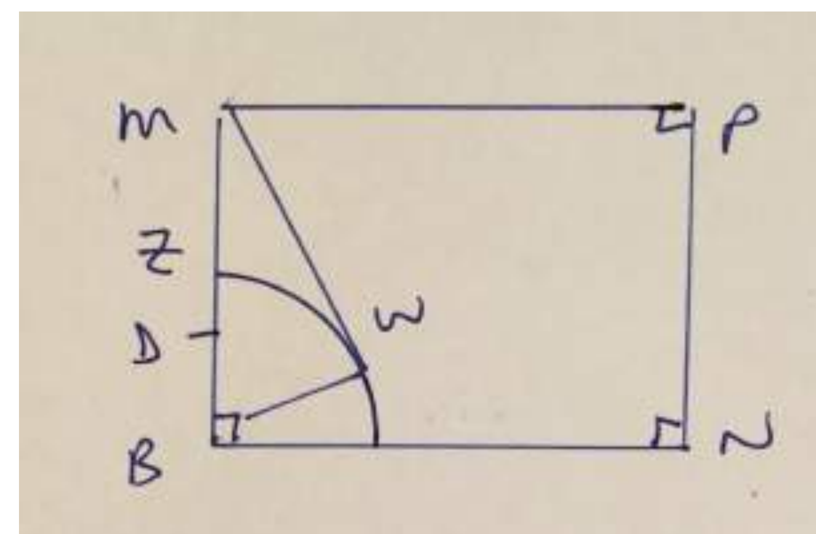
because a non-perpendicular image ray MN is determined by:

MN/DN = BZ/BD



because a non-perpendicular image ray MN is determined by:

MN/DN = BZ/BD



and for the ellipse:

$NQ/NP = BX/BZ$

$BZ^2/NP^2 = BA^2/(BA^2 - BN^2)$

$(BZ^2 - NP^2)/NP^2 = BN^2/(BA^2 - BN^2)$

$(BZ^2 - NP^2)/BN^2 = NP^2/(BA^2 - BN^2)$

$(BZ^2 - NP^2)/BN^2 = NP^2/NQ^2$
$= BZ^2/BG^2$

$BZ^2/BG^2 = BE^2/BG^2 = ED^2/BD^2$
$= (BD^2 - BZ^2)/BD^2$

and for the hyperbola:

$MW/MP = BZ/BS$

$MW^2/MP^2 = (MB^2 - ZB^2)/BN^2$

$BZ^2/BS^2 = EZ^2/EB^2$
$= (ZB^2 - DB^2)/DB^2$

$(MB^2 - ZB^2)/BN^2$
$= (ZB^2 - DB^2)/DB^2$

Since BD > BZ > BM

$(NP^2 - BZ^2)/BN^2$
$= (BZ^2 - BD^2)/BD^2$

$(MN^2 - BZ^2)/BN^2 = BZ^2/BD^2$

$(MN^2 - BZ^2)/BZ^2 = BN^2/BD^2$

$MN^2/BZ^2 = (BN^2 + BD^2)/BD^2$

$MN^2/DN^2 = BZ^2/BD^2$

$MN/DN = BZ/BD$

$(MB^2 - ZB^2 + BN^2)/BN^2$
$= BZ^2/BD^2$

$(MN^2 - BZ^2)/BZ^2$
$= BN^2/BD^2$

$MN^2/ZB^2 = DN^2/DB^2$

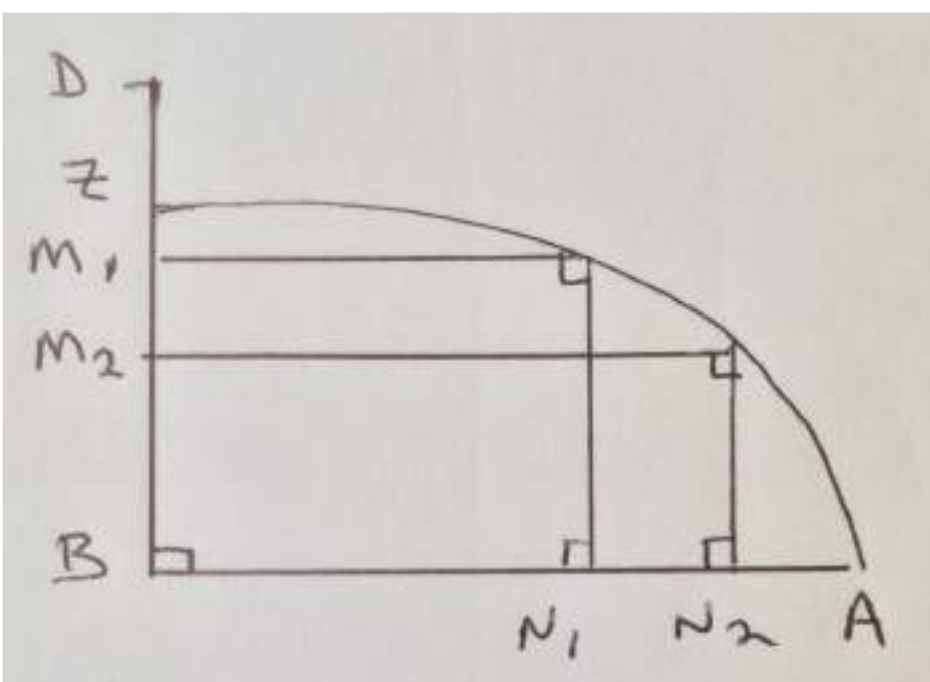
$MN^2/DN^2 = BZ^2/BD^2$

$MN/DN = BZ/BD$

Locating a tangential image ray through a point *not* on the perpendicular containing the object

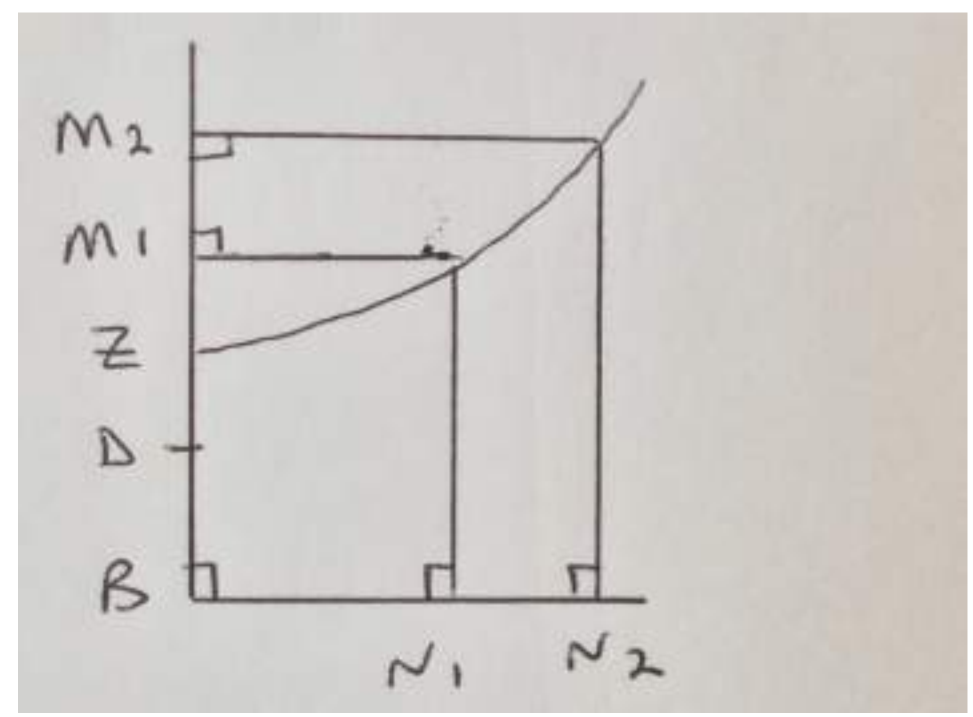
Given object D and axial image Z:

$\mathbb{R} = BD/BZ$
$\mathbb{R} = N_1D/N_1M_1$
$\mathbb{R} = N_2D/N_2M_2$

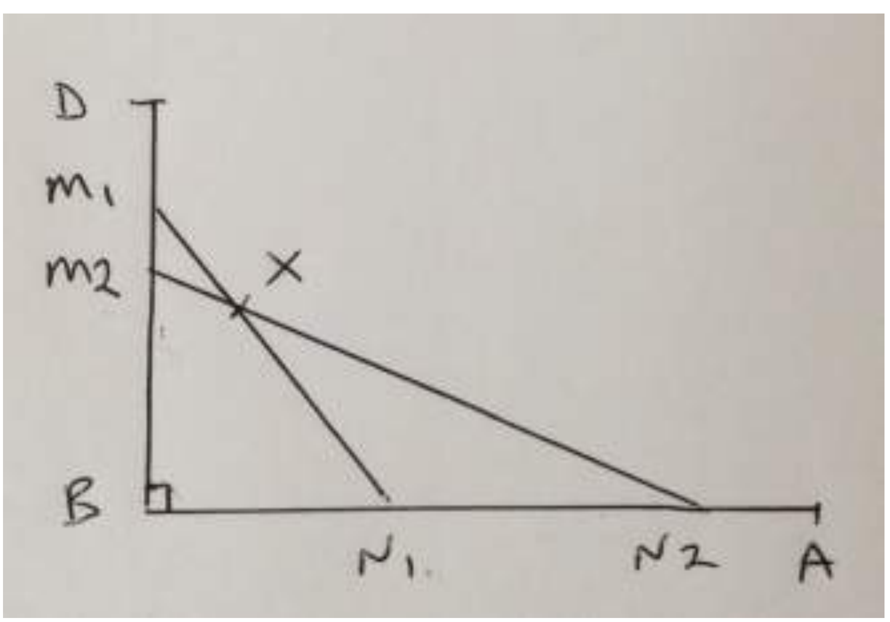


Given object D and axial image Z:

