Crossed Parabolic Cylinder Meridional Maximum Refraction

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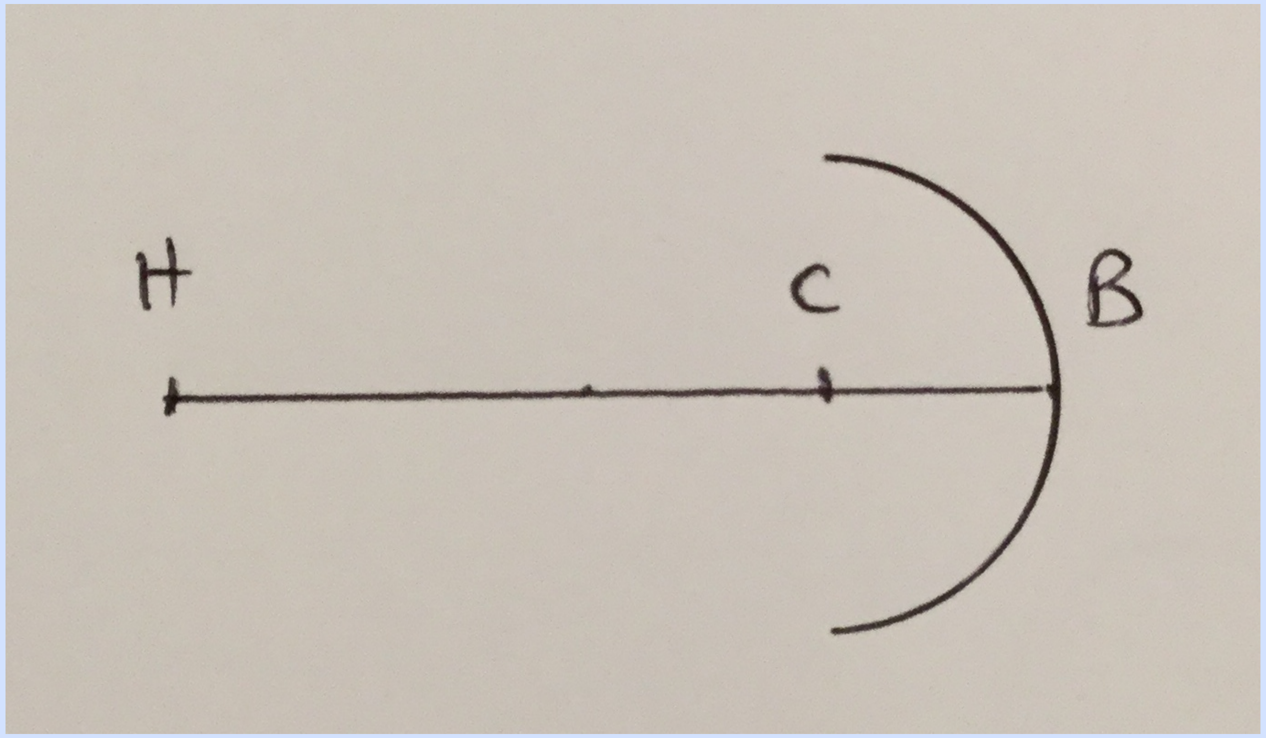
It is often useful to know the meridian of maximum axial refraction when combining the effects of two spherical cylinders at an oblique axis. To do this, we need to describe how their axial radii of curvature change with various meridional cross sections, and find expressions of those axial radii of curvature that are additive in terms of refraction. We then need to find the maximum sum of those expressions in terms of the meridional axis.

Meridional cross sections of a spherical cylinder are ellipses, (until they become parallel lines along the cylinder axis). Finding the axial radii of these ellipses would be difficult. Assuming a spherical cylinder is a parabolic cylinder, (and assuming cross sections of parabolas are parabolas until they become a line along the cylinder axis), allows for a much simpler determination of the axial radii of curvature of meridional cross sections.

This course works with these assumptions in order to provide approximations of axial radii of curvature for meridional cross sections of spherical cylinder. It also then uses expressions of these axial radii of curvature that are additive in terms of refraction, and demonstrates how to find the maximum sum of those expressions in terms of the meridional axis.

With any axial radius of curvature **CB**, and index of refraction ℝ, the axial image of a distant object lies at **H** when:



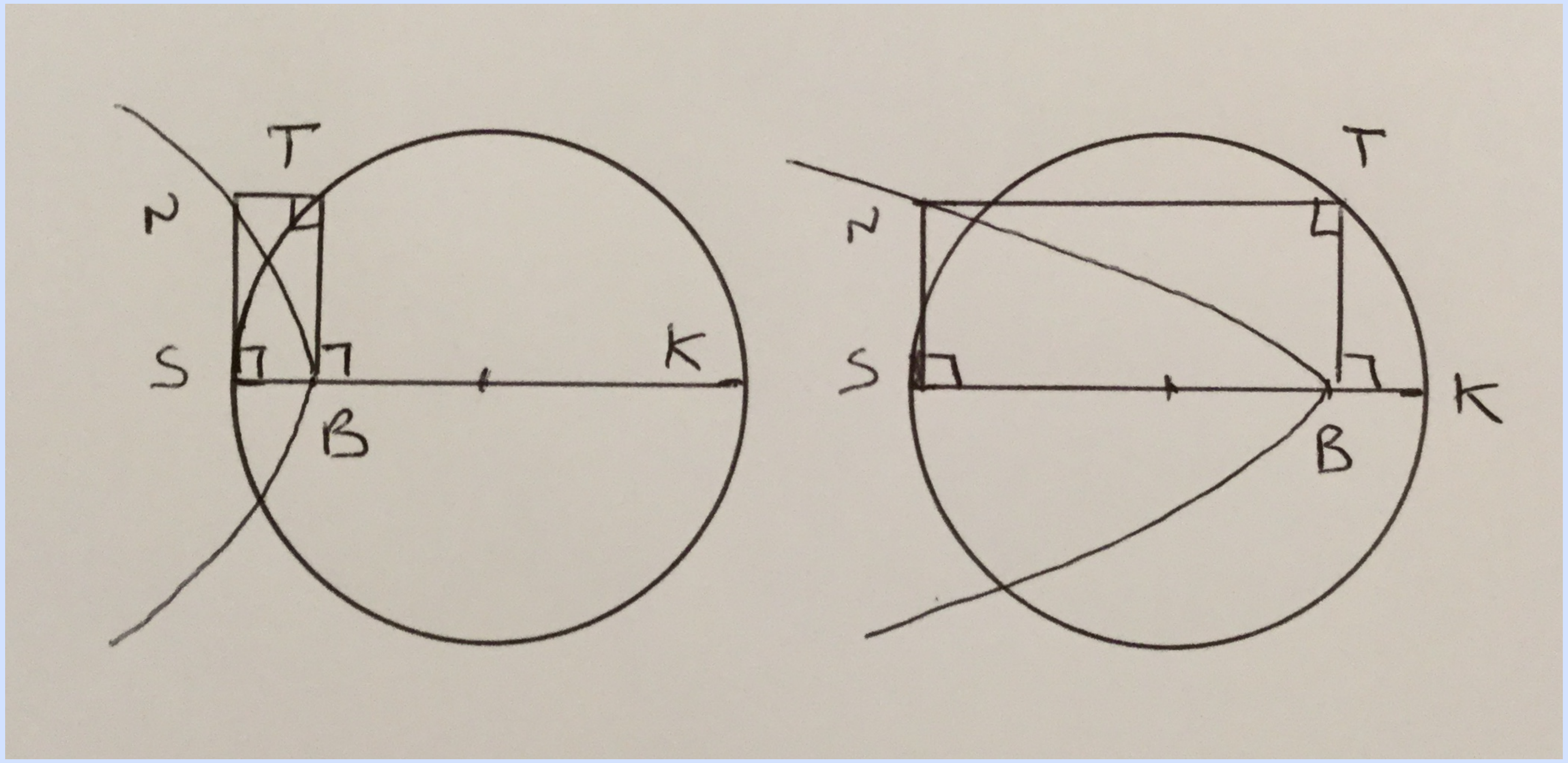
ℝ = HB/HC

The axial refractive effects of compound refractive surfaces at **B** are additive only as their refractive "powers," which equal:

$$\frac{\mathbb{R}}{HB} = \frac{1}{HC} = \frac{(HB - HC)/HC}{CB} = \frac{(\mathbb{R} - 1)}{CB}$$

All parabolas have the same shape, in the same way that all circles have the same shape. However, while circles have a single (internal) determining constant, the radius of curvature, parabolas have both a determining constant internal and external to the curve, and can be defined by either.

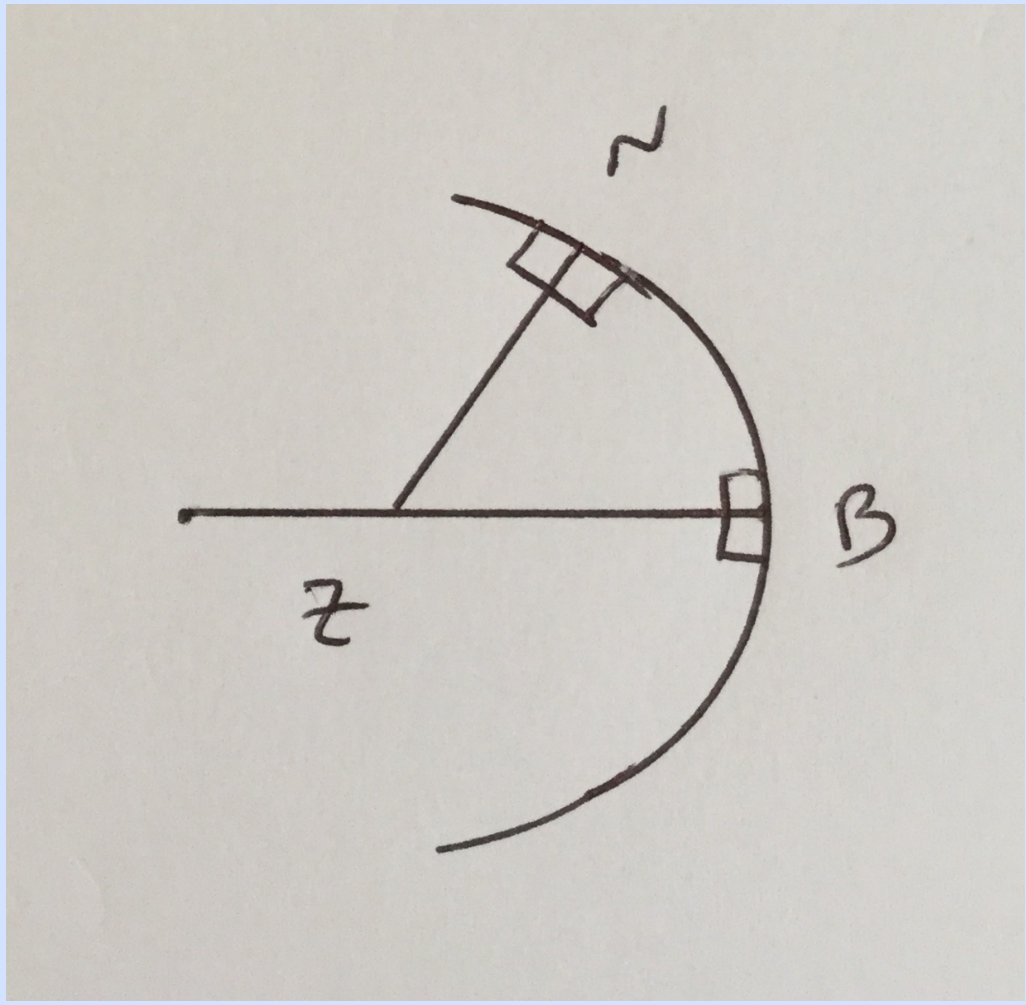
For example, a parabola's external determining constant equals **BK** when:



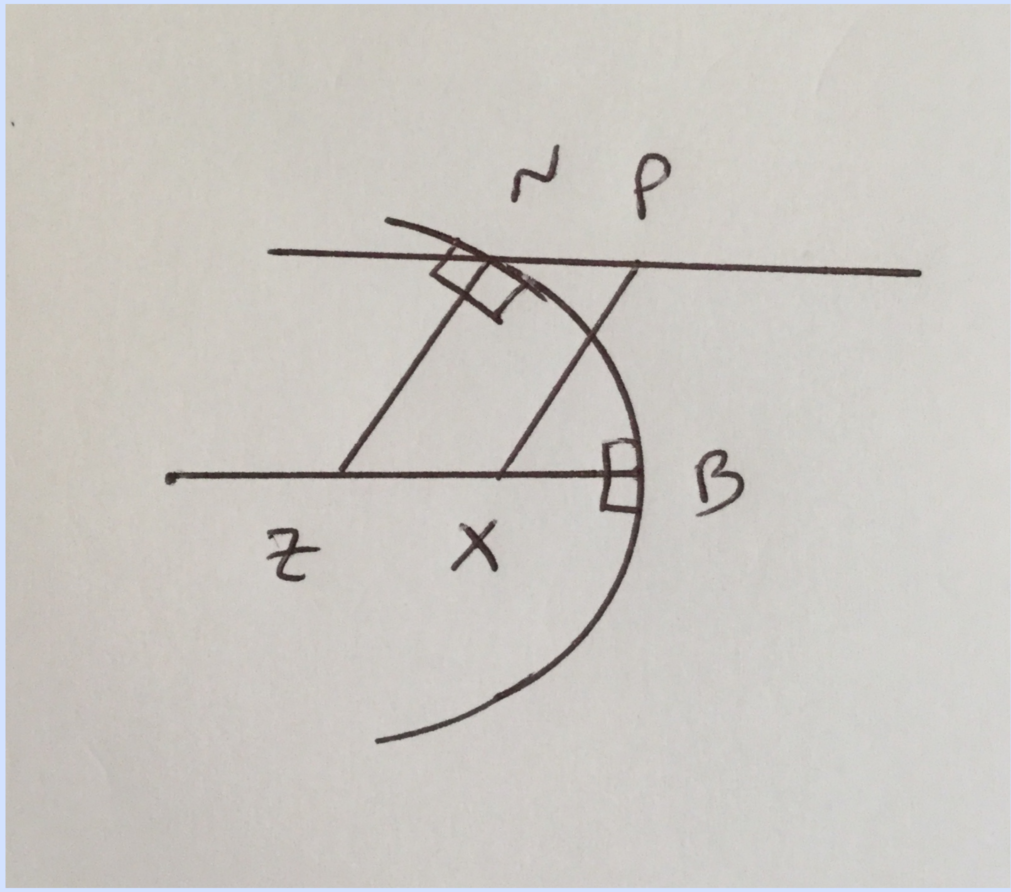
$$\frac{SB}{BT} = \frac{BT}{BK}$$

[2(**SN**) equals the sagitta corresponding to the sagittal depth **SB**].

We can set up the necessary off-axis conditions to determine a parabola's axial center of curvature in terms of its internal determining constant **XB**, by involving **ZN** in the geometric solution for **XB**.



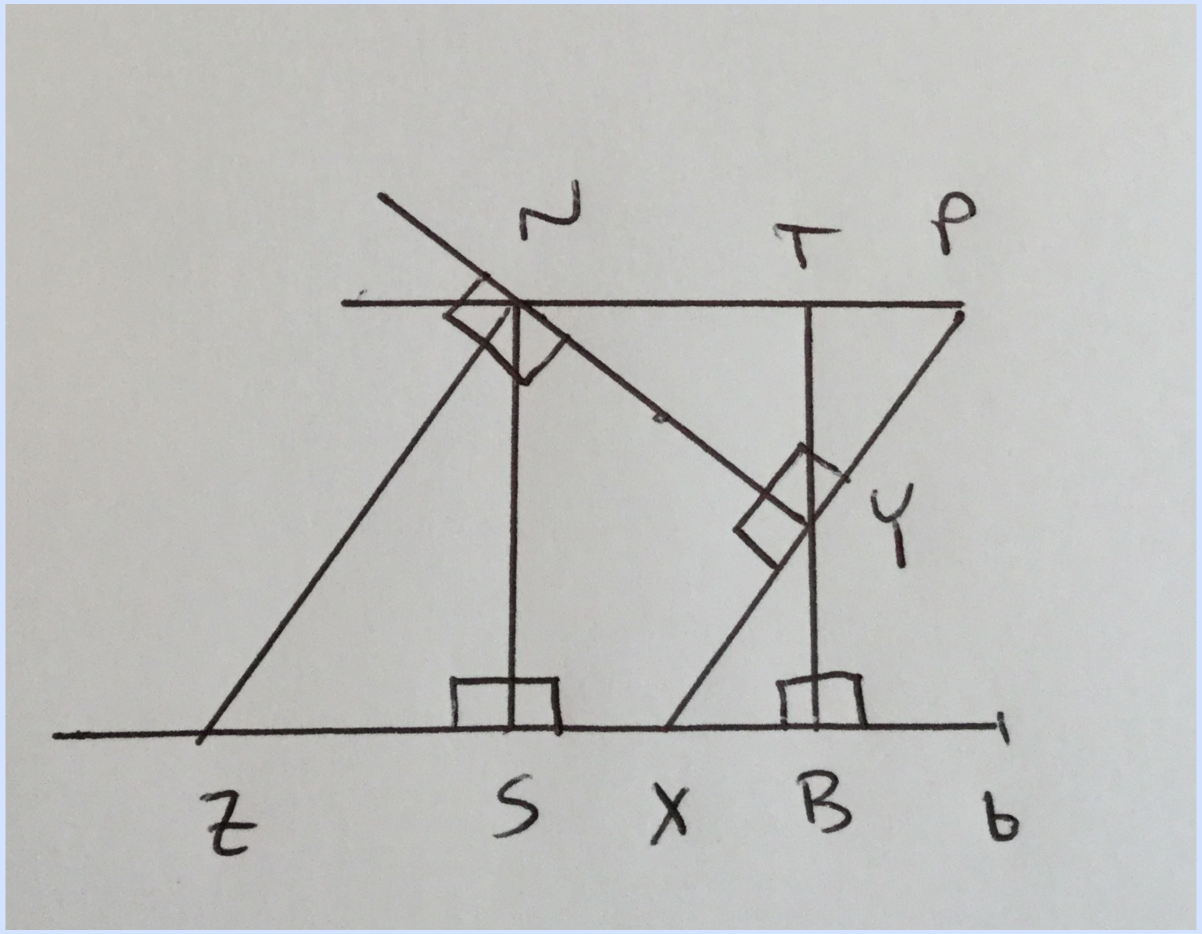
In order to keep the determining geometrical relationships axial as **N** ⇒ **B**, they should also depend on line **NP** being parallel to the axis, and **XP** being parallel to **ZN**.



We know **X** lies between **Z** and **B**, since parabolas flatten in their periphery.

Since as **N** $\Rightarrow$ **B**, **Z** $\Rightarrow$ **C** by definition, and since **XP** = **ZN**, **P** will remain external to the curve, and **X** can therefore not be its axial center of curvature, but must instead lie somewhere along **CB**.

In order to maintain **ZN** perpendicular to the parabola at **N** as **N** ⇒ **B**, the same geometrical relationships must exist that allow for that when **N** lies at **B**.



In other words:

YP = YX and
Bb = BX so
CB = 2(XB)

Since:

$$\frac{TN}{TB} = \frac{TN}{2(TY)} = \frac{YB}{2(XB)} = \frac{YB}{CB} = \frac{TB}{2(CB)}$$

We know the external determining constant **BK** equals 2(**CB**), and the internal determining constant **XB** equals (**CB**)/2.

Axial refracting power equals $\dfrac{(\mathbb{R} - 1)}{CB}$

Since for a parabola:

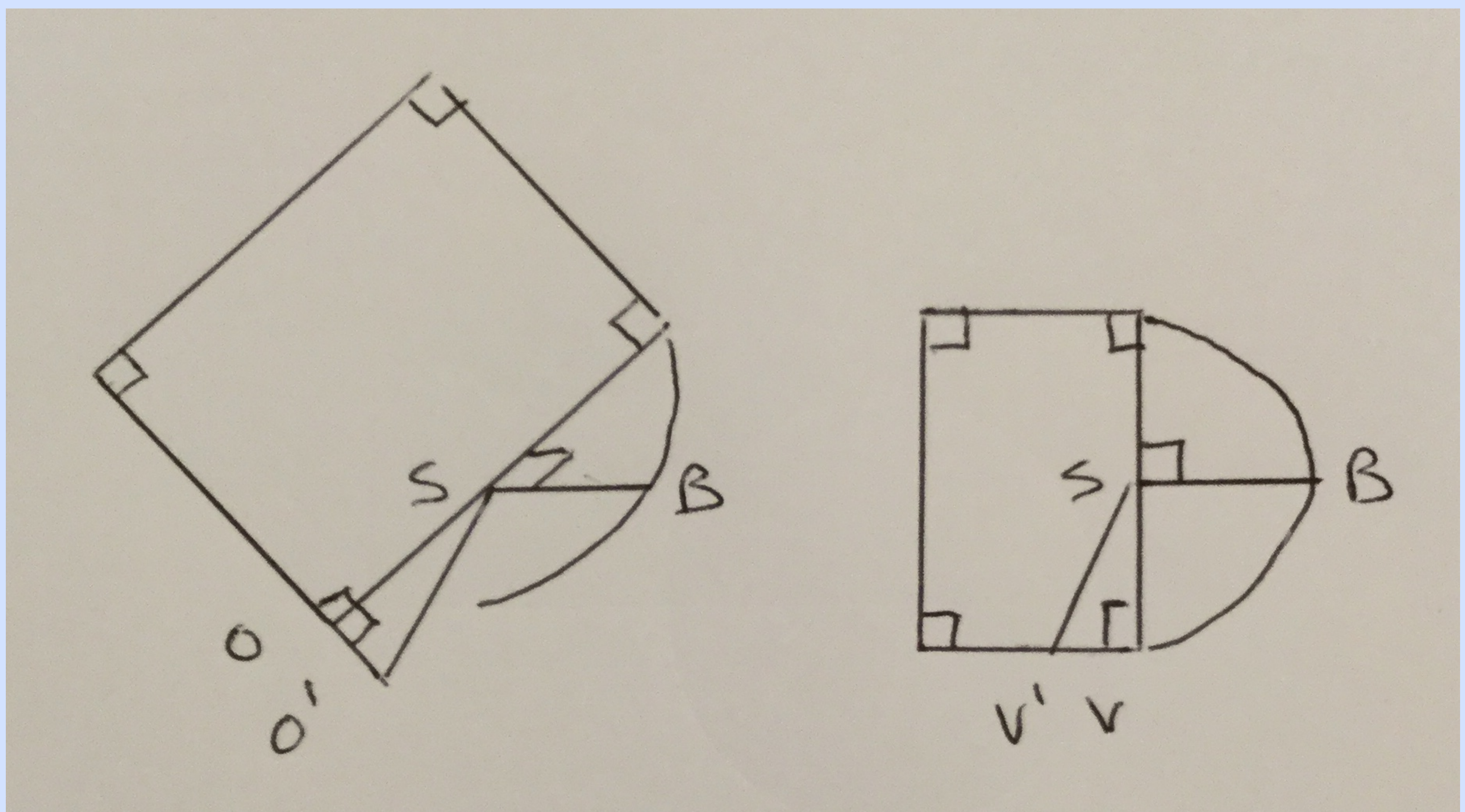
$$\frac{SB}{SN} = \frac{SB}{TB} = \frac{TB}{2(CB)}$$

