Geometrical Optics
2020

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Reference:

Isaac Barrows Optical Lectures, 1667;
Translated by H.C. Fay
Edited by A.G. Bennett
Publisher: The Worshipful Company of Spectacle Makers;
London, England; 1987
ISBN # 0-951-2217-0-1

Friedrich Schiller, in his, “Twenty Seven Letters on the Aesthetic Education of Mankind,” stated that *play* is the act of balancing abstract thoughts about what *could be*, which what actually *is*. He stated that it is necessary for the determination of beauty, which he defined as the connection between the actual and the ideal. It was with this sense of play that William Brown, PhD, introduced geometrical optics during my freshman year of optometry school in 1979. This aesthetic education provided for the continued construction of context out of the free interplay of content and form, as well as over four decades of fun.

Section 1

Geometry of the Circle

I begin with the circle, because we are already filled with ideas about how its pieces fit. For example, we may easily believe that parallel lines intersect it across equal arcs. From that we can show that equal arcs along a circle subtend equal angles, and that certain triangles within a circle therefore can be shown to have the same shape, with their sides forming ratio equalities. Quadrilaterals with corners along the same circle can then describe equalities with multiple ratios. In 1667 Isaac Barrow used this approach to find triangles using other triangles, and describe refraction along a line and a circle.

Figure 1:

Given a circle with diameter EU and center N:

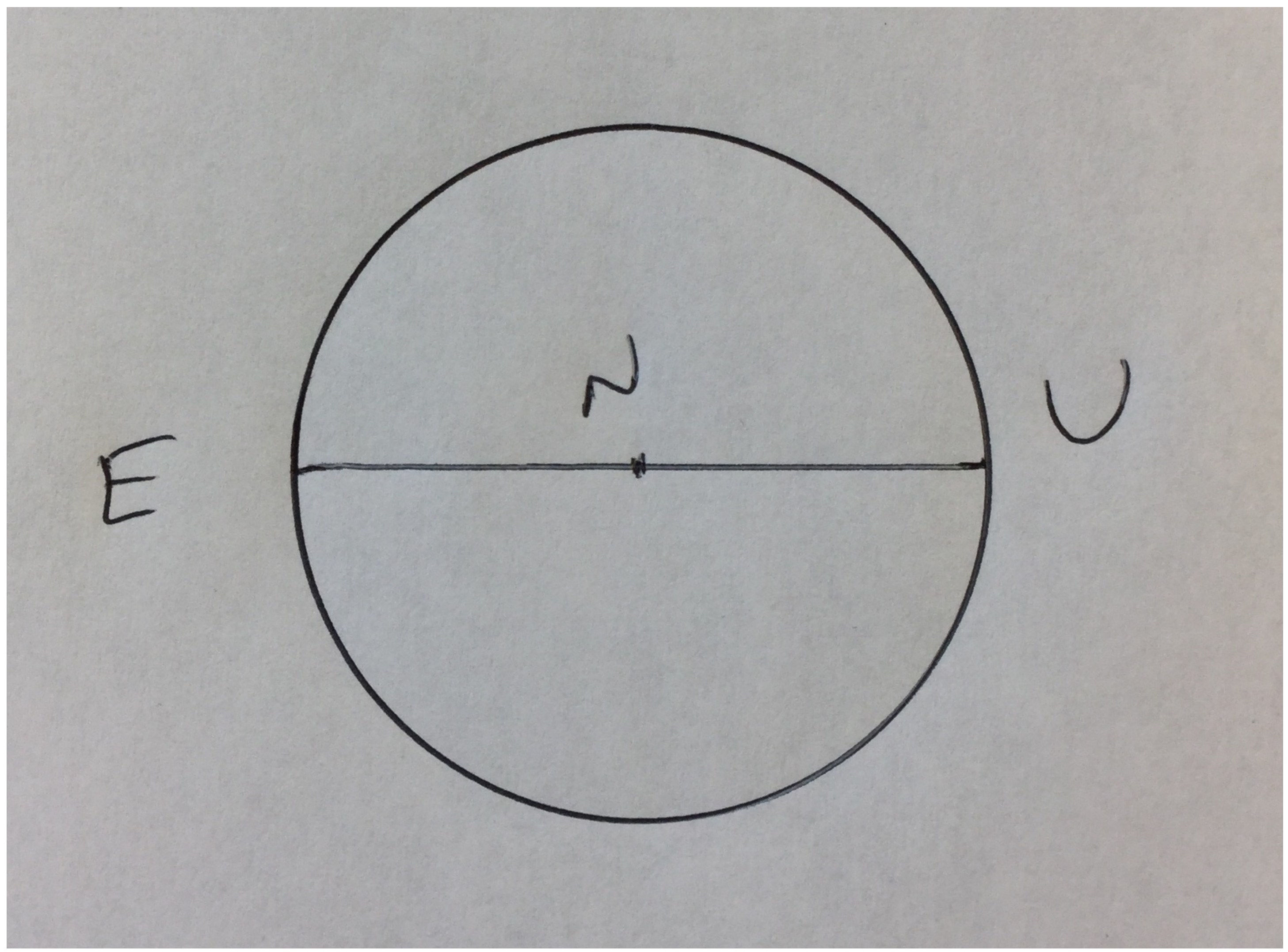
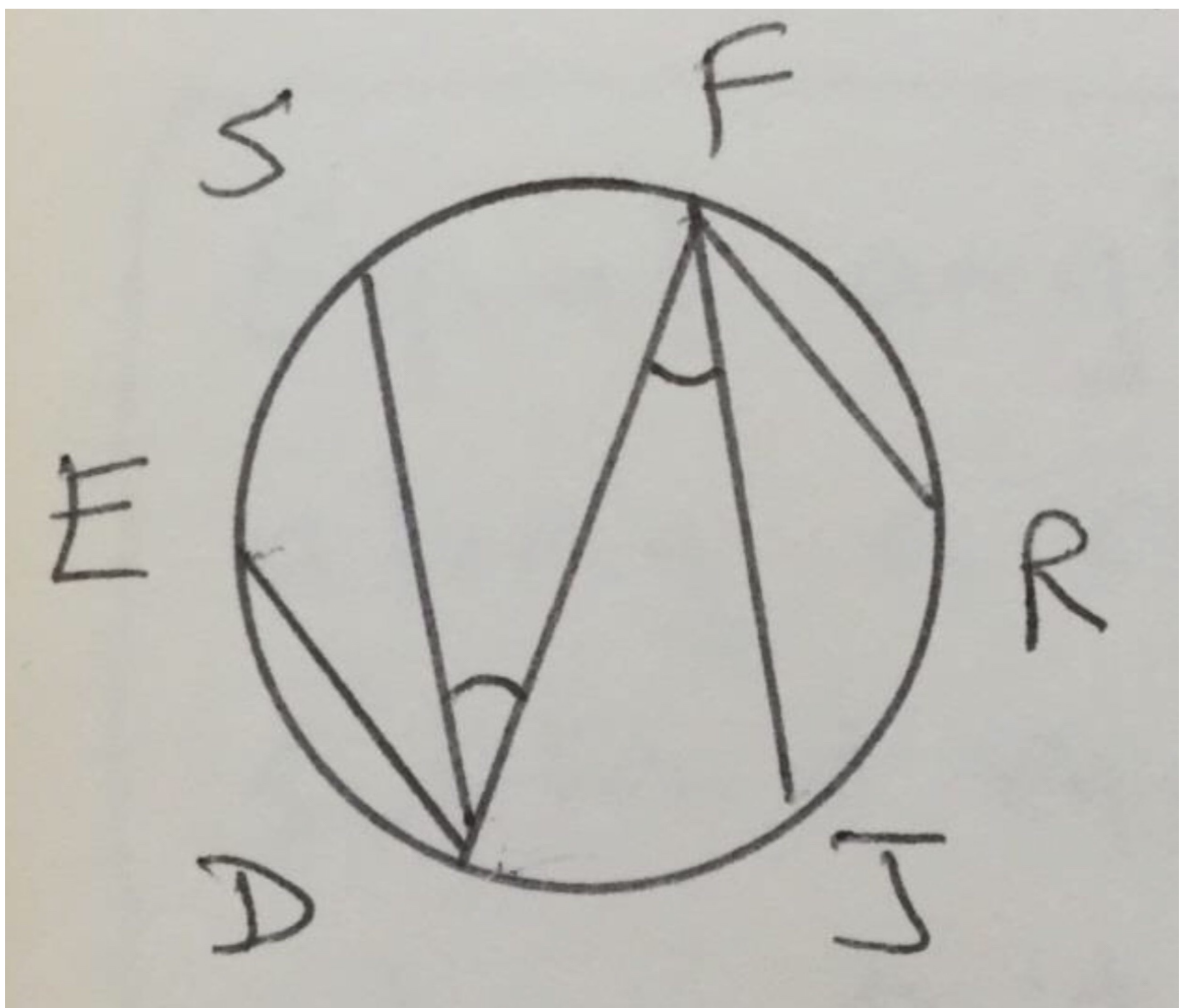