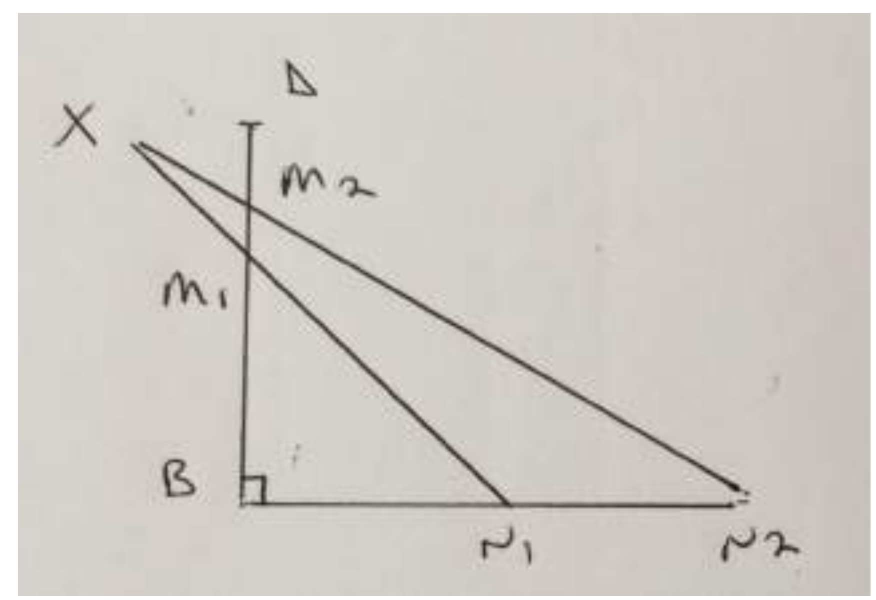
$\mathbb{R} = BZ/BD$
$\mathbb{R} = N_1M_1/N_1D$
$\mathbb{R} = N_2M_2/N_2D$



Because the refraction is described by a reference ellipse through Z and A, $BM_1 > BM_2$ and N_1M_1 crosses N_2M_2 at X within the right angle $\angle DBA$.

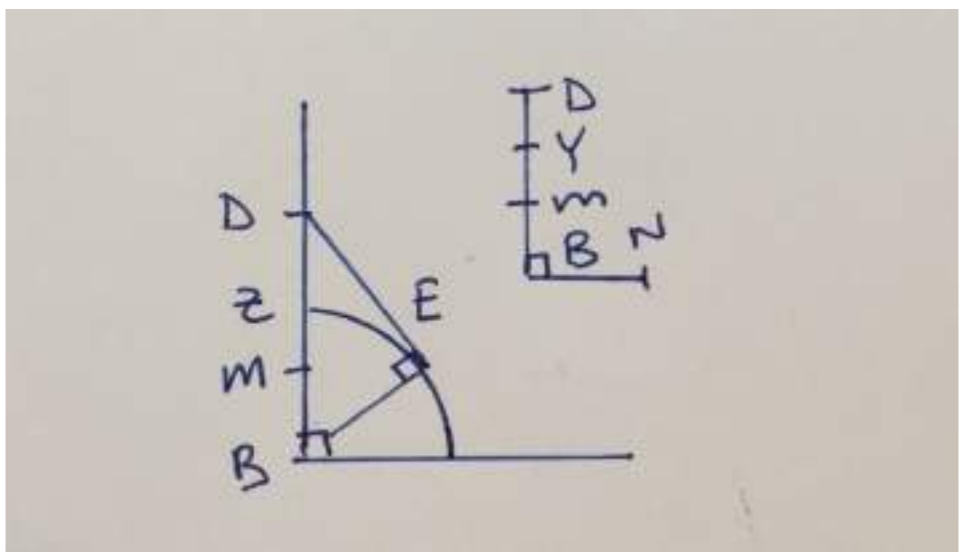


Because the refraction is described by a reference hyperbola through Z, $BM_2 > BM_1$ and N_1M_1 crosses N_2M_2 at X outside the right angle $\angle DBN_2$.



Given object D and image Z:

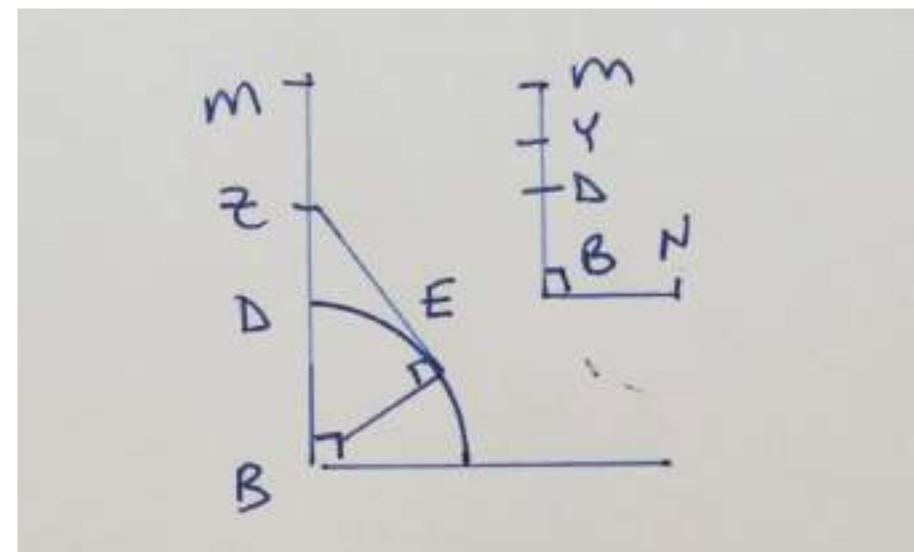
$\mathbb{R}$ = BD/BZ = ND/NM



We need to find a way to locate point N.

Given object D and image Z:

$\mathbb{R}$ = BZ/BD = NM/ND



We need to find a way to locate point N.

Of course, we can use the reference ellipse previously described, but we will need a different method when given a point on the image ray X, but not (M).

Of course, we can use the reference hyperbola previously described, but we will need a different method when given a point on the image ray X, but not (M).

To accomplish that, the following ratio manipulations prove that if:

BY/MB = DB/DE

then:

DB/YN = ED/EB

To accomplish that, the following ratio manipulations prove that if:

BY/DB = BZ/EZ

then:

MB/YN = EZ/EB

$MB^2 = MN^2 - BN^2$

$MB^2 = MN^2 - YN^2 + BY^2$

$BY^2/(MN^2 - YN^2 + BY^2)$
$= DB^2/(DB^2 - BZ^2)$
$= DN^2/(DN^2 - MN^2)$

$BY^2/(YN^2 - MN^2) = DN^2/MN^2$

$BY^2 = YN^2 - BN^2$

$BY^2/DB^2 = BZ^2/(BZ^2 - EB^2)$

$BY^2/(BY^2 - DB^2) = BZ^2/DB^2$

$BZ^2/DB^2 = MN^2/DN^2$

$BY^2/MN^2 = (BY^2 - DB^2)/DN^2$

$(BY^2 + MN^2)/MN^2$
$= (BY^2 - DB^2 + DN^2)/DN^2$

$BY^2 = YN^2 - DN^2 + DB^2$

$(YN^2 - DN^2 + DB^2)/(YN^2 - MN^2)$
$= DN^2/MN^2$

$a/b = c/d; (a + c)/(b + d) = c/d$

$(YN^2 + DB^2)/YN^2 = DN^2/MN^2$

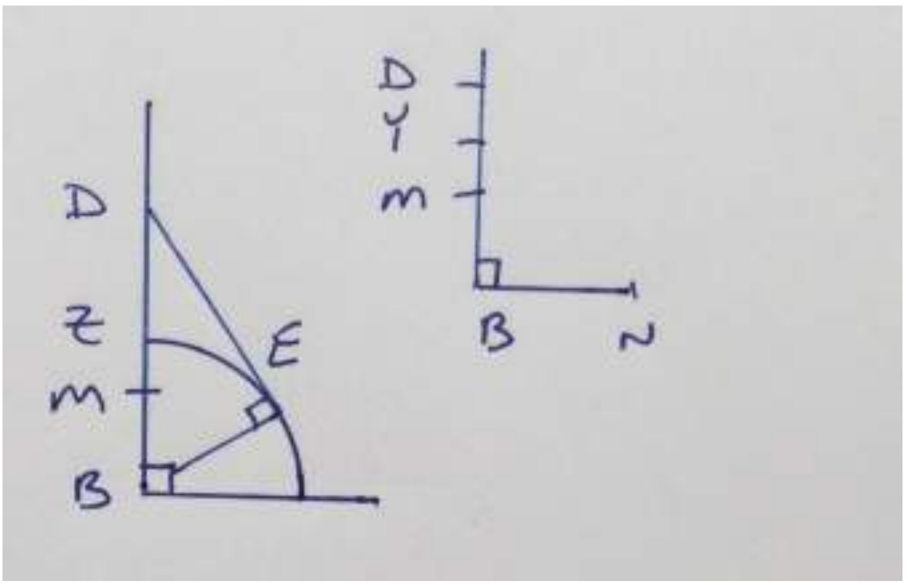
$DB^2/YN^2 = (DN^2 - MN^2)/MN^2$
$= (DB^2 - BZ^2)/DB^2 = ED^2/EB^2$