If $\mathbb{R} = 1.5$

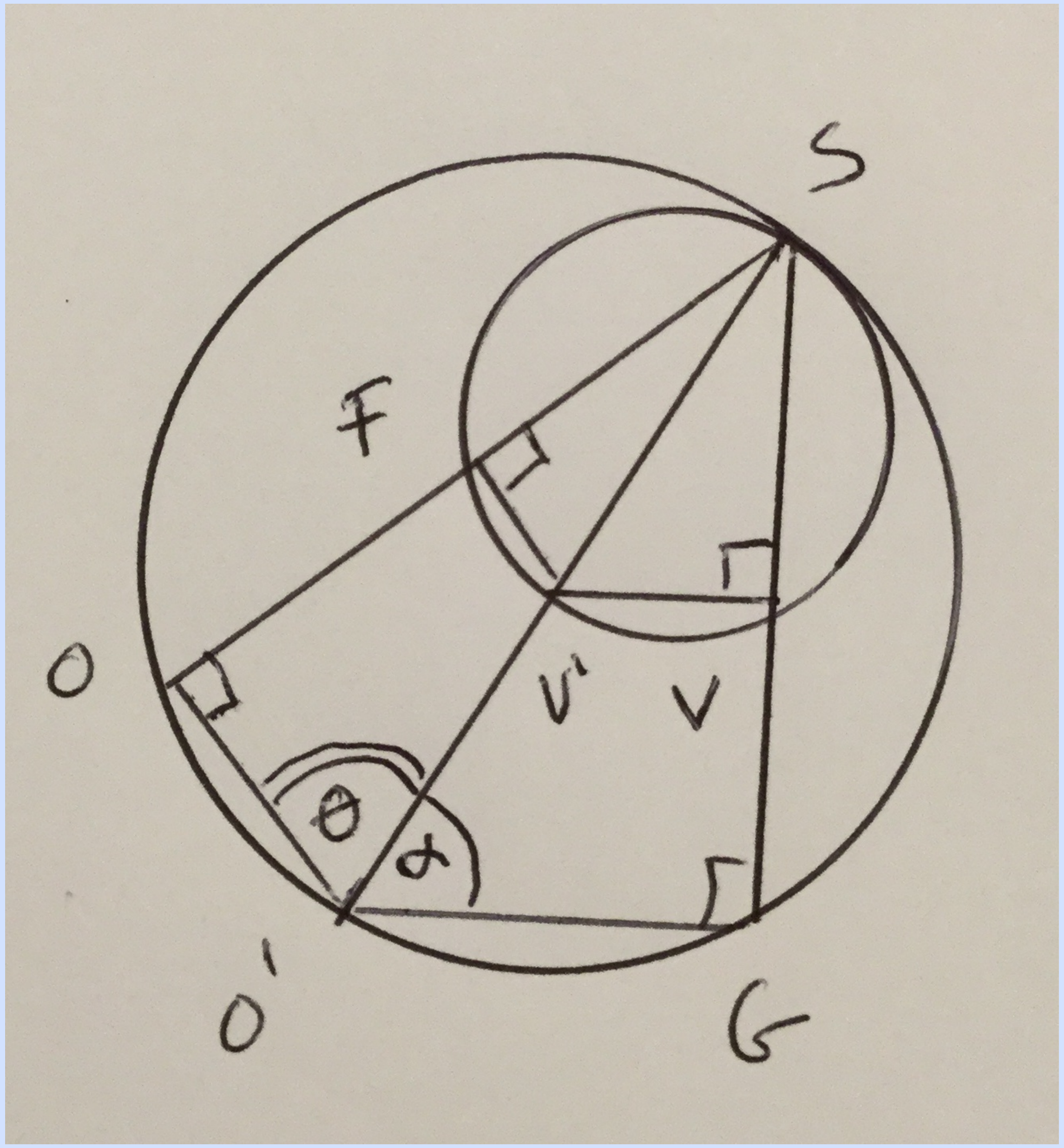
The axial refracting power of a parabola equals:

$$\frac{1}{2(CB)} = \frac{SB}{SN^2} = \frac{1}{BK}$$

When 2(**SO**) equals the minimum sagitta of an oblique parabolic cylinder, and when with equal sagittal depth **SB**, 2(**SV**) equals the minimum sagitta of a more highly curved parabolic cylinder with a horizontal axis:



Keeping Δ**OSV** constant, as we rotate circle **SOG** with variable diameter **SV'O'** around point **S**:

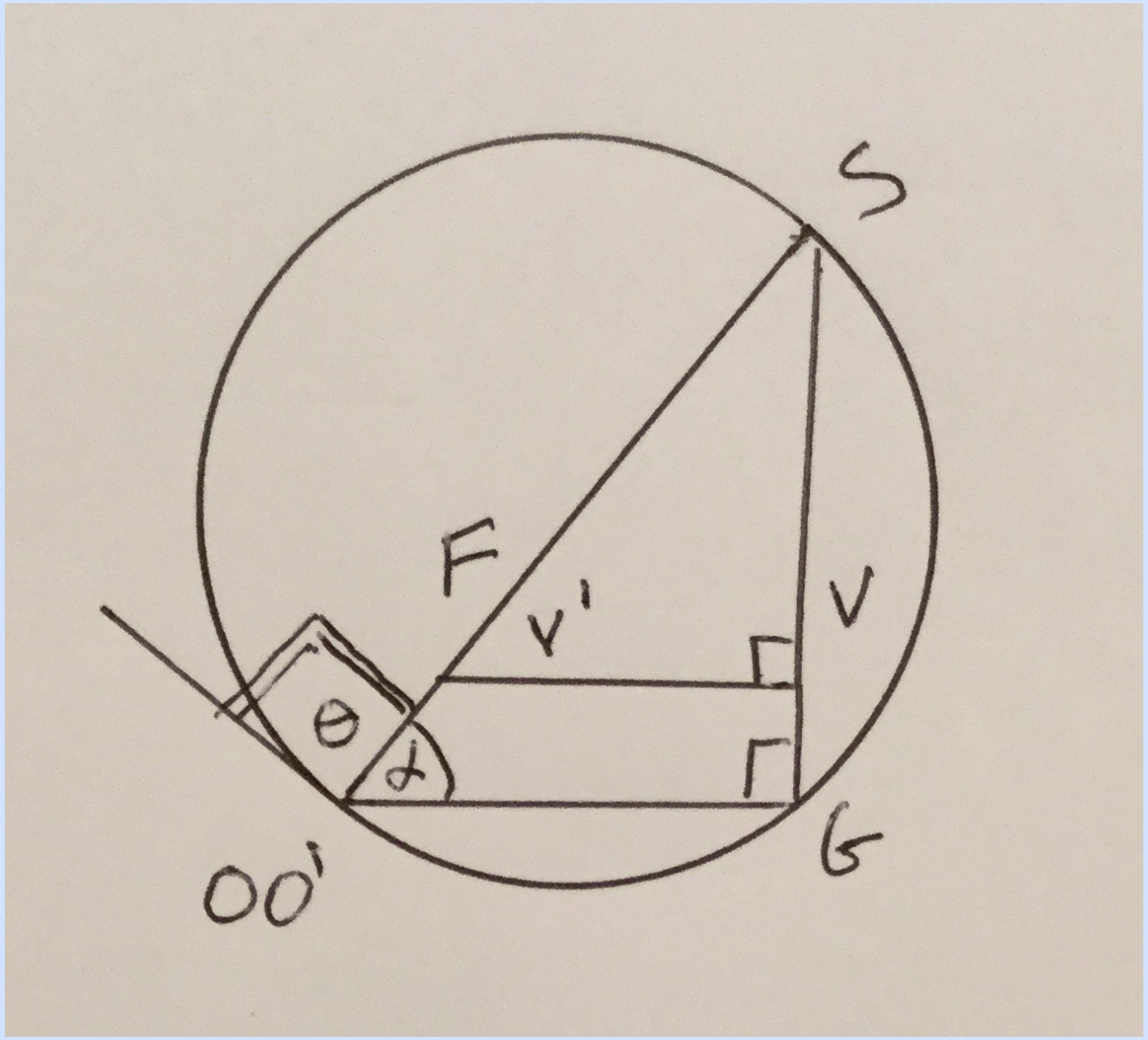


$\angle OO'G$ is constant because $\angle OSG$ is constant,

so $\Delta\theta = -\Delta\alpha$

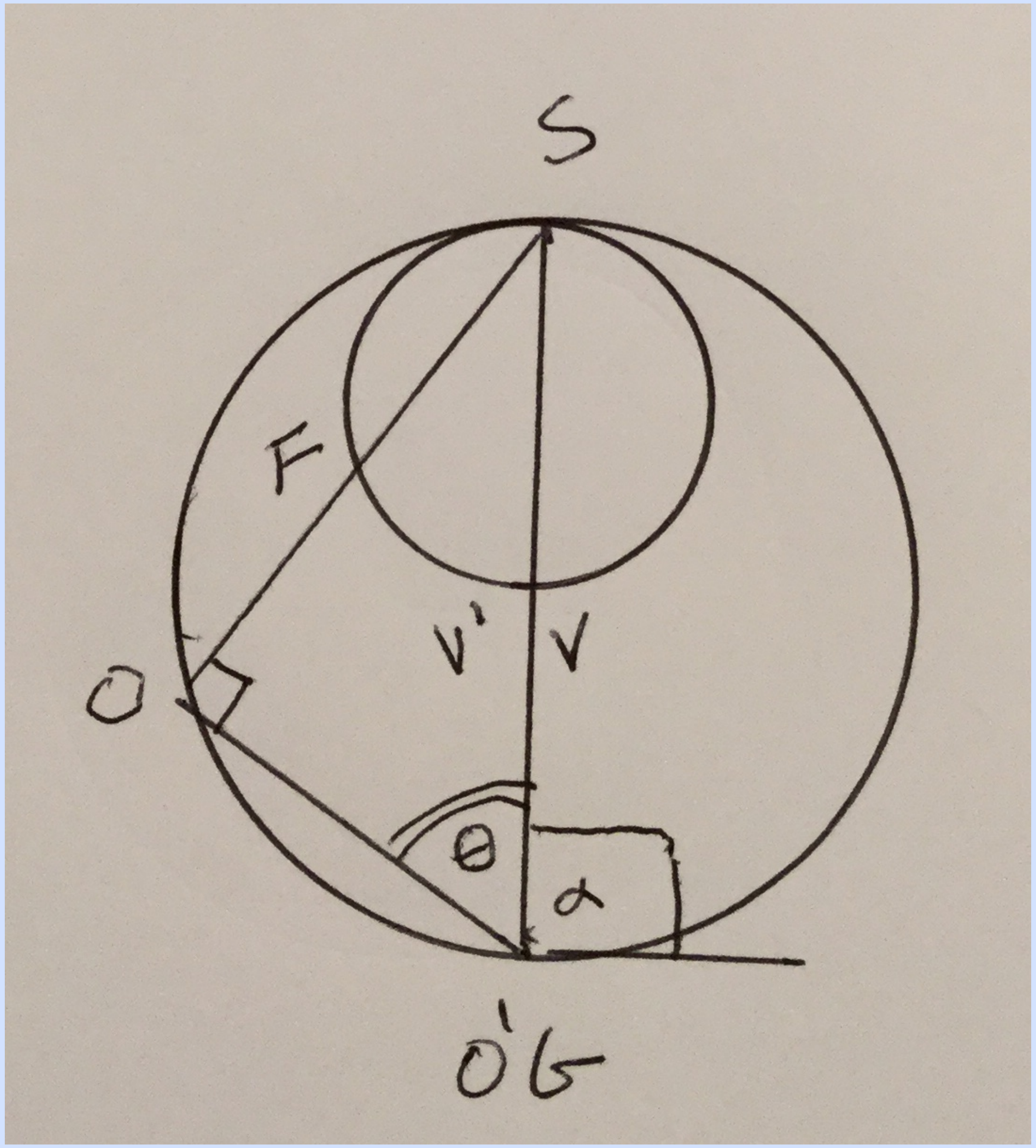
As O' ⇒ O

SV' increases more than SO' decreases



As V' ⇒ V

SO' increases more than SV' decreases



Since the sum (**SO' + SV'**) increases when either:

O' ⇒ **O**, or **V'** ⇒ **V**

there must be a specific **SV'O'** within **ΔOSV** producing a minimum sum (**SO' + SV'**),
which must be near where small rotations produce only minimal changes in (**SO' + SV'**).