Figure 2:

Any two equal arcs ~ES and ~RJ can be shown to subtend equal angles by drawing any two parallel lines SD and JF:



~SF = ~JD

~ES + ~SF = ~RJ + ~JD

~EF = ~RD

ED || RF

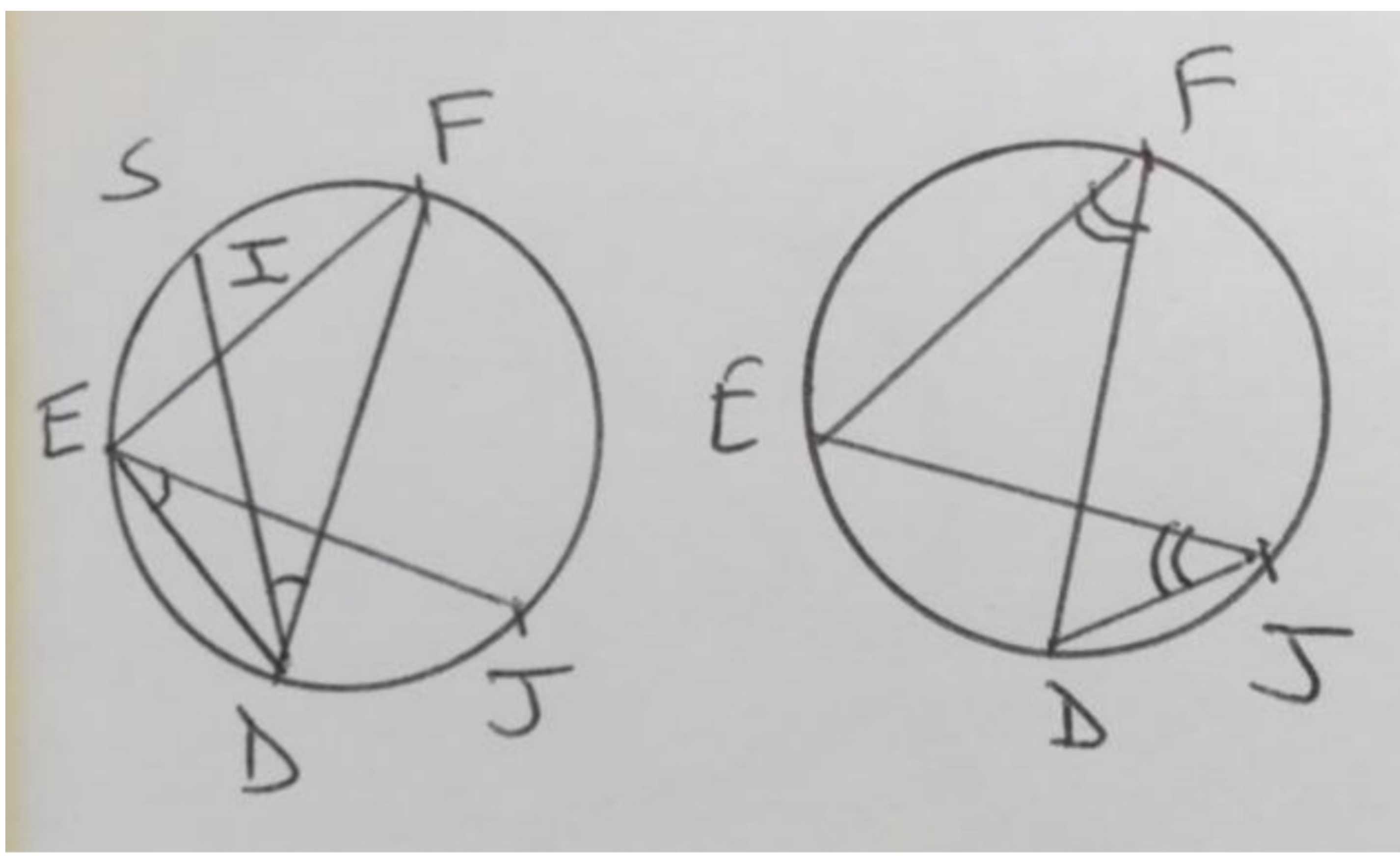
Since equal angles along a circle therefore subtend equal arcs, any angle along any circle can be defined in terms of its subtended arc and the circle's diameter. For example:

$$\angle RFJ = \sim\frac{RJ}{EU}$$

Figure 3:

Triangles need only two equal angles to be the same shape, (or ≅).

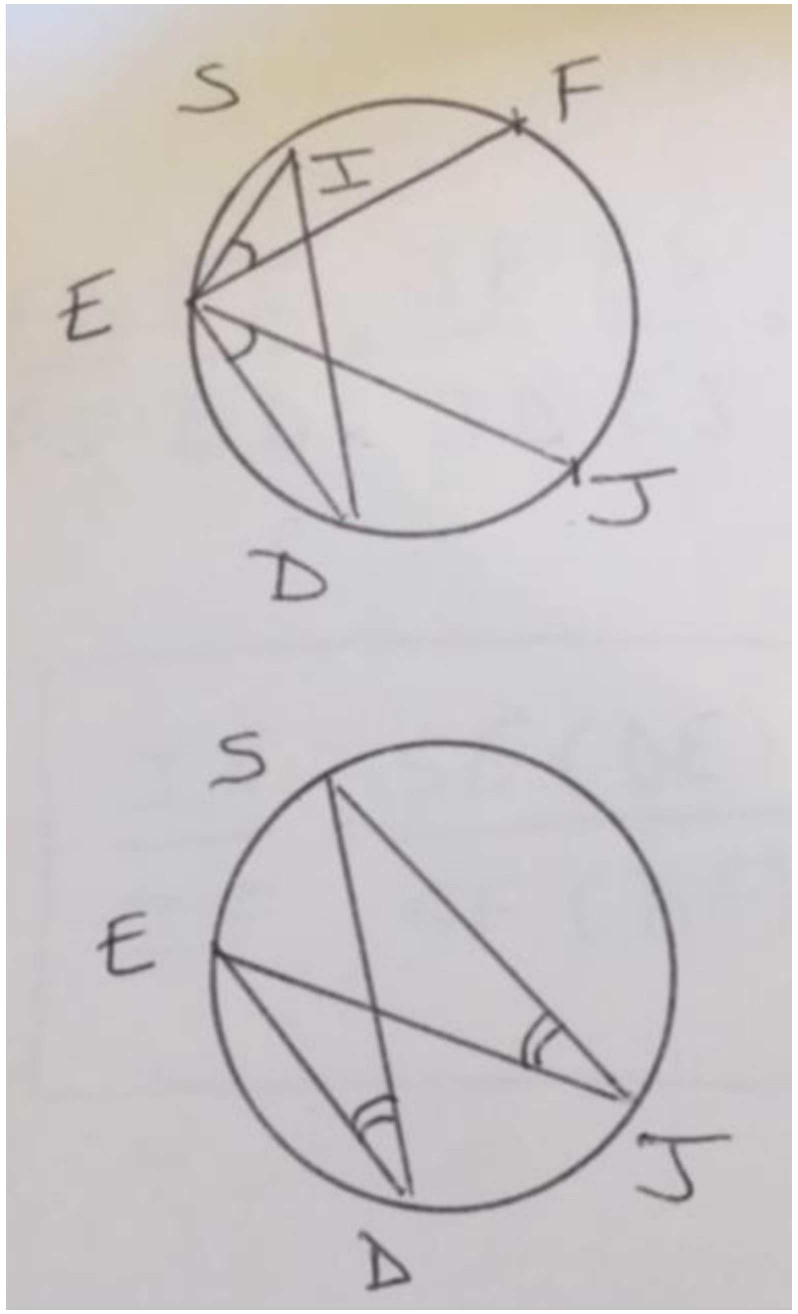
Since equal arcs subtend equal angles along a circle:



ΔEJD ≅ ΔDFI

$$\frac{FD}{FI} = \frac{JE}{JD}$$

Figure 4:



$$\sim SJ = \sim FD$$

$$\Delta EJS \cong \Delta EDI$$

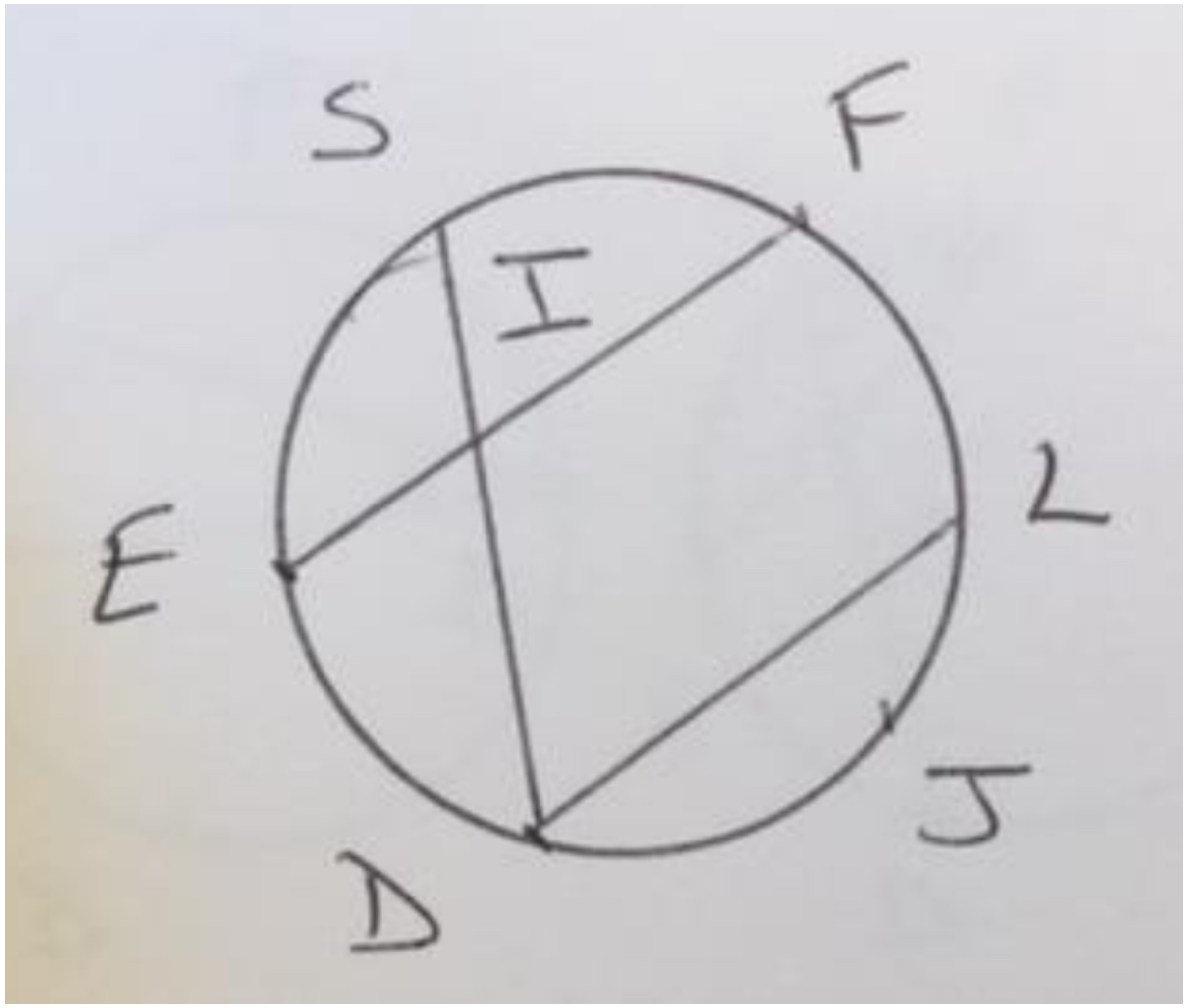
$$\frac{EI}{ED} = \frac{ES}{EJ}$$

$$\frac{FD.EI}{FI.ED} = \frac{JE.ES}{JD.EJ} = \frac{SE}{SF}$$

$$\mathbf{\frac{IE}{IF} = \frac{SE.DE}{SF.DF}}$$

which describes an important property of any cyclic quadrilateral SEDF.