$(BY^2 + MN^2)/(BY^2 + BN^2)$
$= MN^2/DN^2$

$(MN^2 - BN^2)/NY^2$
$= (MN^2 - DN^2)/DN^2$

$MB^2/YN^2 = (BZ^2 - DB^2)/DB^2$
$= EZ^2/EB^2$

Therefore, given object D and image Z:



$\mathbb{R}$ = DB/BZ = ND/NM

if: BY/MB= DB/DE

then: DB/YN = ED/EB

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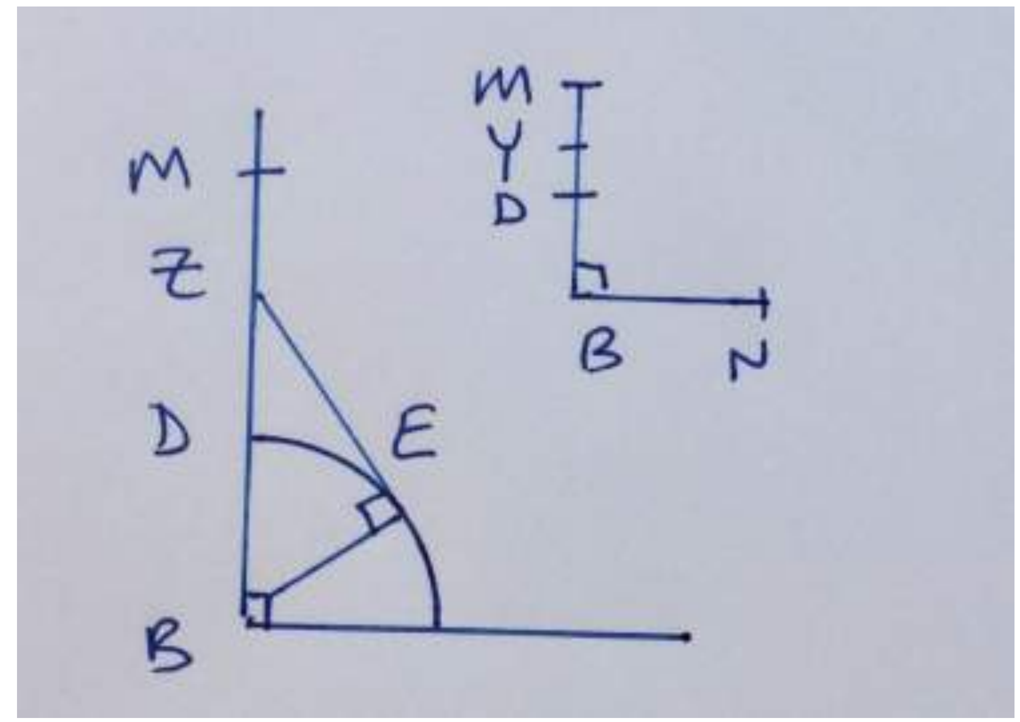
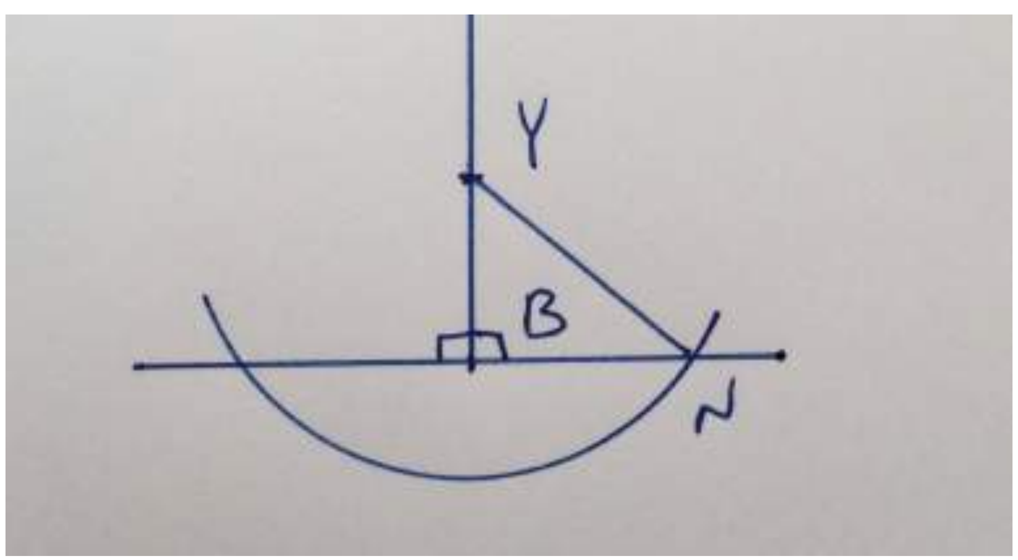


$\mathbb{R}$ = BZ/DB= NM/ND

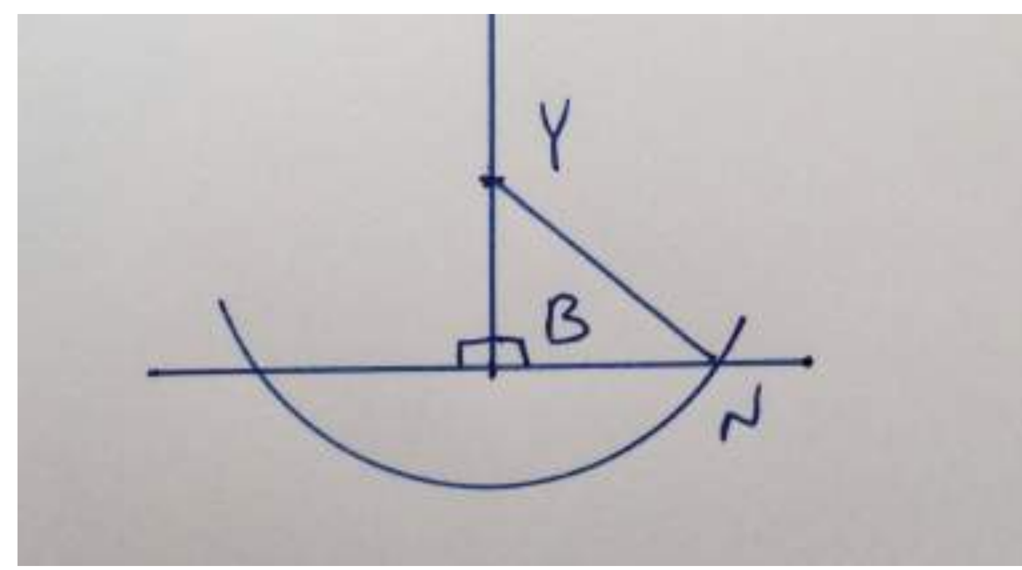
if: BY/DB= ZB/EZ

then: MB/YN= EZ/EB

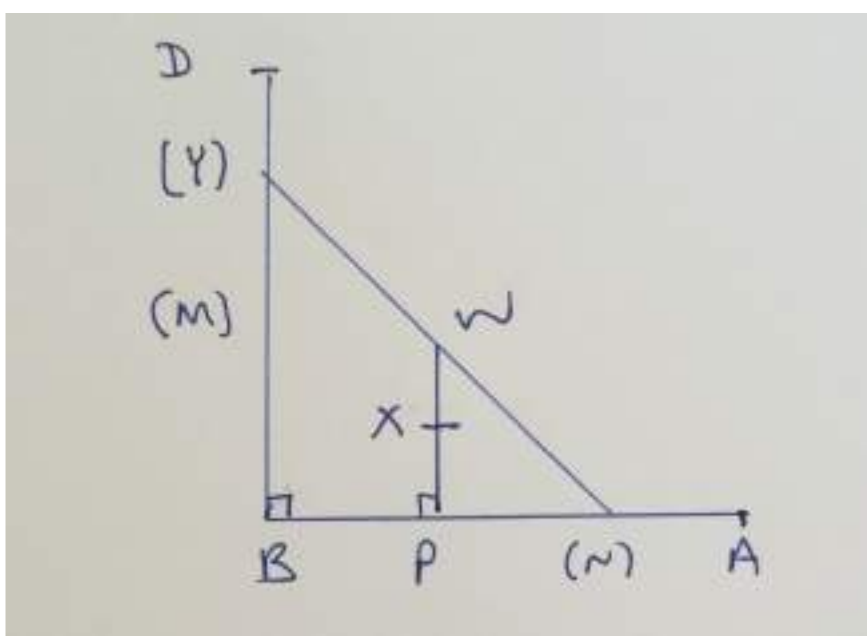
After calculating BY with known BM, (as well as known DB/DE); we can use known DB, (as well as known ED/EB), to calculate YN and use that as a radius about Y to find N:



After calculating BY with known DB, (as well as known ZB/ZE); we can use known MB, (as well as known EZ/EB), to calculate YN and use that as a radius about Y to find N:



This method gives us no advantage over the previously described method. However, it will allow us to develop a way to find the line segment (M)X(N) without knowing (M), (or N).