Since as when one term of the sum (**SO' + SV'**) increases, the other always decreases, this process can be taken to its limits to determine the meridian with minimum (**SO' + SV'**) using:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta(SO') \quad = \quad \lim_{\Delta\alpha \Rightarrow 0} \Delta\ (SV')$$

However, the combined effects of refraction are additive only as refractive powers,
which, when $\mathbb{R} = 1.5$, equal:

$$\frac{SB}{(SO')^2} \quad \text{and} \quad \frac{SB}{(SV')^2}$$

Therefore, the meridian with the maximum combined effects of this refraction can be found using:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta \frac{SB}{(SO')^2} = \lim_{\Delta\alpha \Rightarrow 0} \Delta \frac{SB}{(SV')^2}$$

To solve this equation, all variables must be expressed in terms of the variables approaching zero, so:

$$\underset{\Delta\theta \Rightarrow 0}{\text{Limit}} \quad \Delta\, \frac{SB(SO/SO')^2}{(SO)^2} \quad = \quad \underset{\Delta\alpha \Rightarrow 0}{\text{Limit}} \quad \Delta\, \frac{SB(SV/SV')^2}{(SV)^2}$$

$$\underset{\Delta\theta \Rightarrow 0}{\text{Limit}} \quad \Delta\, \frac{(SB)\sin^2}{(SO)^2}\,\theta \quad = \quad \underset{\Delta\alpha \Rightarrow 0}{\text{Limit}} \quad \Delta\, \frac{(SB)\sin^2}{(SV)^2}\,\alpha$$

$$\frac{SB}{SO^2} \underset{\Delta\theta \Rightarrow 0}{\text{Limit}} \quad \Delta \sin^2\theta \quad = \quad \frac{SB}{SV^2} \underset{\Delta\alpha \Rightarrow 0}{\text{Limit}} \quad \Delta \sin^2\alpha$$

$$\frac{\lim\limits_{\Delta\theta \Rightarrow 0} \Delta \sin^2 \theta}{\lim\limits_{\Delta\alpha \Rightarrow 0} \Delta \sin^2 \alpha} = \frac{SO^2}{SV^2}$$

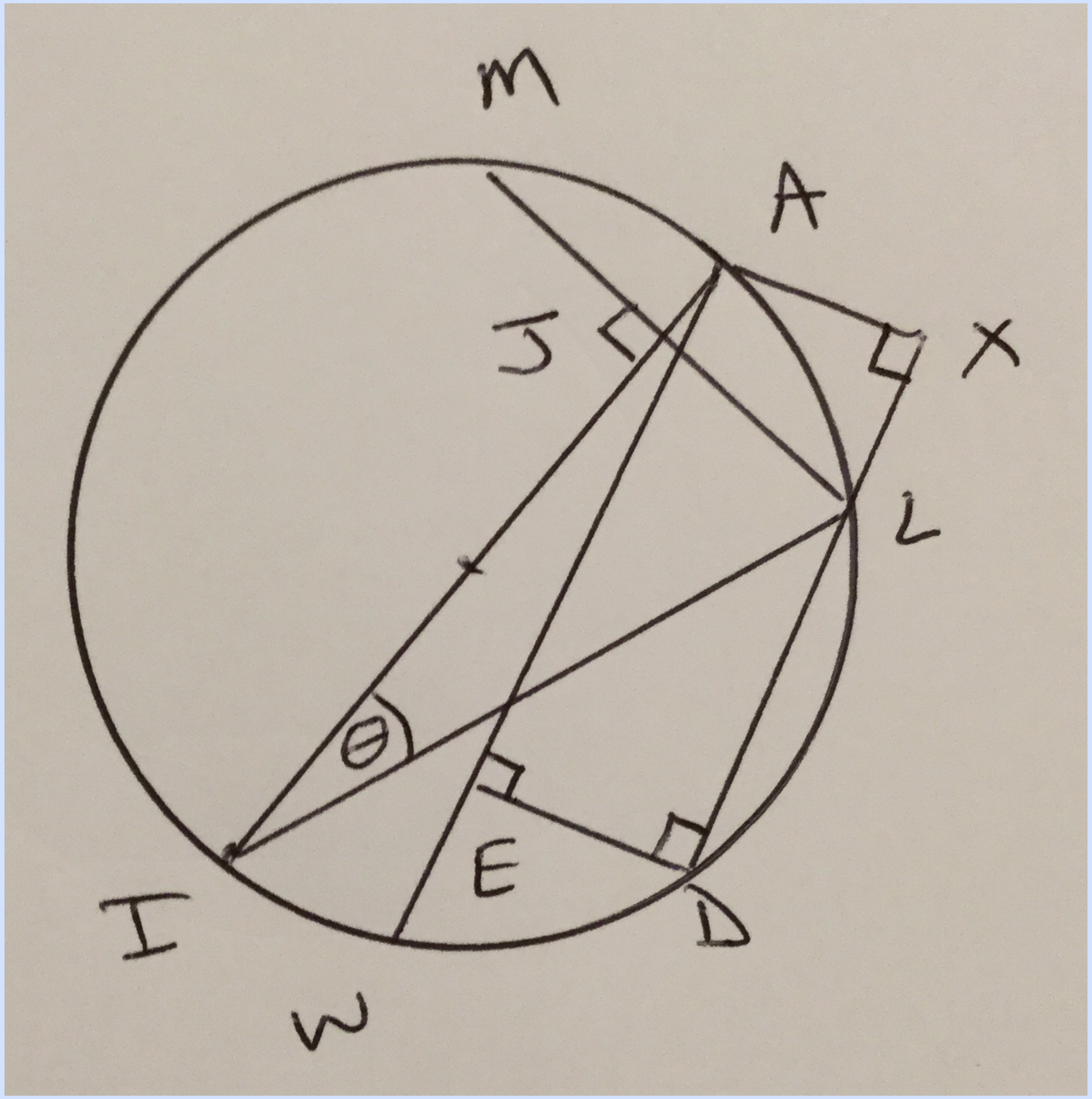
Solve for

$$\lim_{\Delta\theta \Rightarrow 0} \Delta \sin^2 \theta$$

on the reference circle:

AW ≥ LD // AW

$$\angle ALD = \sim\frac{\underline{AID}}{AI} \geq \sim\frac{\underline{AI}}{AI} = \pi$$



Establish the necessary functions of θ in terms of line segments and chords.

$$\theta = \sim\frac{AL}{AI} \quad ; \quad \sin^2\theta = \frac{AL^2}{AI}$$

$$\Delta\theta = \sim\frac{LD}{AI} \quad ; \quad \sin^2\Delta\theta = \frac{LD^2}{AI}$$

$$(\theta + \Delta\theta) = \sim\frac{ALD}{AI} \quad ; \quad \sin^2(\theta + \Delta\theta) = \frac{AD^2}{AI}$$

$$\cos\theta = \frac{IL}{AI} \quad ; \quad \cos(\theta + \Delta\theta) = \frac{DI}{AI}$$

$$\sin\theta = \frac{AL}{AI} = \frac{JL}{IL} \quad ; \quad \sin\theta\cos\theta = \frac{JL}{IL}\,\frac{IL}{AI}$$

$$2(\sin\theta\cos\theta) = \frac{ML}{AI} = \sin 2\theta$$

Then consider the following property of the cyclic quadrilateral circle **ALDW**: AD(LW) = AL(DW) + LD(AW)

$\Delta DIA \cong \Delta EWD \cong \Delta XLA$; $AD^2 = AL^2 + LD(AW)$

$AW = LD + 2(AL)\dfrac{LX}{LA}$; $AW = LD + 2(AL)\dfrac{ID}{IA}$

$$AD^2 - AL^2 = LD^2 + 2(LD)(AL)\frac{ID}{IA}$$

$AI\,[\sin^2(\theta + \Delta\theta) - \sin^2\theta] =$

$AI\,[\sin^2\Delta\theta] + 2(LD)(AL)\cos(\theta + \Delta\theta) =$

$AI\,[\sin^2\Delta\theta] + 2(LD)\,[(AI)\sin\theta]\,\cos(\theta + \Delta\theta)$

Divide both sides by **AI**:

$\sin^2(\theta + \Delta\theta) - \sin^2\theta = \sin^2\Delta\theta + 2(LD)\,\sin\theta\,\cos(\theta + \Delta\theta)$

$$\lim_{\Delta\theta \Rightarrow 0} \frac{\Delta(\sin^2\theta)}{\sim LD} = 2\sin\theta\,(\cos\theta) = \sin 2\theta$$

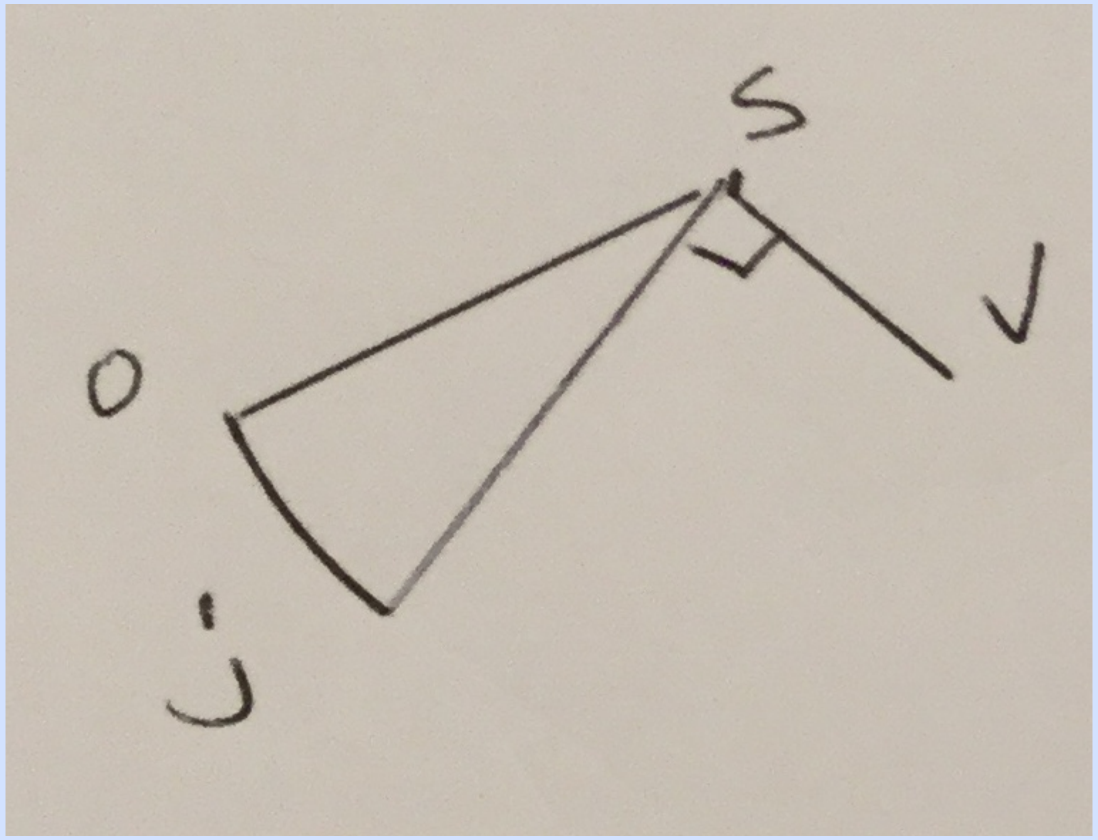
Therefore, the meridian with the maximum combined effects of refraction can be found using:

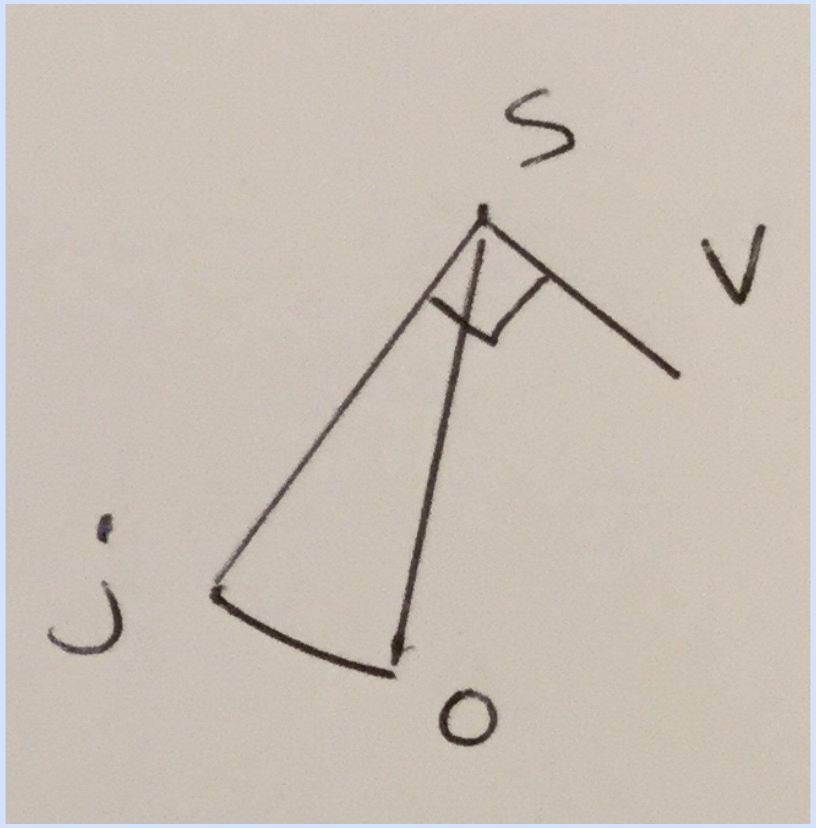
$$\frac{\sin 2\theta}{\sin 2\alpha} = \frac{SO^2}{SV^2}$$