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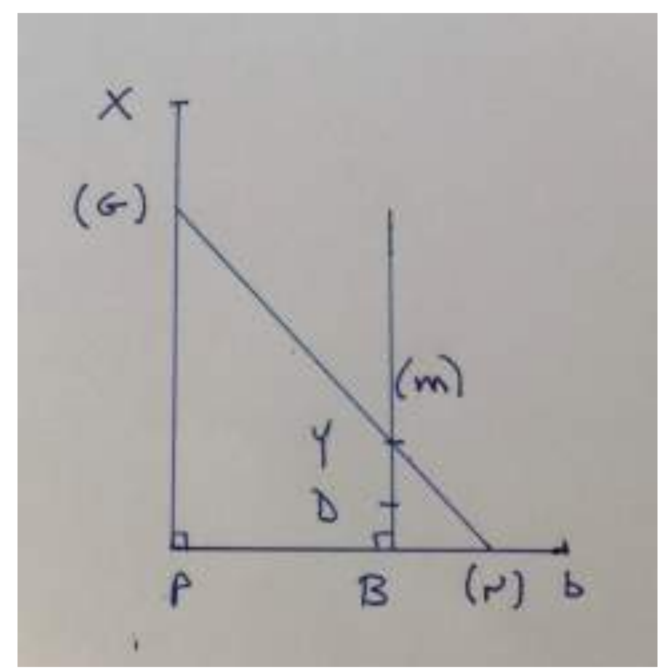


After calculating PW with known PX and DB/DE using:

PW/PX = (BY/MB) = DB/DE

since DB and ED/EB are also known:

DB/YWN = (WP/XN) = ED/EB

allows us to calculate the length of YWN, and we can then find (N) by inserting the calculated length YWN within the right angle ∠DBA through W.

After calculating BY with known DB and ZB/ZE using:

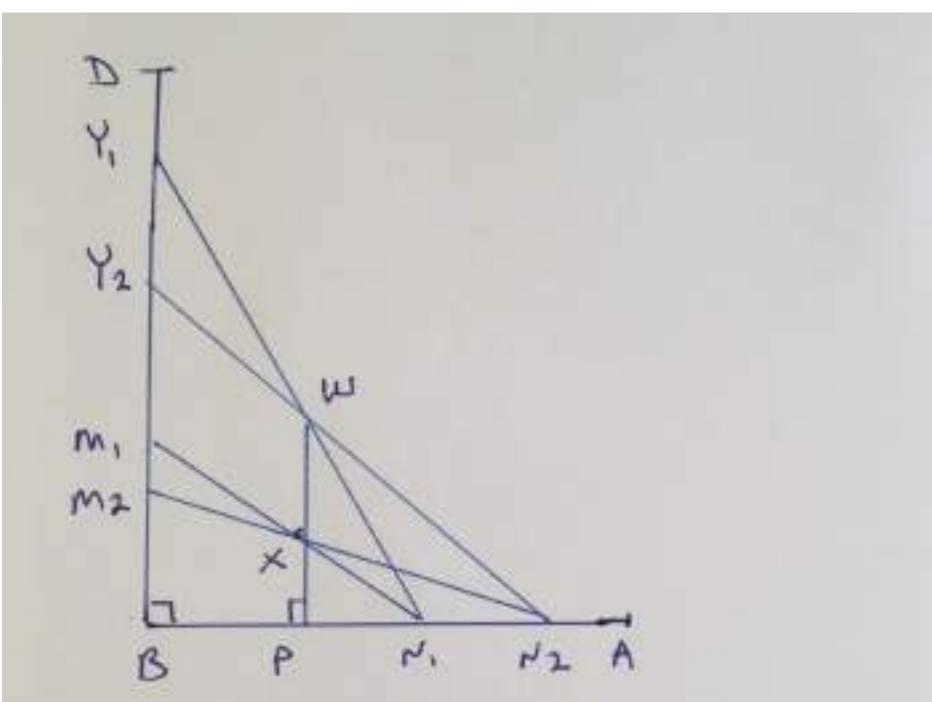
BY/DB = ZB/EZ

since PX and EZ/EB are also known:

PX/GYN = (MB/YN) = EZ/EB

allows us to calculate the length of GYN, and we can then find (N) by inserting the calculated length GYN within the right angle ∠XPb through Y.

For any given calculated value of YWN, draw the maximum number of line segments, (two),
$Y_1WN_1 = Y_2WN_2$
though W, within the right angle ∠DBA to find both N_1 and N_2 for the image rays through X.



For any given calculated value of GYN, draw the maximum number of line segments, (two),
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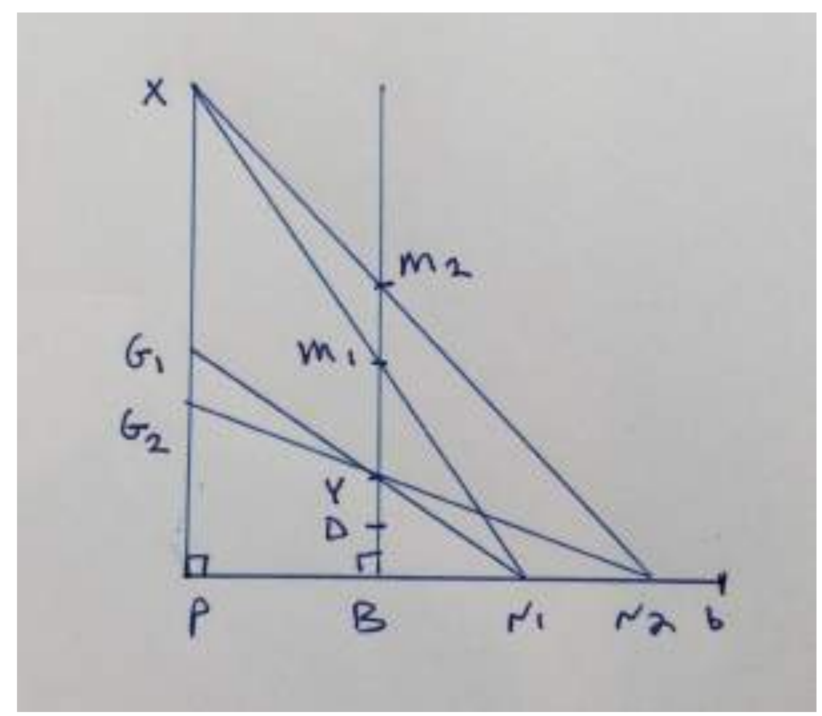
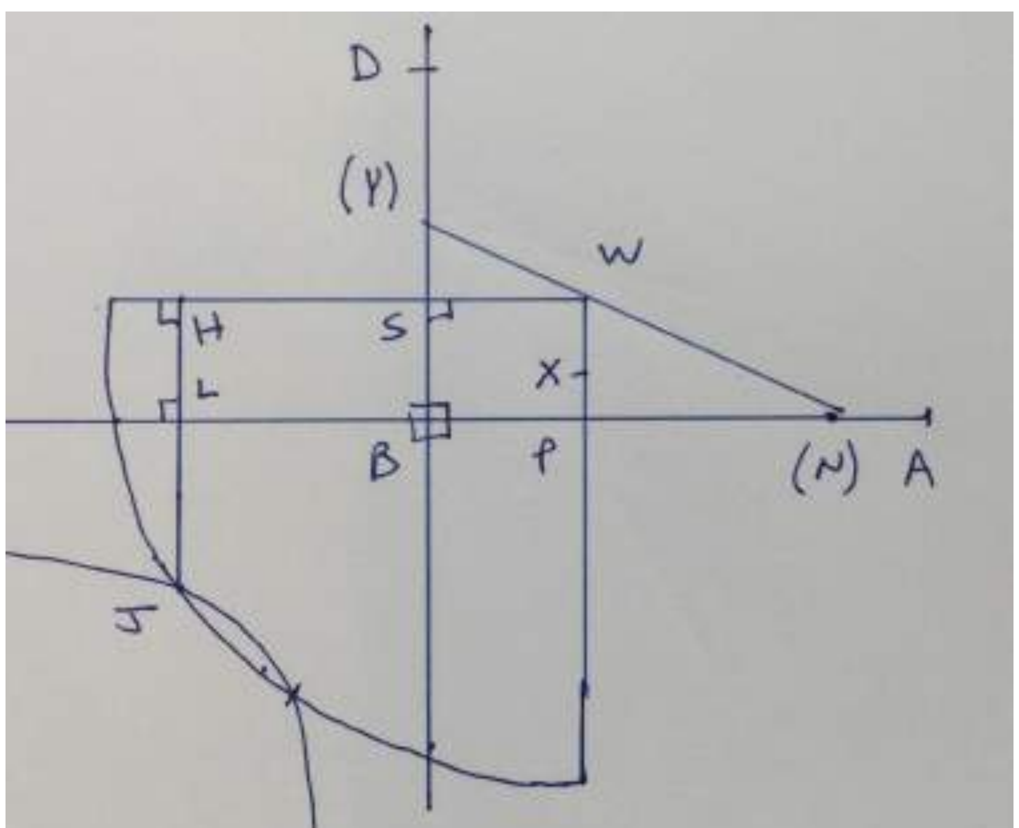


To do this, consider the given point W to be on a reference hyperbola where:

(LB)LJ = (BP)PW

LB/PW= BP/LJ

and draw its opposite arm as shown:

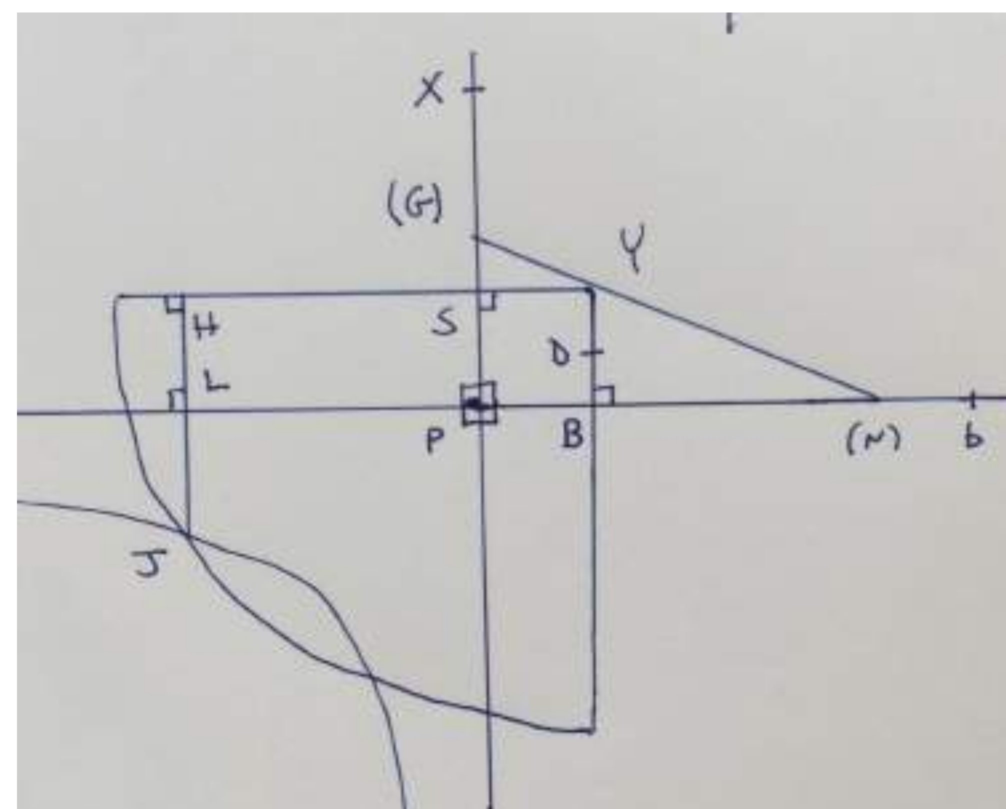


To do this, consider the given point Y to be on a reference hyperbola where:

(LP)LJ = (BP)BY

PL/BY= BP/LJ

and draw its opposite arm as shown:



If we make radius WJ
equal the (calculated)
YWN, and construct:
P(N) = BL,   then:
(Y)W(N) = WJ = YWN
because:

by construction:
P(N)/PW = BP/LJ

but:
P(N)/PW = SW/S(Y) so:
SW/S(Y) = BP/LJ.

If we make radius YJ
equal the (calculated)
GYN, and construct:
B(N) = PL,   then:
(G)Y(N) = YJ = GYN
because:

by construction:
B(N)/BY = BP/LJ

but:
B(N)/BY = SY/S(G) so:
SY/S(G) = BP/LJ.

And since SW = BP:
S(Y) = LJ

S(Y) + SB = LJ + HL
B(Y) = HJ

and by construction:
B(N) = LP

since LP = HW:
B(N) = HW.

Therefore the right triangle Δ(N)B(Y) equals the right triangle ΔWHJ, so:
WJ = (Y)W(N) = YWN

And since SY = BP:
S(G) = LJ

S(G) + SP = LJ + HL
P(G) = HJ

and by construction:
P(N) = LB

since LB = HY:
P(N) = HY.

Therefore the right triangle Δ(N)P(G) equals the right triangle ΔYHJ, so:
YJ = (G)Y(N) = GYN

Since the radius WJ cuts the hyperbola at a maximum of two points J_1 and J_2, both $(Y_1)W(N_1)$ and $(Y_2)W(N_2)$ can be found using this reference hyperbola.

Since the radius YJ cuts the hyperbola at a maximum of two points J_1 and J_2, both $(G_1)Y(N_1)$ and $(G_2)Y(N_2)$ can be found using this reference hyperbola.

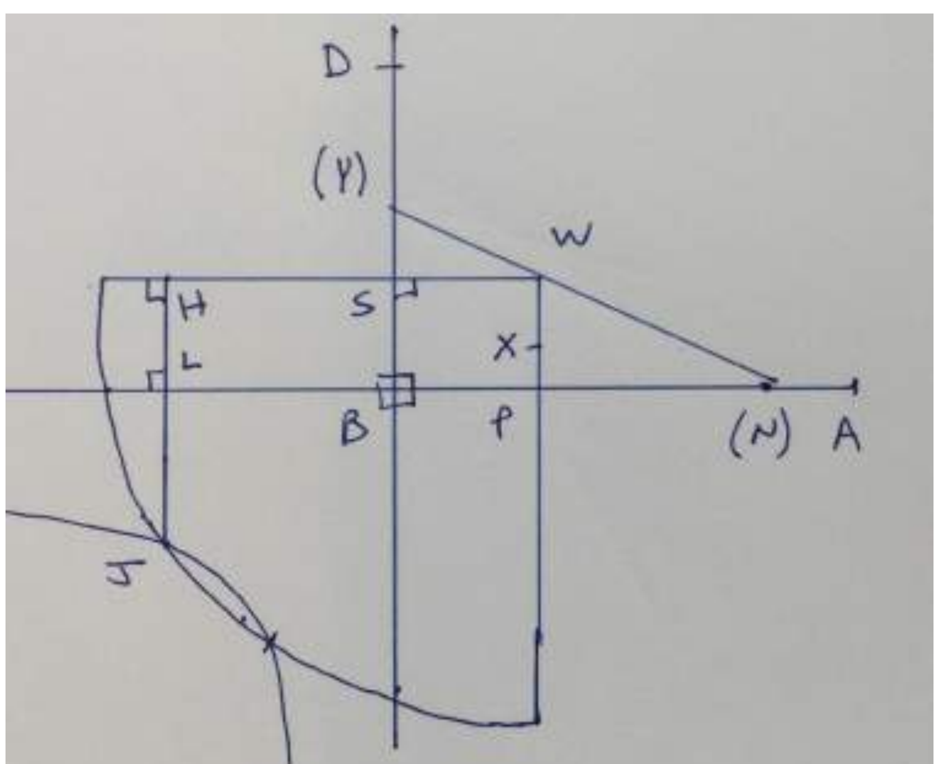
Locating a clear tangential image *not* on a perpendicular containing the object

X is the clear tangential image of object D only when N_1 and N_2 overlap, which occurs when $Y_1WN_1 = Y_2WN_2$ represents the line segment of minimum length through W within the right angle ∠DBA.

X is the clear tangential image of object D only when N_1 and N_2 overlap, which occurs when $G_1YN_1 = G_2YN_2$ represents the line segment of minimum length through Y within the right angle ∠XPb.

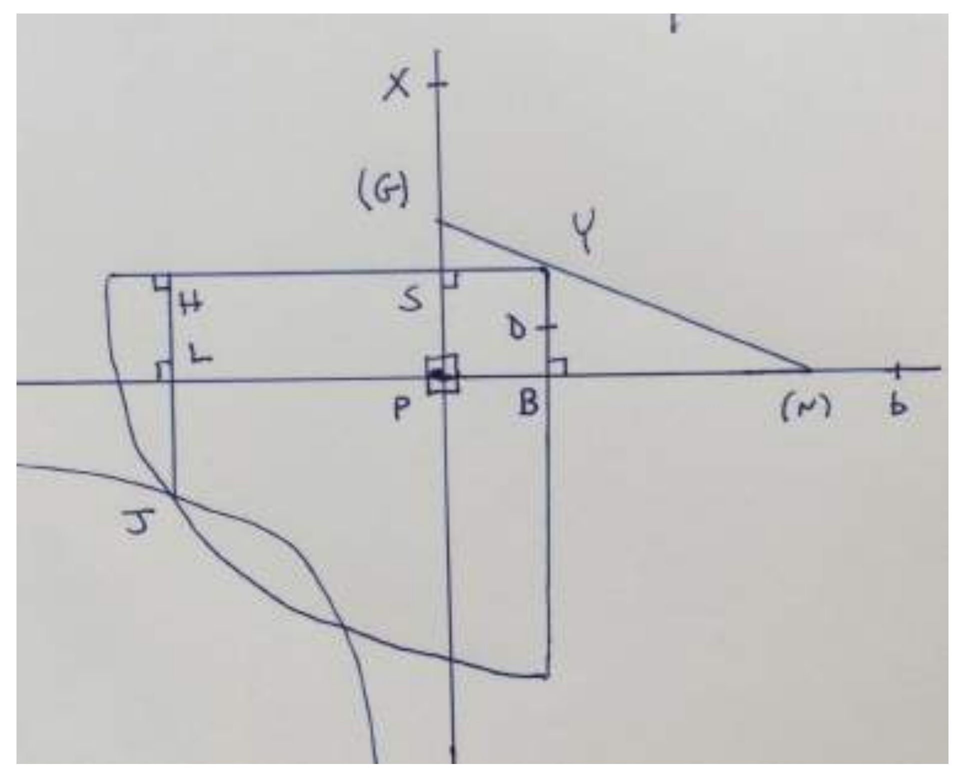
This minimum length equals the radius WJ of the reference circle when it is just tangent to the reference hyperbola. In this way, a position of $(Y_c)W(N_c)$ can be found that produces a clear image X_c, when:

$PW/PX = DB/DE$ and:
$DB/Y_cWN_c = ED/EB$



This minimum length equals the radius YJ of the reference circle when it is just tangent to the reference hyperbola. In this way, a position of $(G_c)Y(N_c)$ can be found that produces a clear image X_c, when:

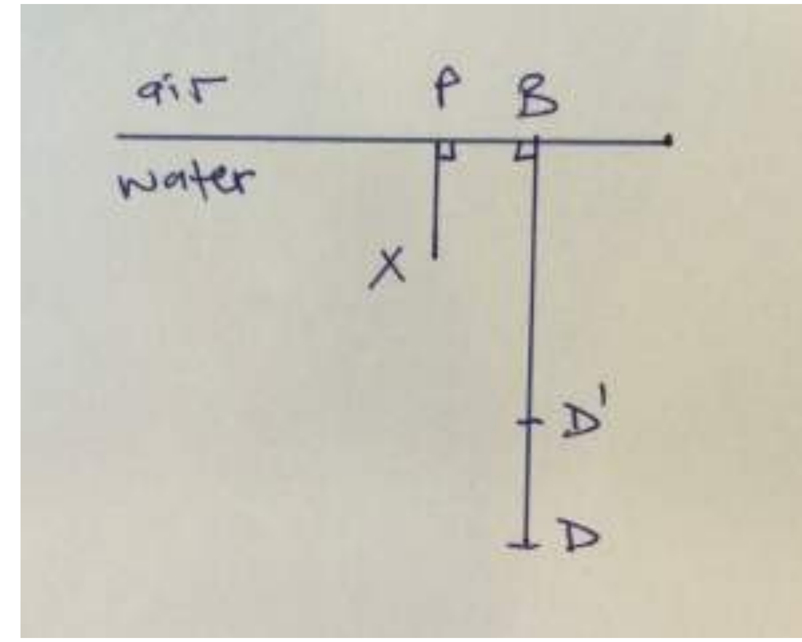
$BY/DB = ZB/EZ$ and:
$PX/G_cYN_c = EZ/EB$



The “Coin-in-fountain” Example

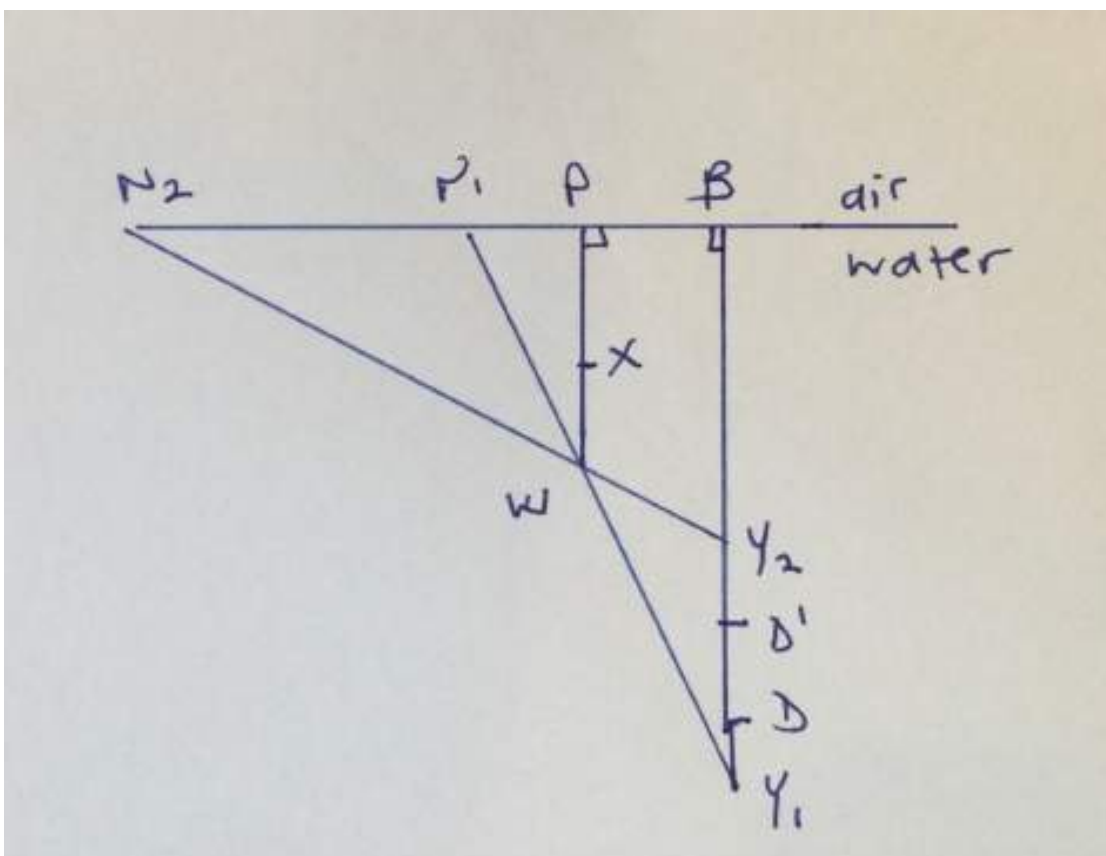
Imagine a coin at the bottom of a fountain in a city park. While sitting on the edge of the fountain looking down and forward towards the refracted image of the coin, we perceive its image location relative to the object using only the image rays within the plane connecting the two, the “tangential plane.” Isaac Barrow showed that this image does not lie directly above the object, but above and slightly towards us.

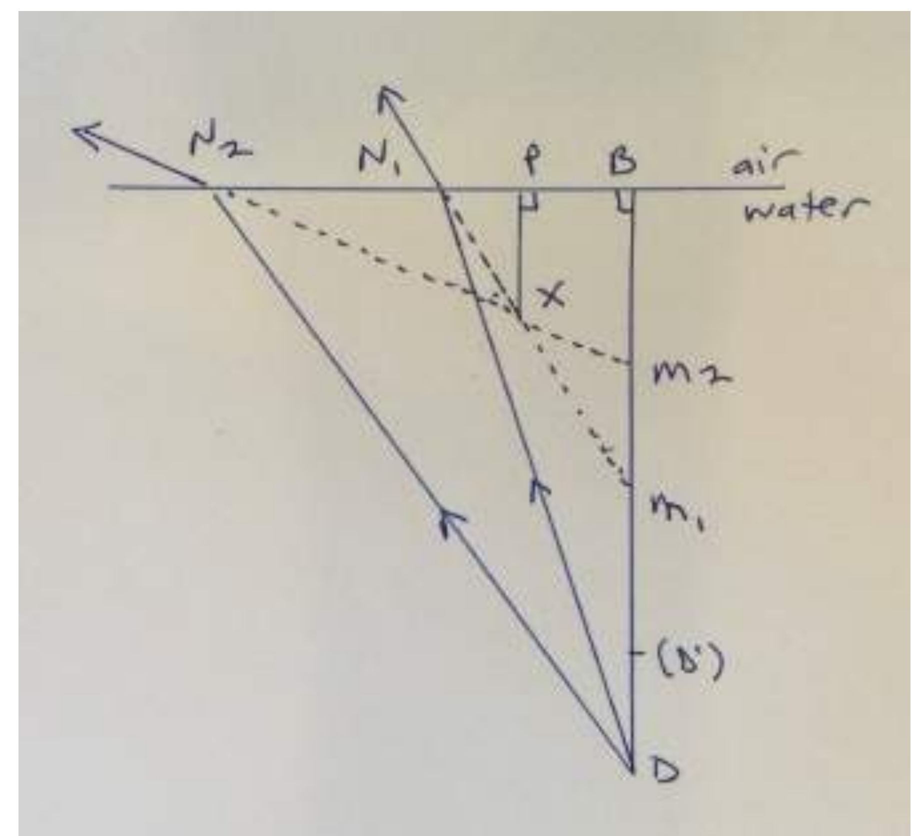
For example, let the object lie at point D; and its perpendicular distance from the water's surface, (DB), equal 4 inches. Let the object's image along the perpendicular be at D'; and its perpendicular distance from the water's surface, (D'B), equal 3 inches. This would occur since the index of refraction of water, (4/3), would have to equal DB/D'B.



As discussed, Isaac Barrow described a method of finding all the possible image rays through a specific image point X, without knowing their points of refraction along the surface of the water, or their intersection with the perpendicular DB.

He also showed that there can be a maximum of *two* image rays through X, since only two segments equaling his calculated constant YWN fit through W within the right angle at B.