The first step to solve this problem is to divide **SV** into **SaV** so that:

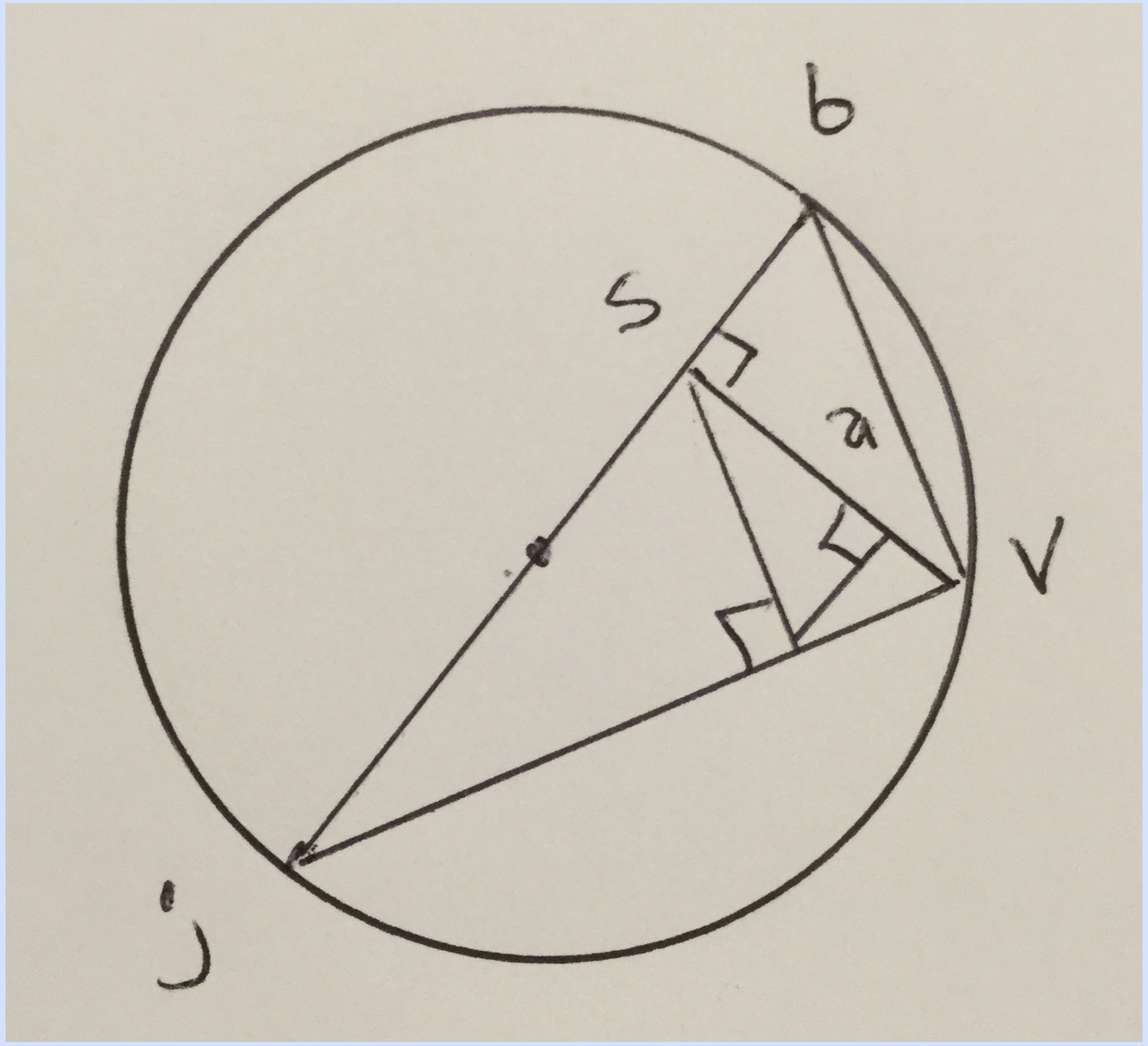
$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

Make **SO** $=$ **Sj** $\perp$ **SV** to construct:





$$\frac{Sj}{SV} = \frac{SV}{Sb} \quad ; \quad \frac{Sj^2}{SV^2} = \frac{Sj}{Sb} = \frac{SO^2}{SV^2}$$

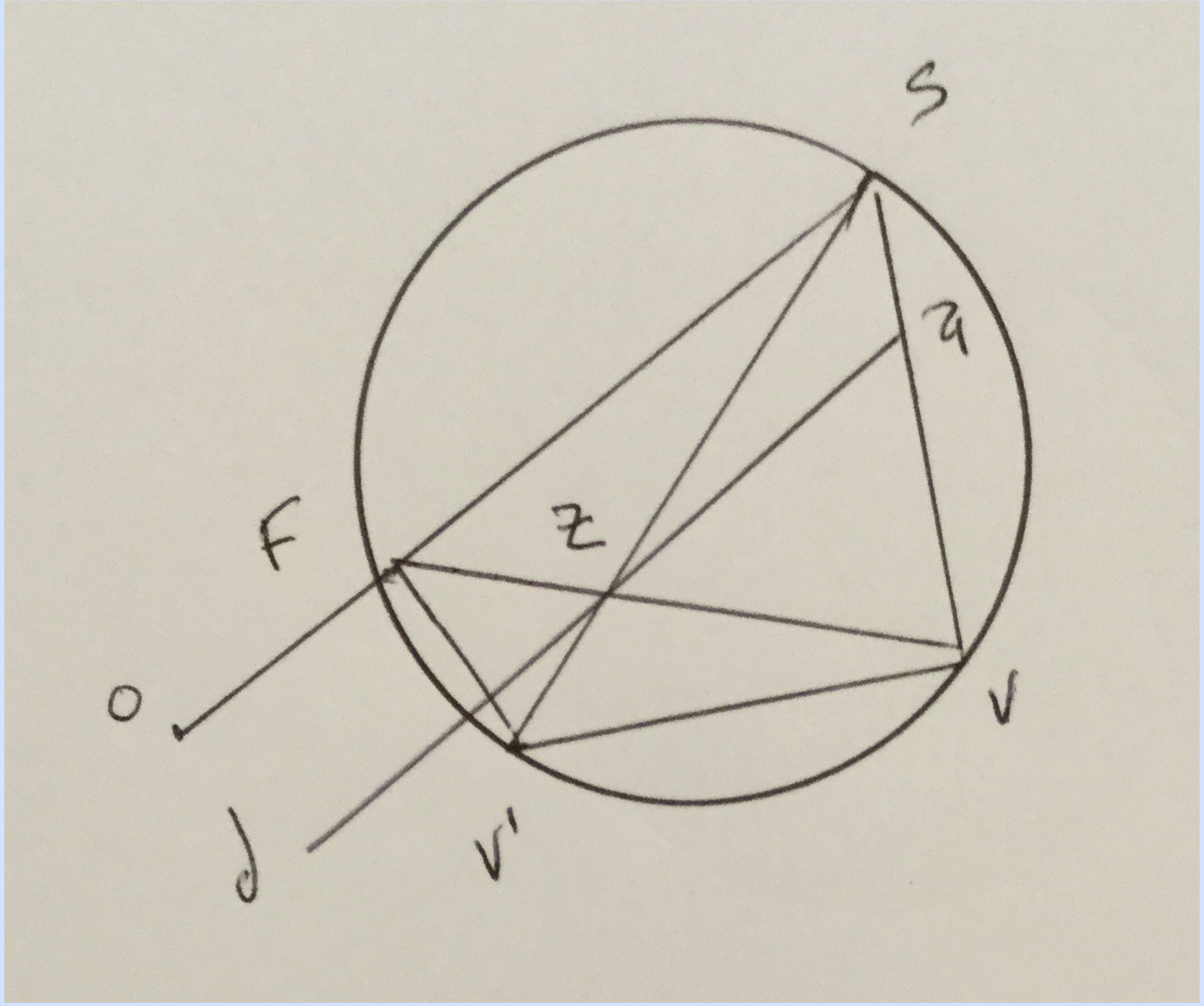


Similar triangles show that:

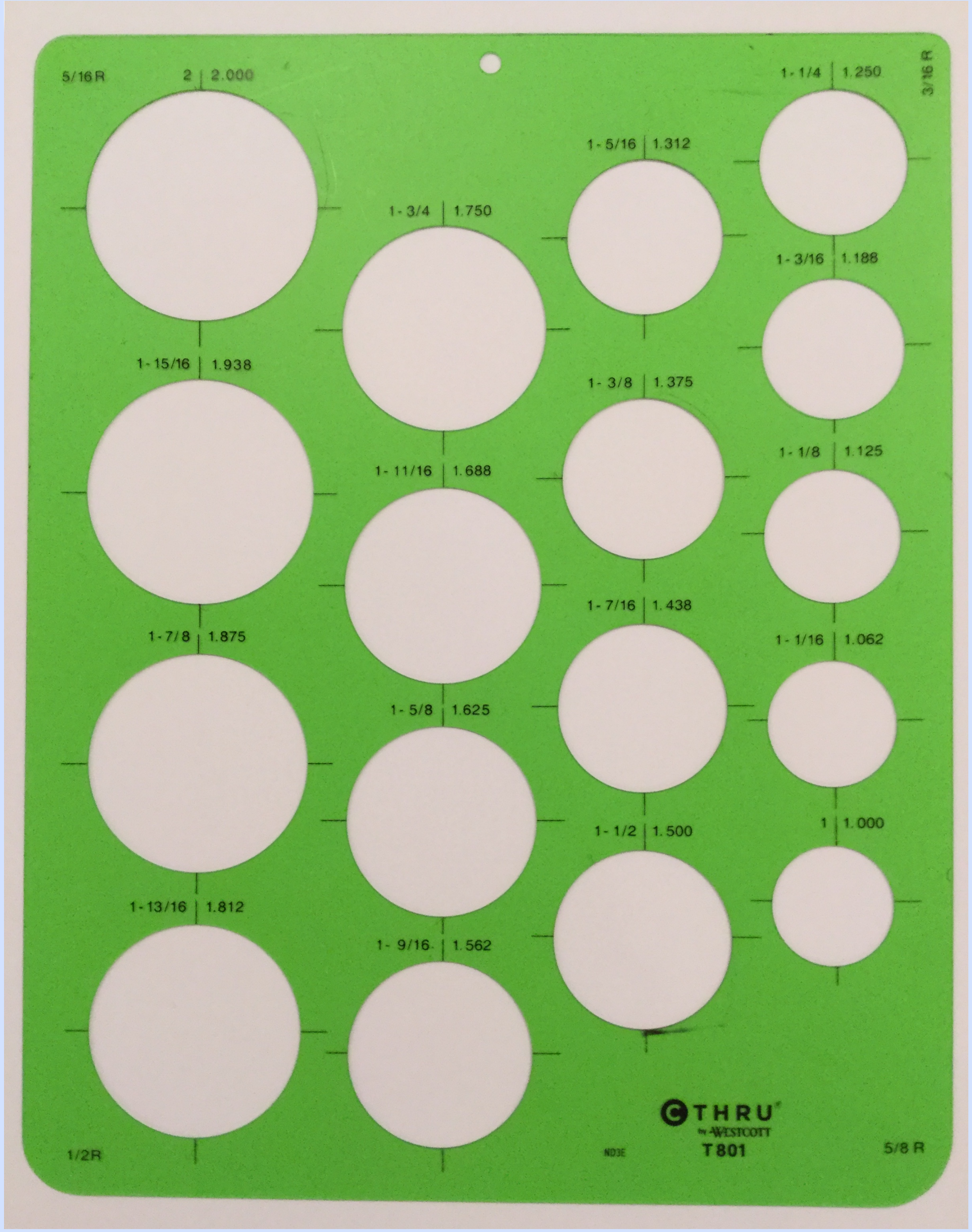
$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