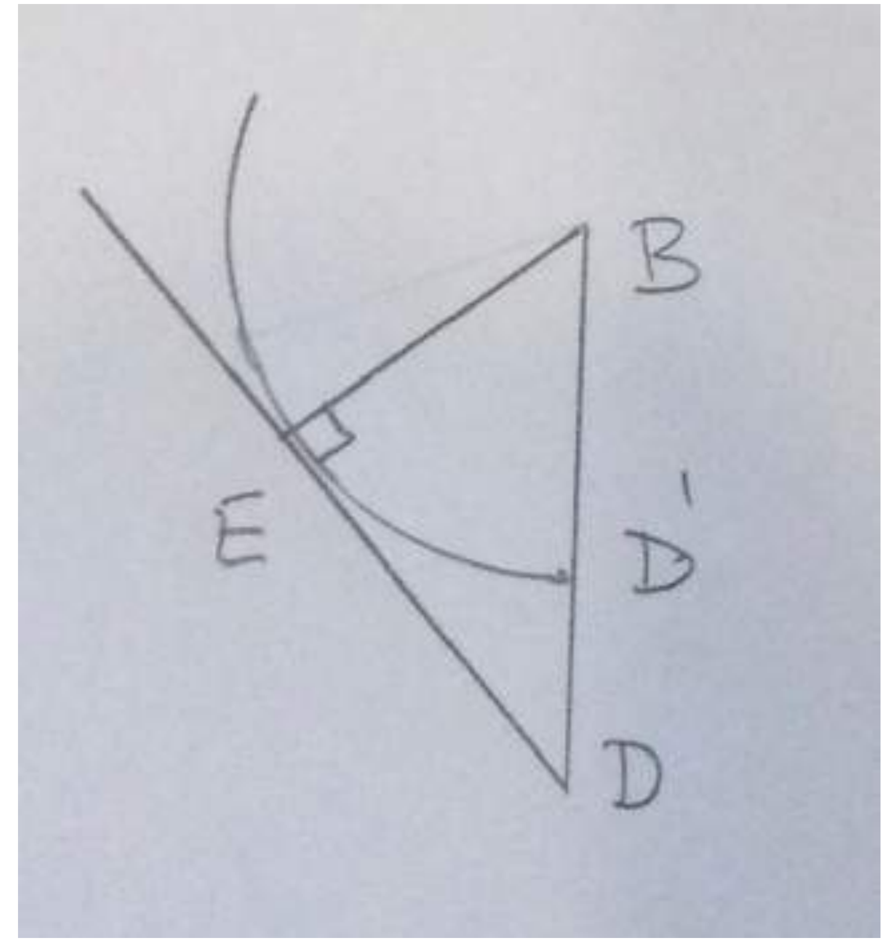


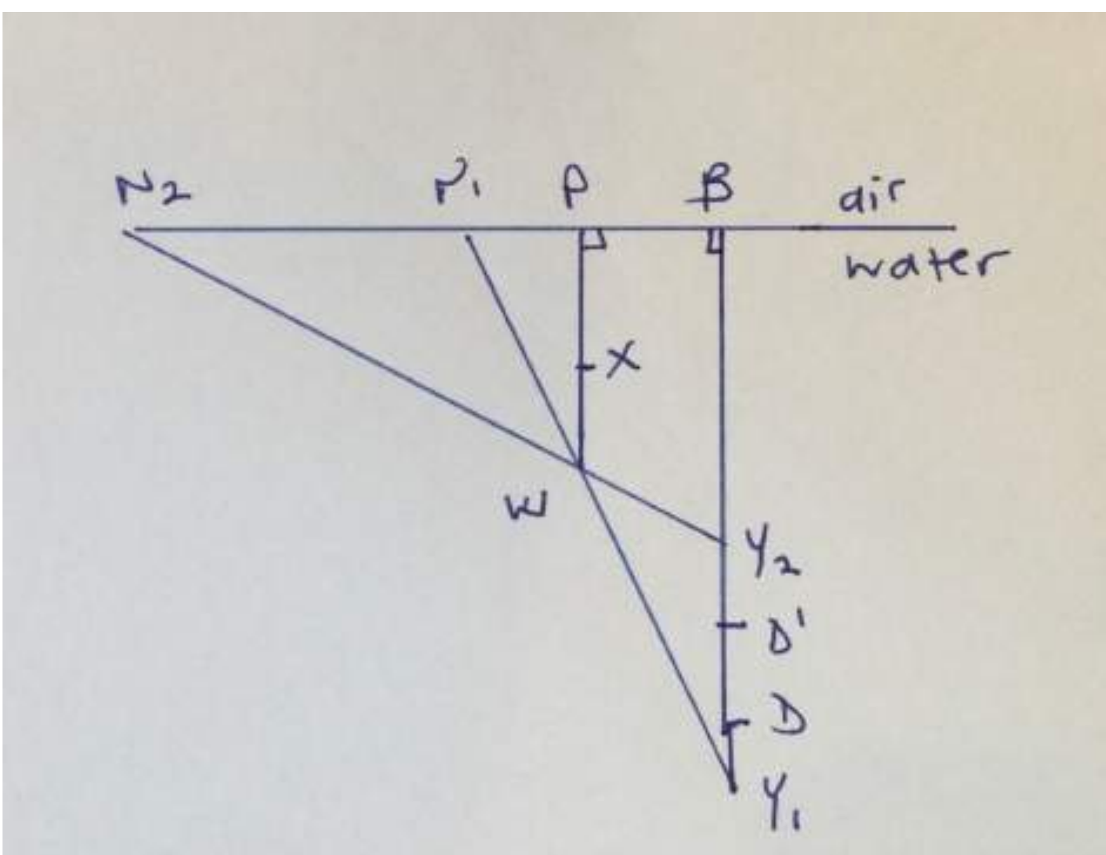


To locate the two possible image rays through X, we first locate point W using his calculation:

PW/PX = DB/DE = 1.5

When PX = 1 inch,
PW = 1.5 inches





$DB/DE = 4/\sqrt{7} = 1.5$

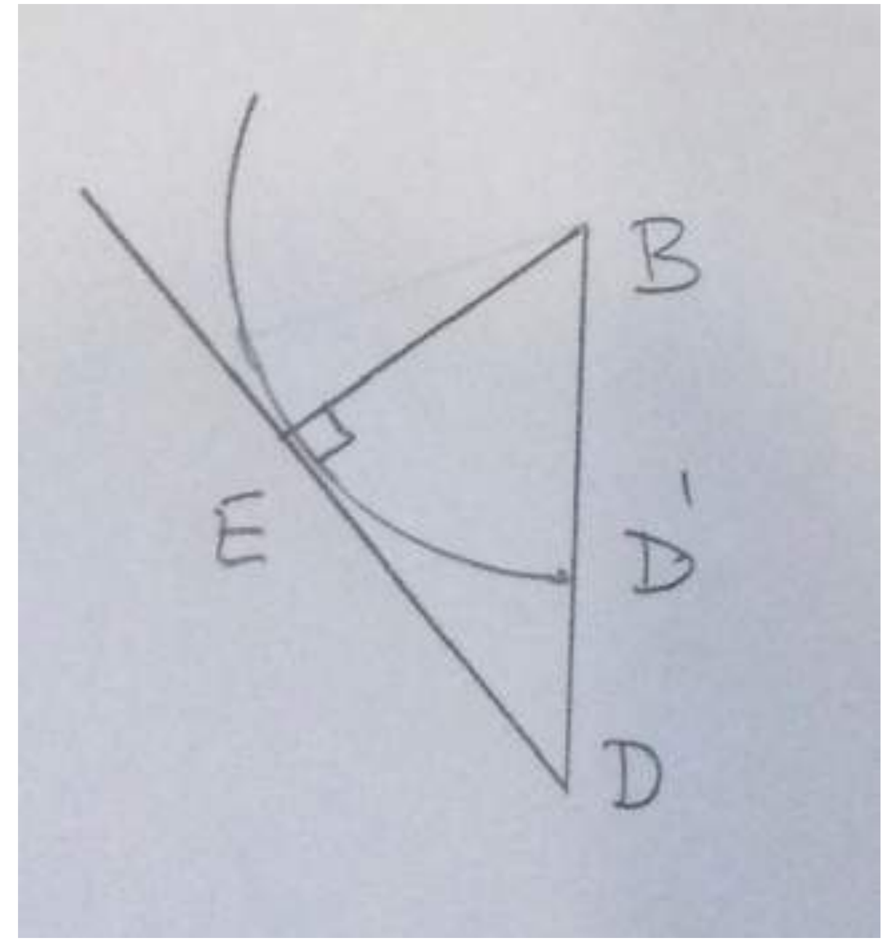
$ED/EB = \sqrt{7}/3 = 0.88$

We then calculate the constant reference line segment length YWN, (based solely on the distance DB and the index of refraction), using his formula:

DB/YWN = ED/EB

So YWN = DB/0.88

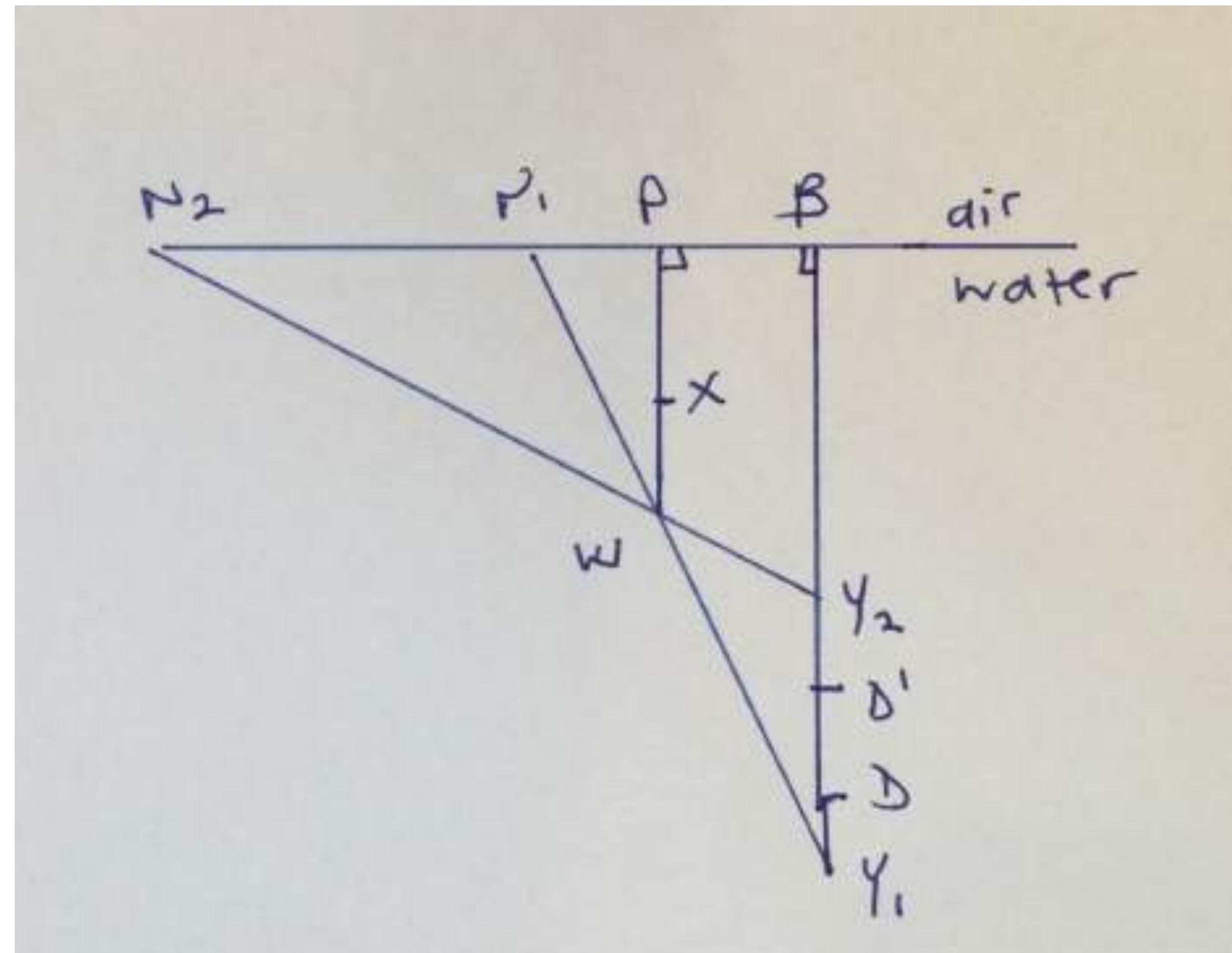
YWN = 4.54 inches



DB/DE = $4/\sqrt{7}$= 1.5

ED/EB = $\sqrt{7}/3$ = 0.88

Remember that Y_1WN_1 and Y_2WN_2 are neither object or image rays, but rather reference line segments that simply allow for the determination of points N_1 and N_2, which are the points of refraction that produce image rays through X.



Once these two image rays are drawn, we can measure them in inches to confirm that:

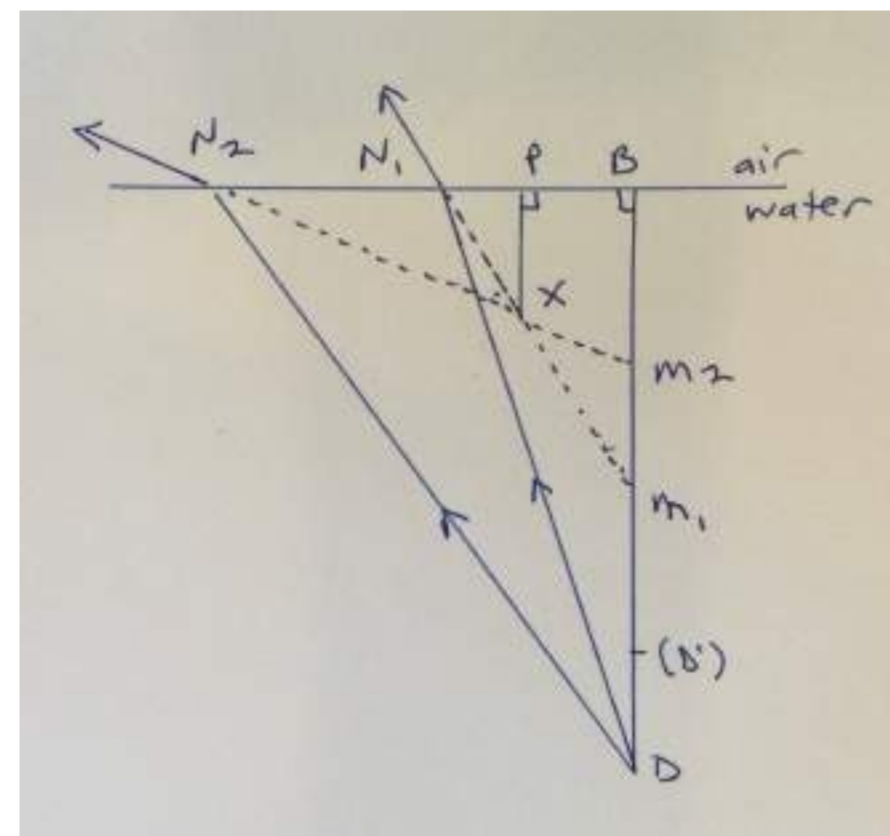
$DB/D'B = DN_1/M_1N_1 = DN_2/M_2N_2 = 4/3$

$DB/D'B = 4/3$

When, for example, PB = 1 inch:

$DN_1/M_1N_1 = 4.25/3.20 = 1.33$

$DN_2/M_2N_2 = 5.60/4.20 = 1.33$

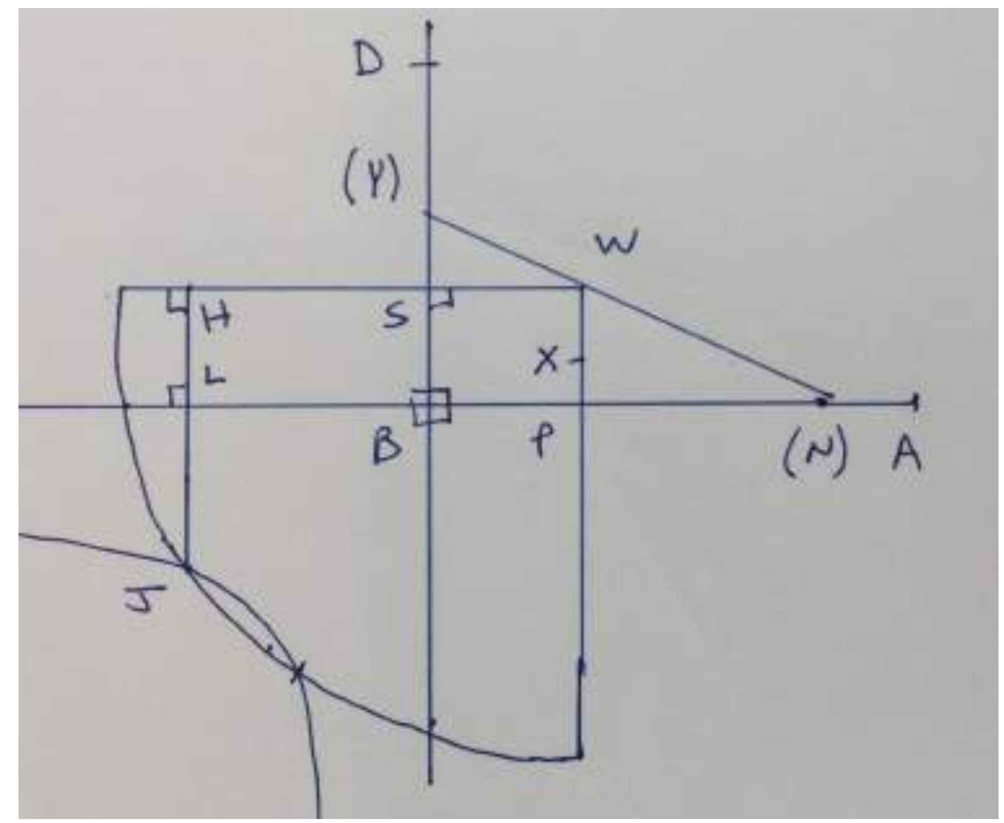


Isaac Barrow draws the line segments of calculated length YWN through the right angle at B by using a reference hyperbola where:

(LB)LJ = (BP)PW

and by making: PN = BL

so that: WJ = YWN



When the reference circle with radius WJ intersects the reference hyperbola at a *single* point J, the reference segment length YWN is at a minimum, N_2 overlaps N_1, and X is a clear image. The position of the minimum YWN can be then found by simply making PN = BL.