Draw ad // SO

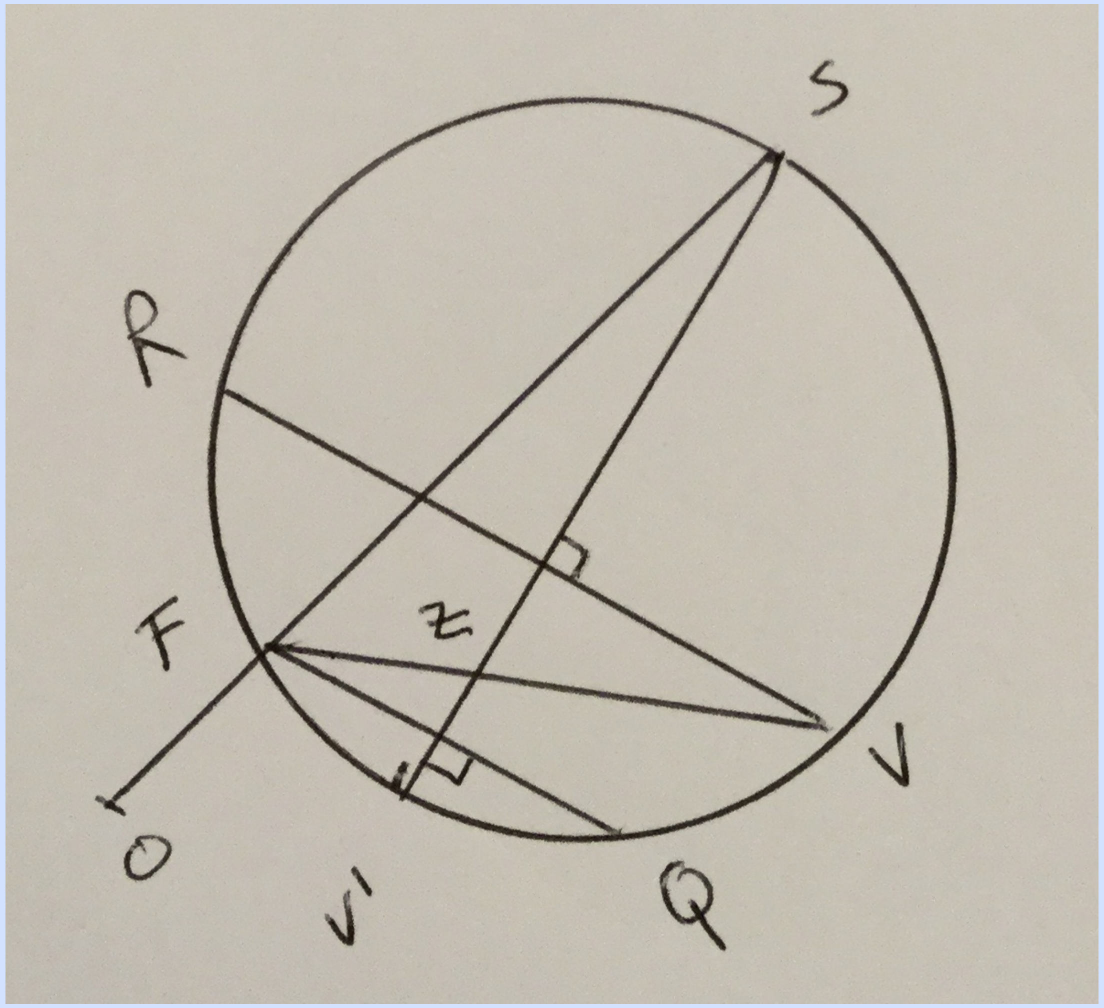
Choose a circle through **S** and **V** with a variable diameter **SV'** so that **FZV** lies on a common chord.



The easiest way to do this involves a template of various circles, each with the location of their diameters already marked.



SV' is the meridian with the maximum combined effects of refraction because:



$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{FZ}{ZV} = \frac{FQ/2}{RV/2} = \frac{FQ}{RV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

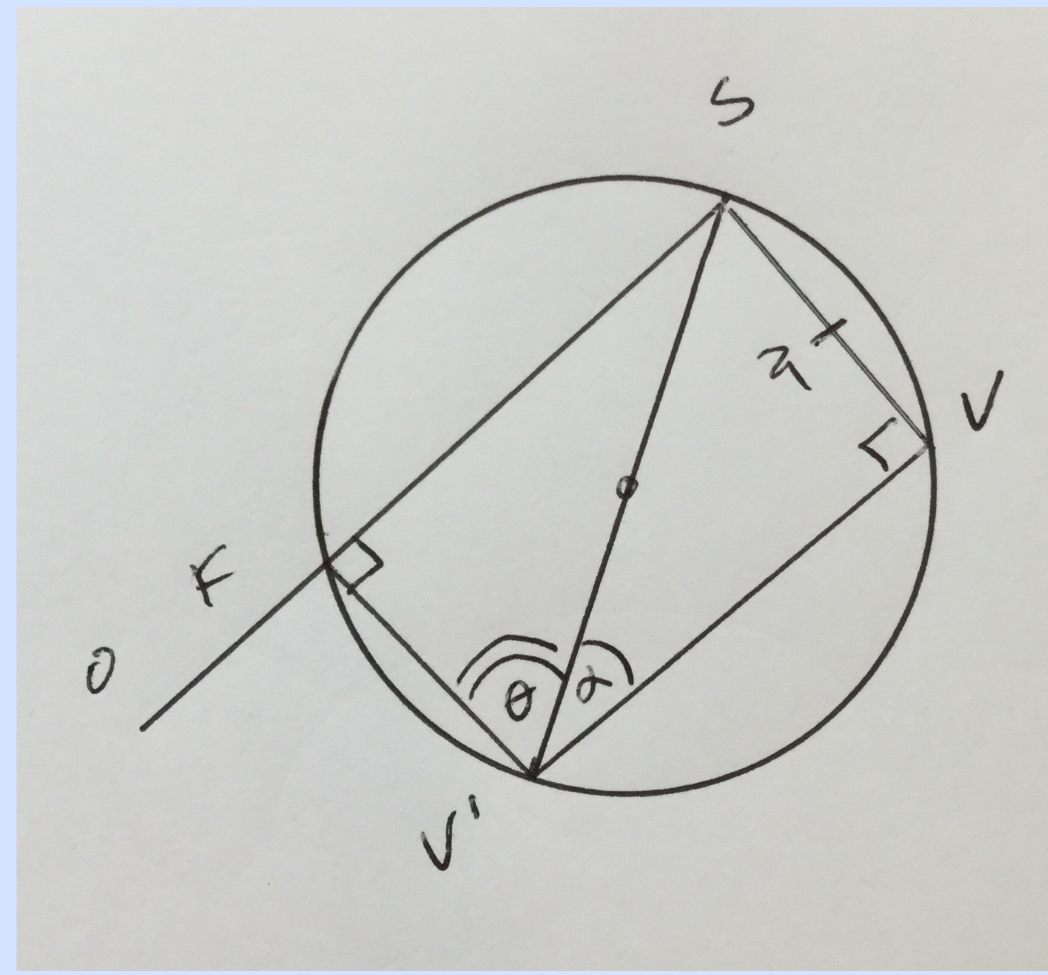
Double-angle Method

Given constant Δ**OSV**:

∠**FSV** is constant

∠**FSV** + (θ + α) = π

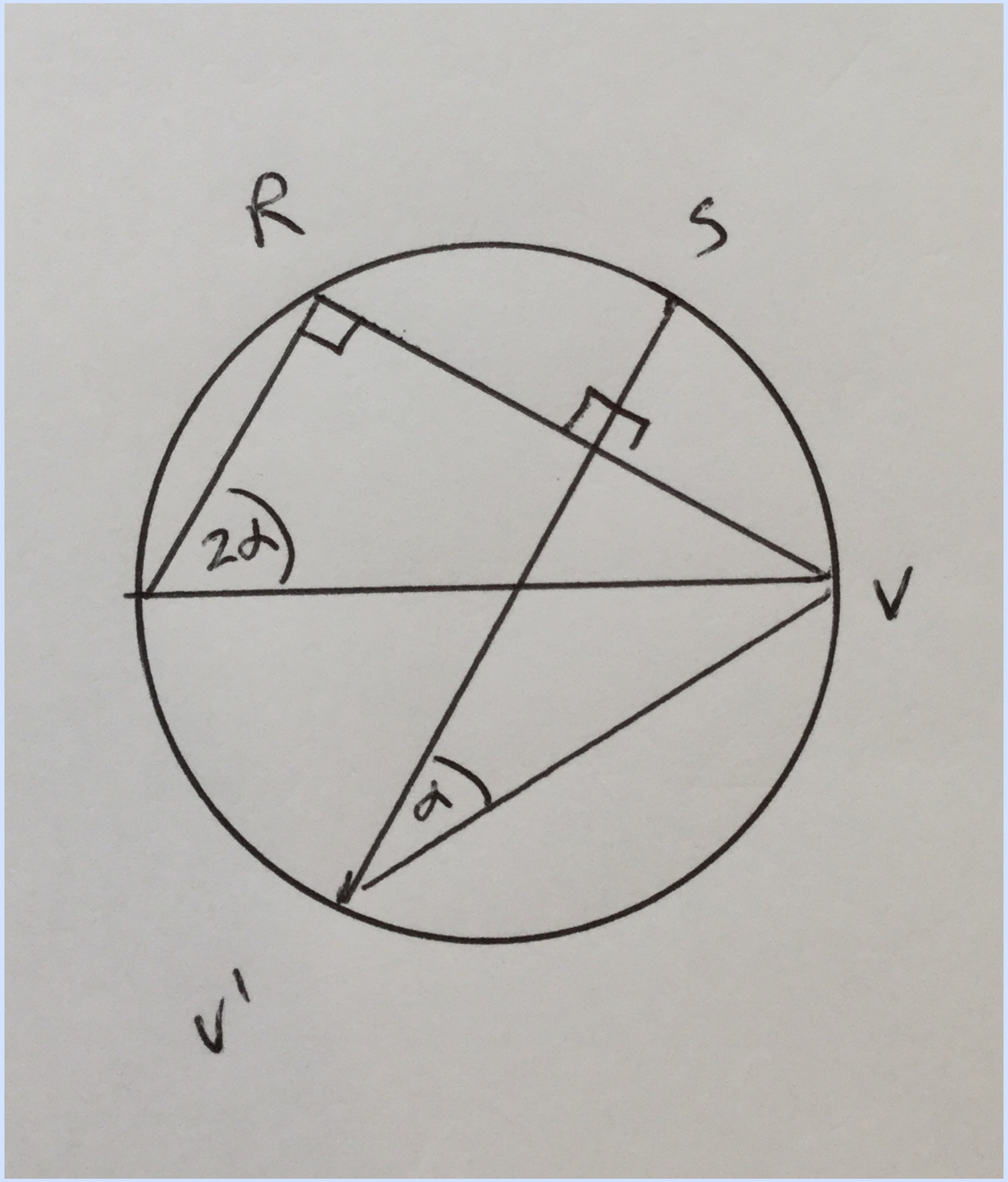
(θ + α) Is constant



We have already shown how to find single angles θ + α so that:

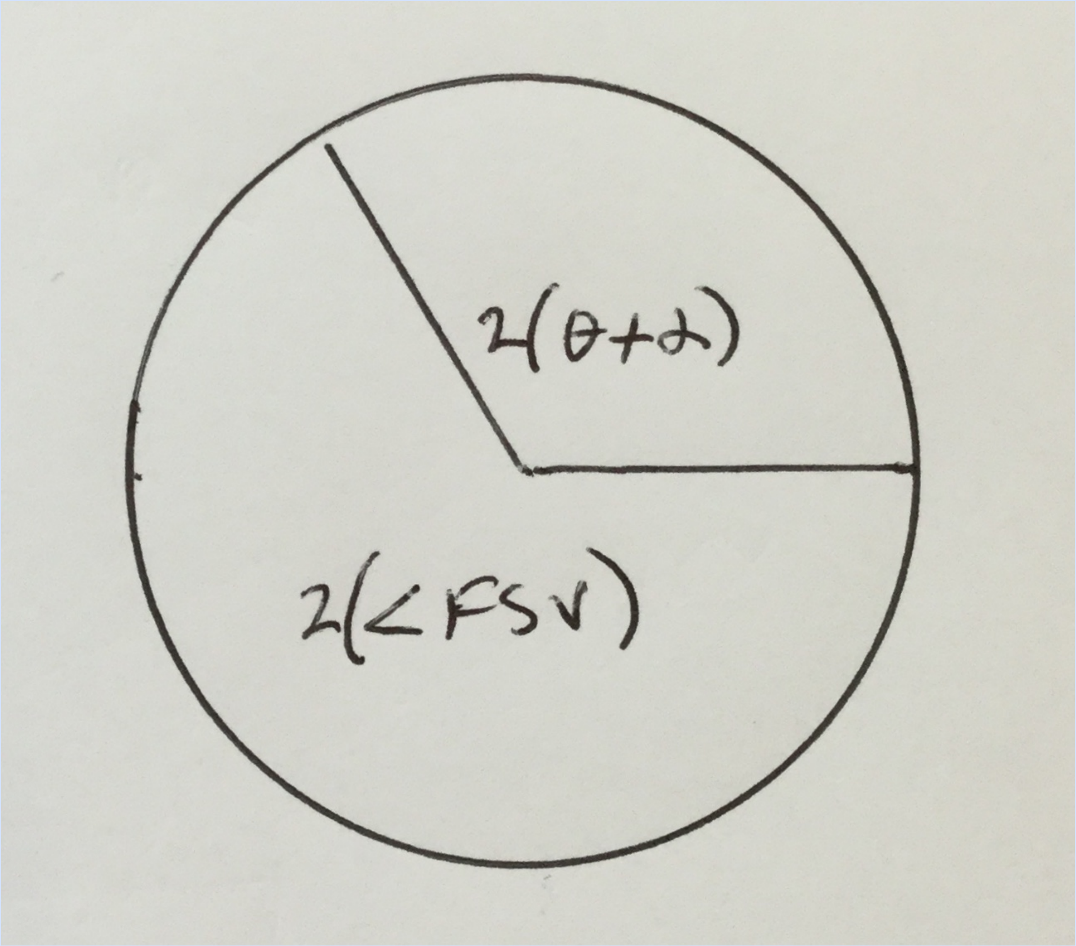
$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

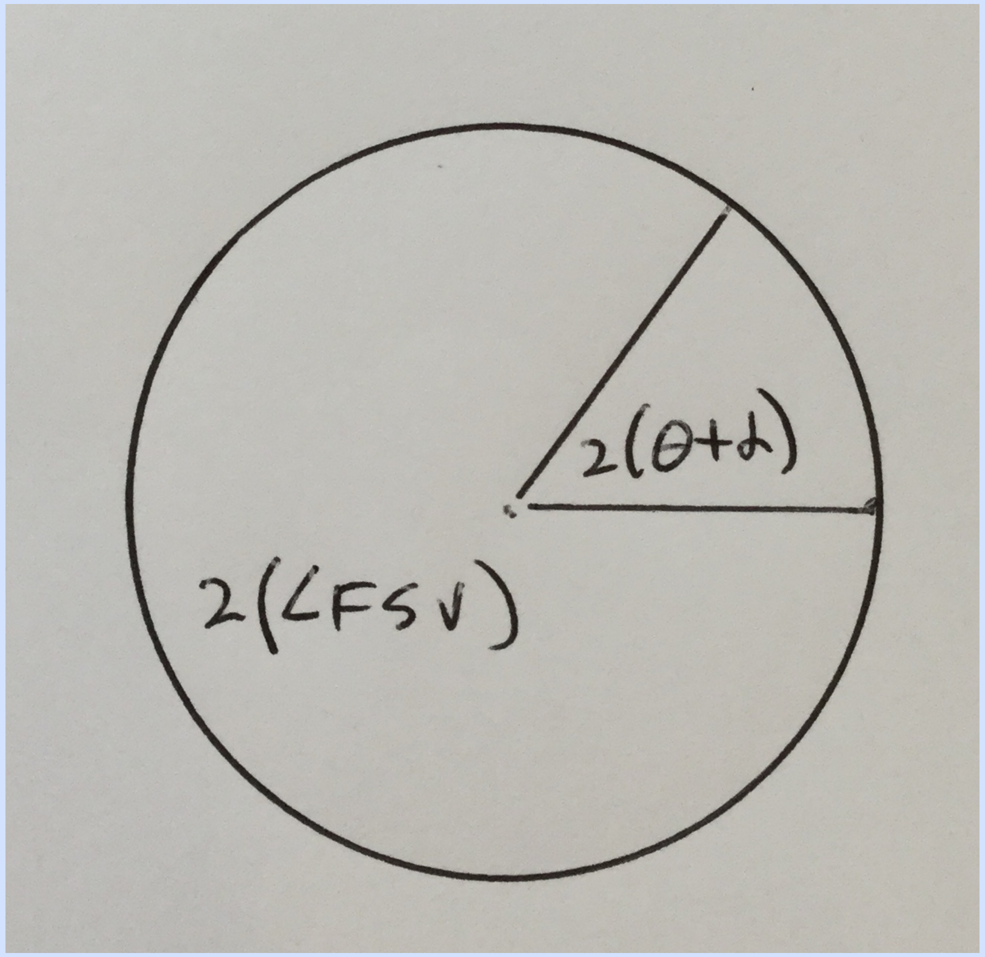
An angle on a circle equals its inscribed arc, divided by the arc's diameter. Since the sum of all angles measured on a circle's circumference add to π, when measured from a circle's center they add to 2π.



Therefore:

$$2(\angle FSV) + 2\,(\theta + \alpha) = 2\pi$$

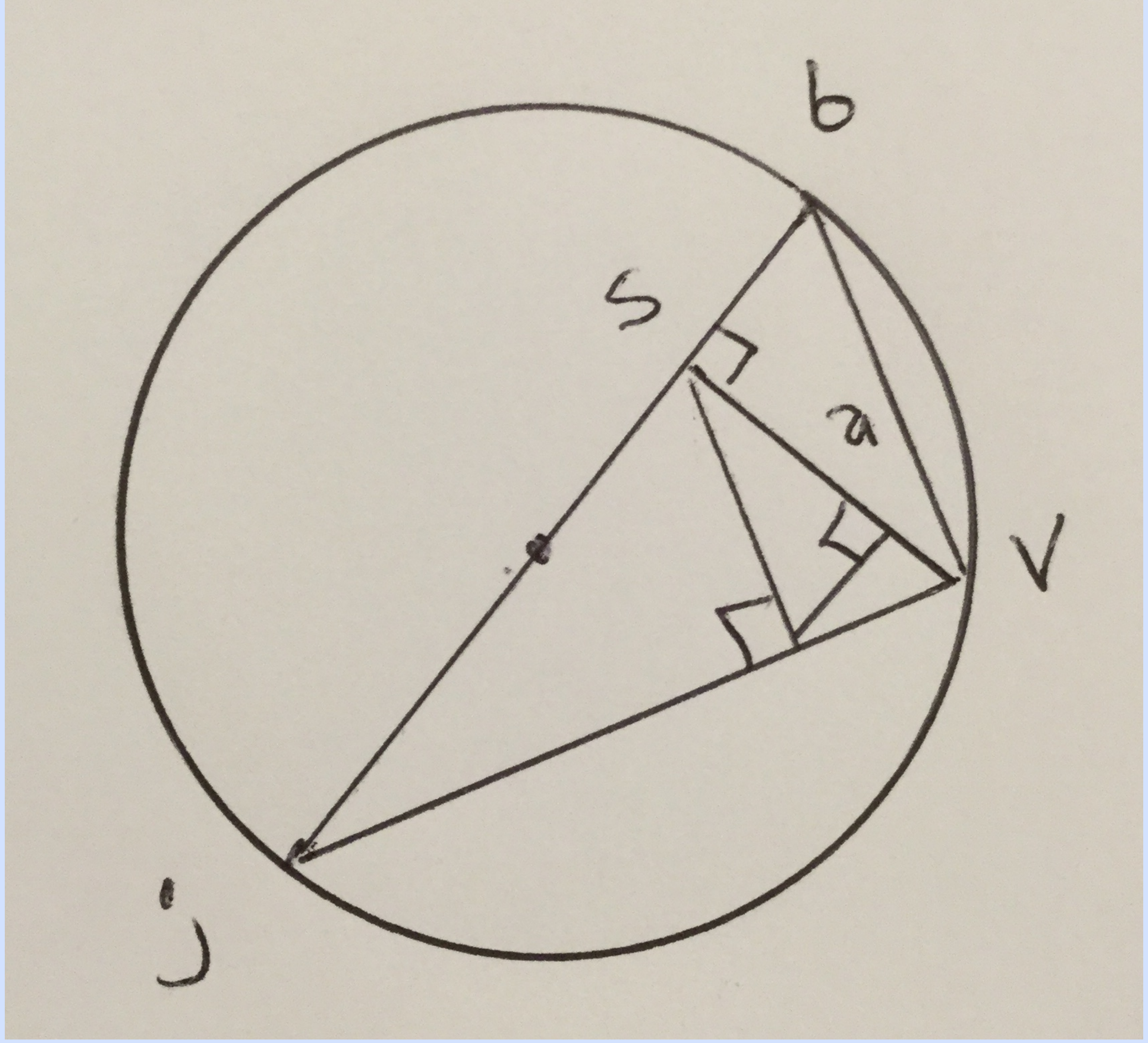




When:

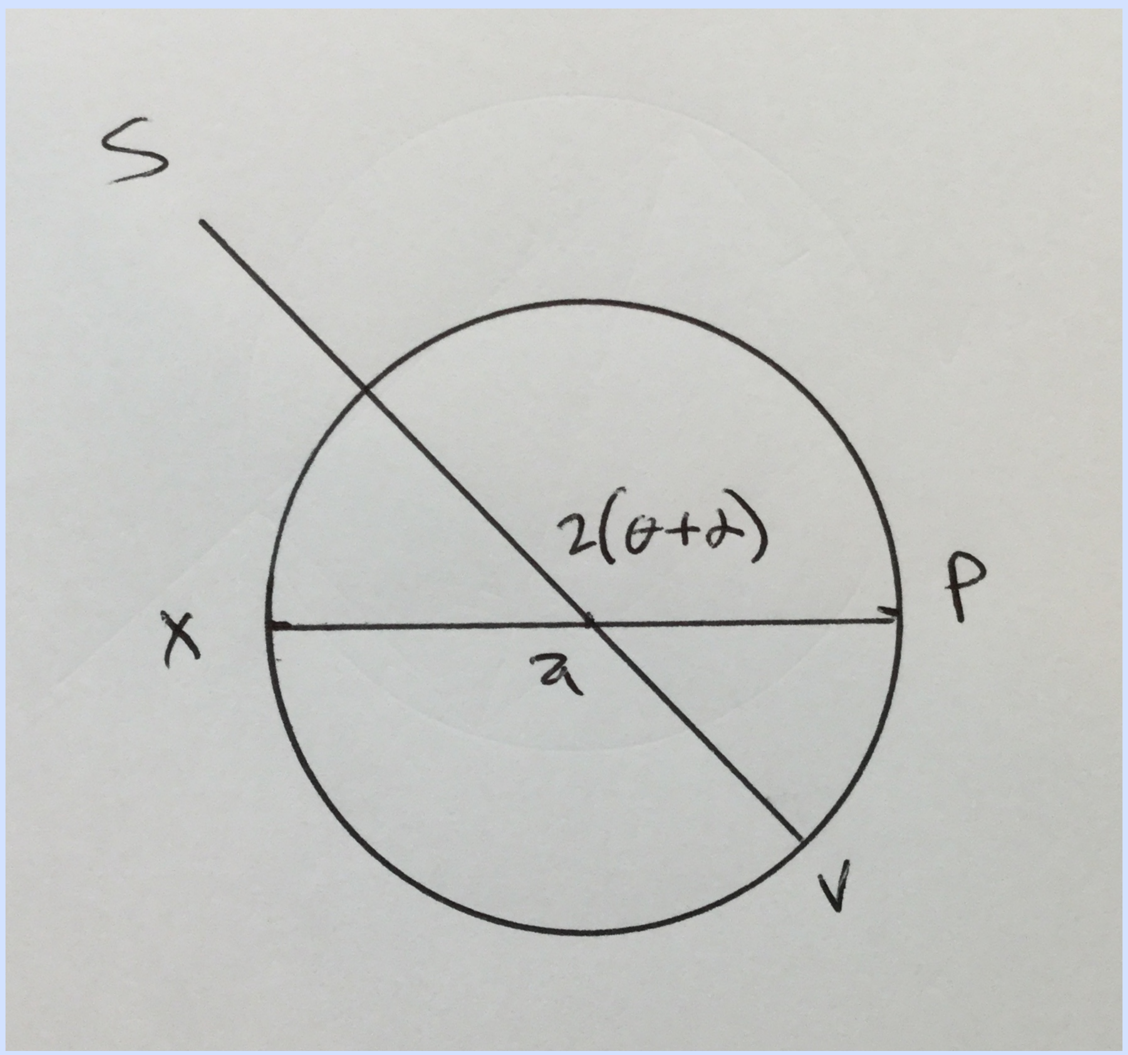
$$\frac{SO^2}{SV^2} = \frac{Sj^2}{SV^2} = \frac{aS}{aV}$$

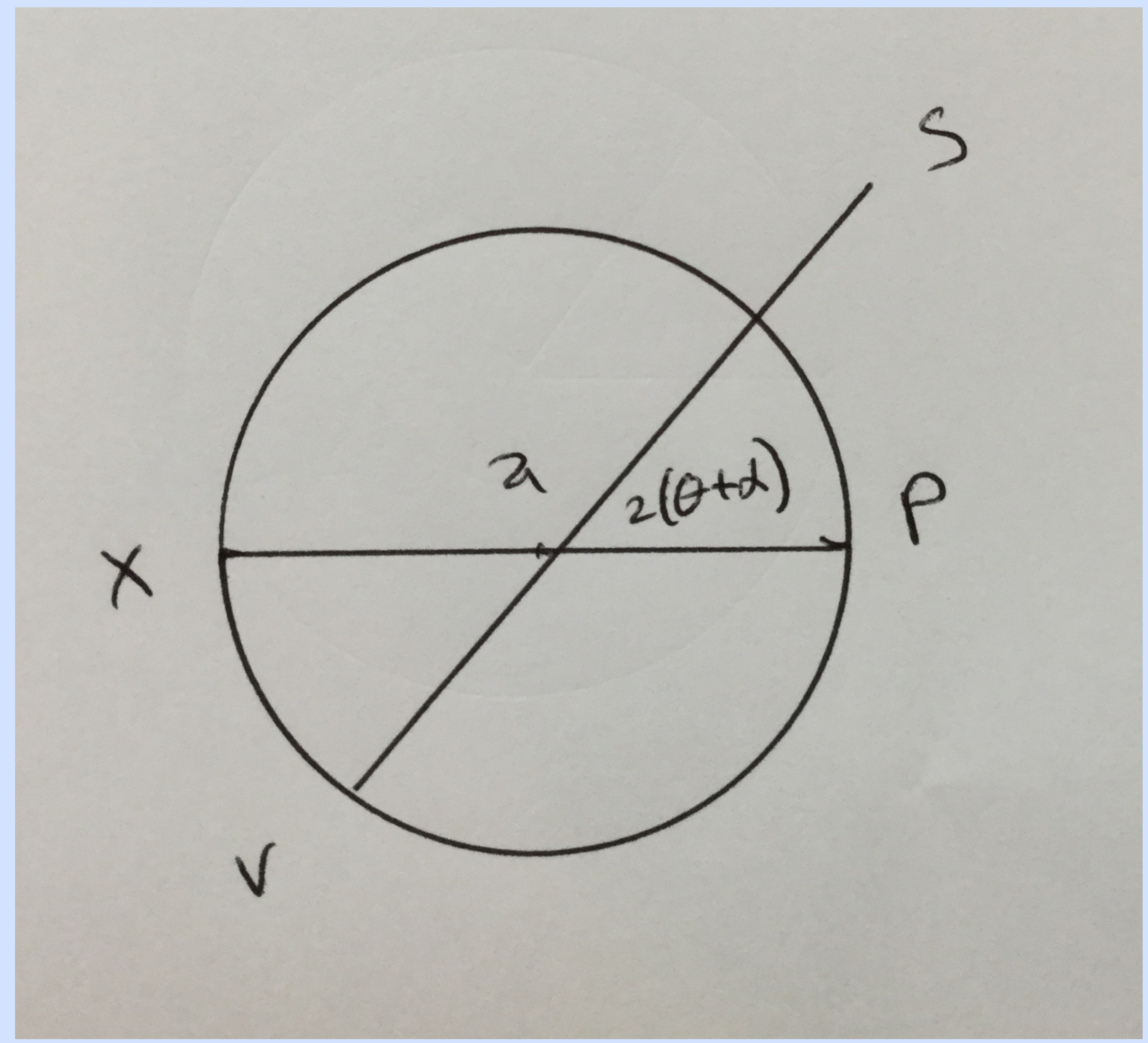
as drawn:



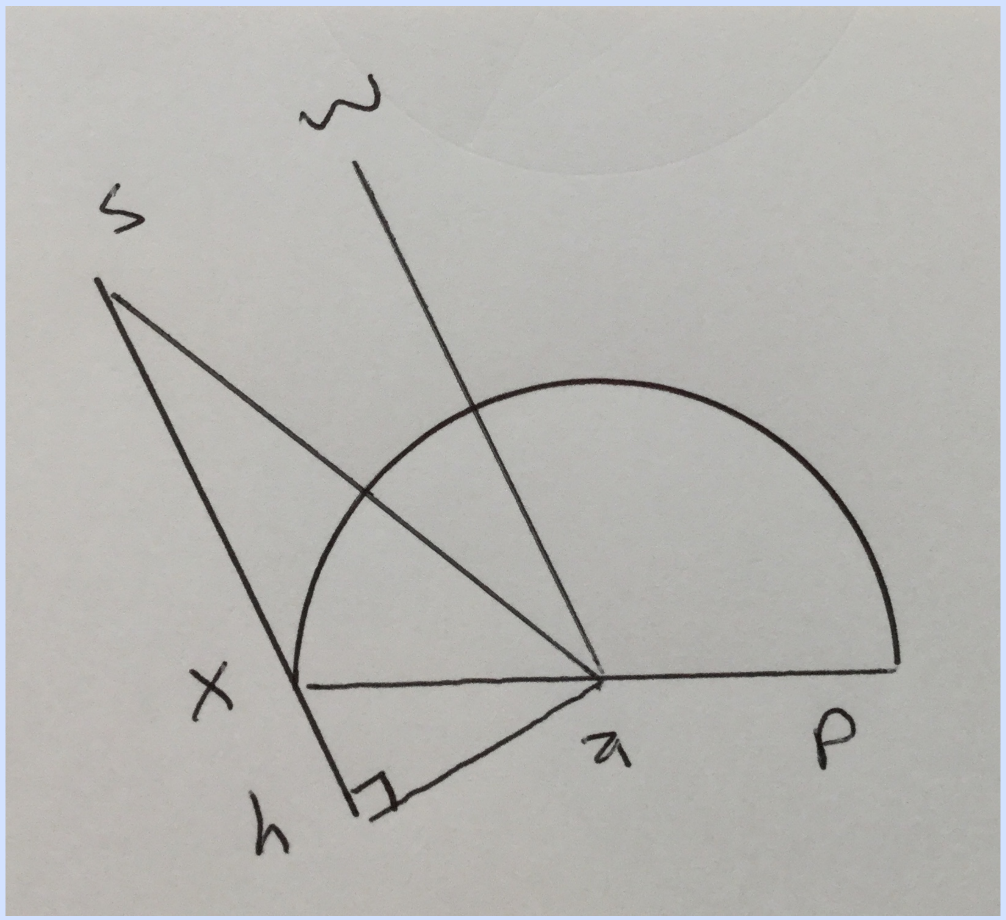
If we draw diameter **XaP** so:

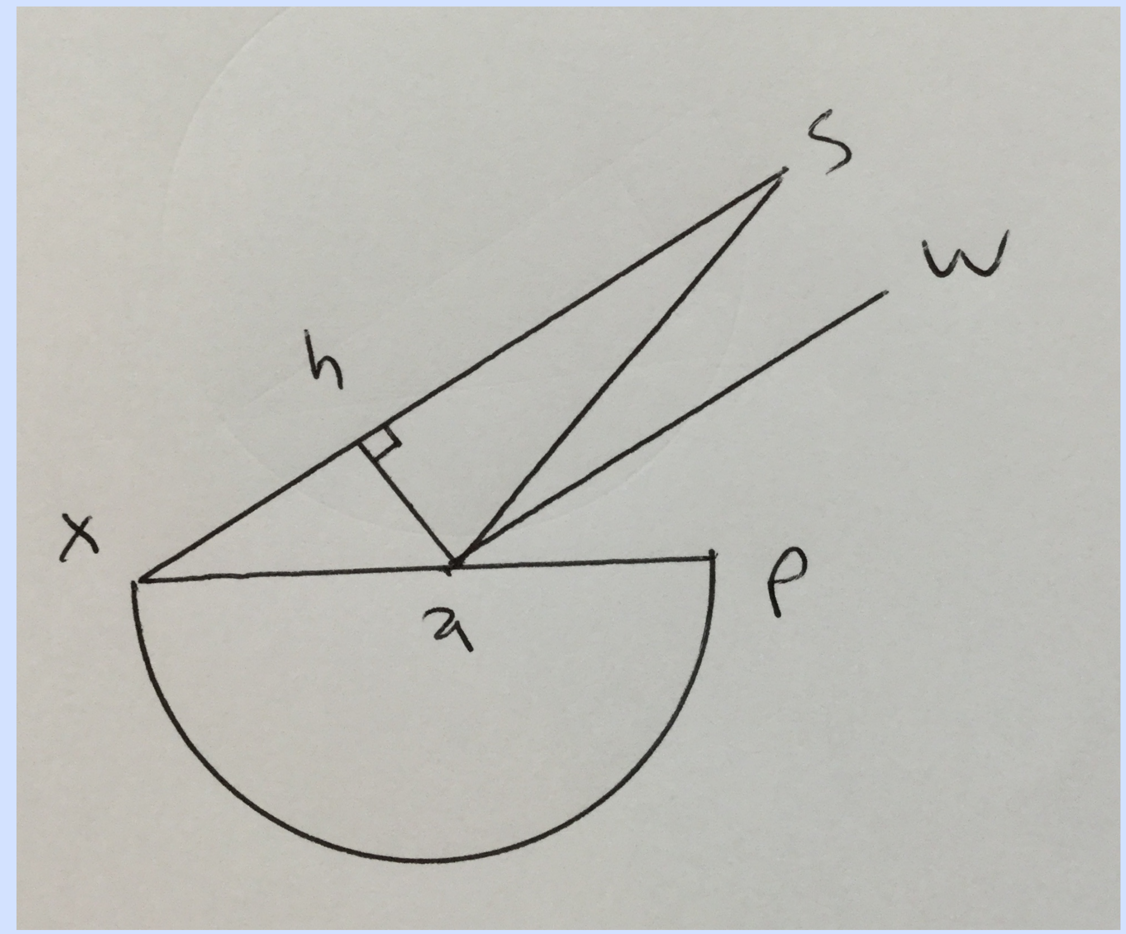
$aX = aV$, and $\angle SaP = 2(\theta + \alpha)$





$$\frac{SO^2}{SV^2} = \frac{aS}{aX} = \frac{ah/aX}{ah/aS} = \frac{\sin 2\theta}{\sin 2\alpha}$$





When aw // sX, we have divided the doubled angle 2 (θ + α) = ∠SaP into 2θ = ∠WaP, and 2α = ∠WaS.

In Conclusion

The approximate meridian of maximum refraction of two crossed spherical cylinders can be visualized by first examining the parabolic sagitta of each component cylinder in various cross meridians using the same sagittal depth **SB**. Although the meridian with the minimum sagittal sum does not represent the meridian of maximum refractive effect, a geometrical determination of that meridian can be determined once axial refractive power is expressed in terms of parabolic sagitta.