Figure 5:



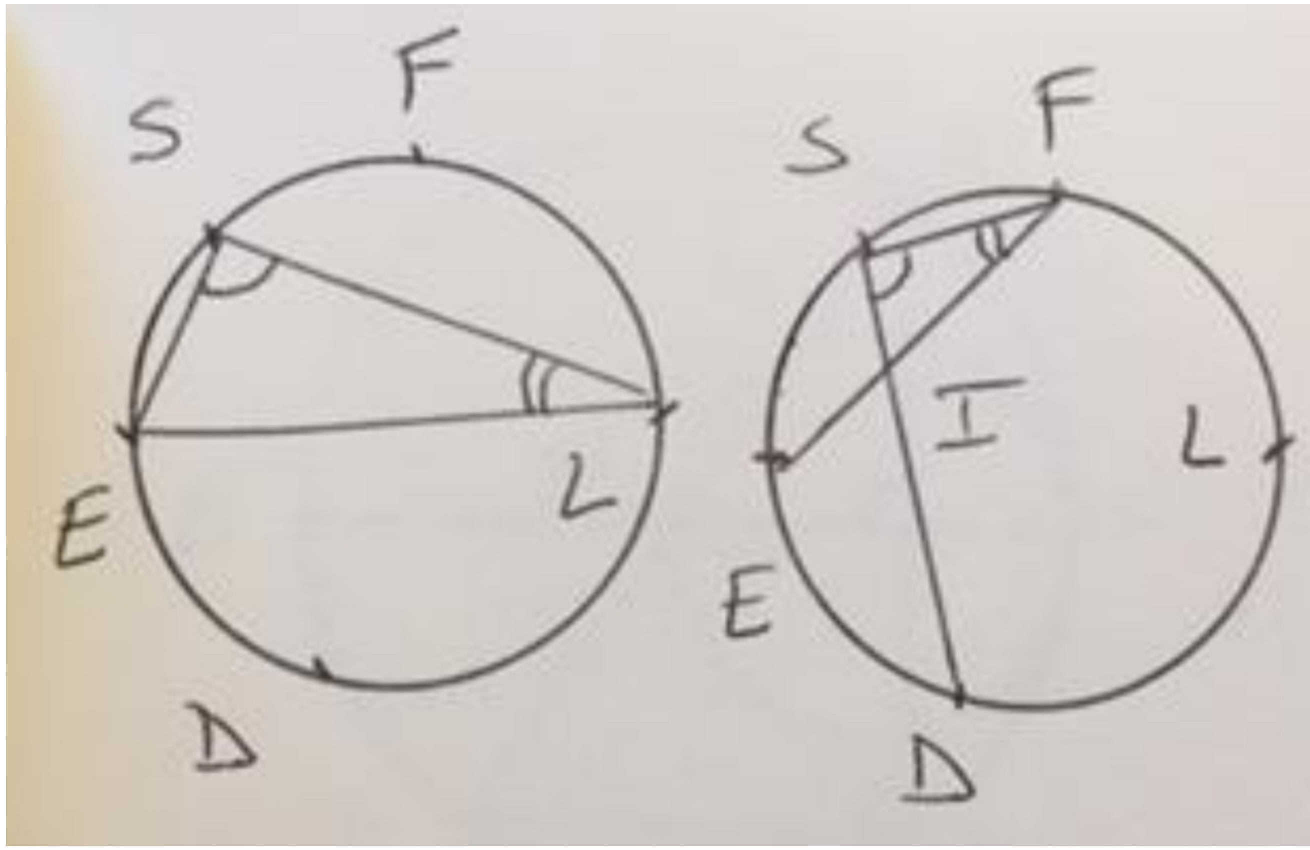
LD || FE

$$\frac{DE}{DF} = \frac{LF}{LE}$$

$$\frac{IE}{IF} = \frac{SE.LF}{SF.LE}$$

$$\frac{FE}{FI} = \frac{SE.LF + SF.LE}{SF.LE}$$

Figure 6:



LD || FE

~EL = ~FD

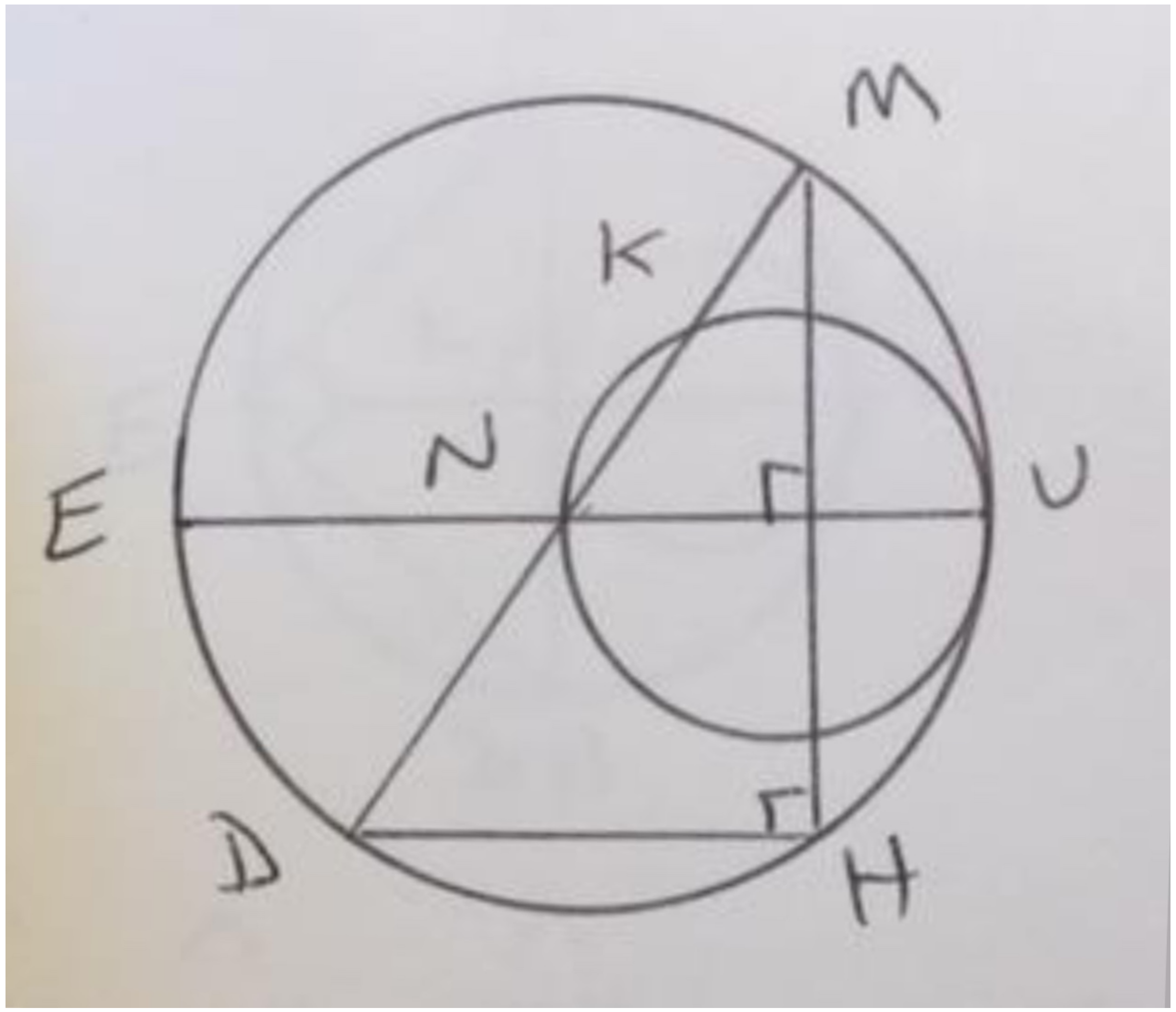
ΔLSE ≅ ΔFSI

$$LS = \frac{FS.LE}{FI}$$

FE.LS = SE.LF + SF.LE

which describes an important property of any cyclic quadrilateral SELF.

Figure 7:



$$\angle KNU = \angle MDH$$

$$\frac{\sim UK}{UN} = \frac{\sim MH}{MD} = \frac{\sim MH}{UE}$$

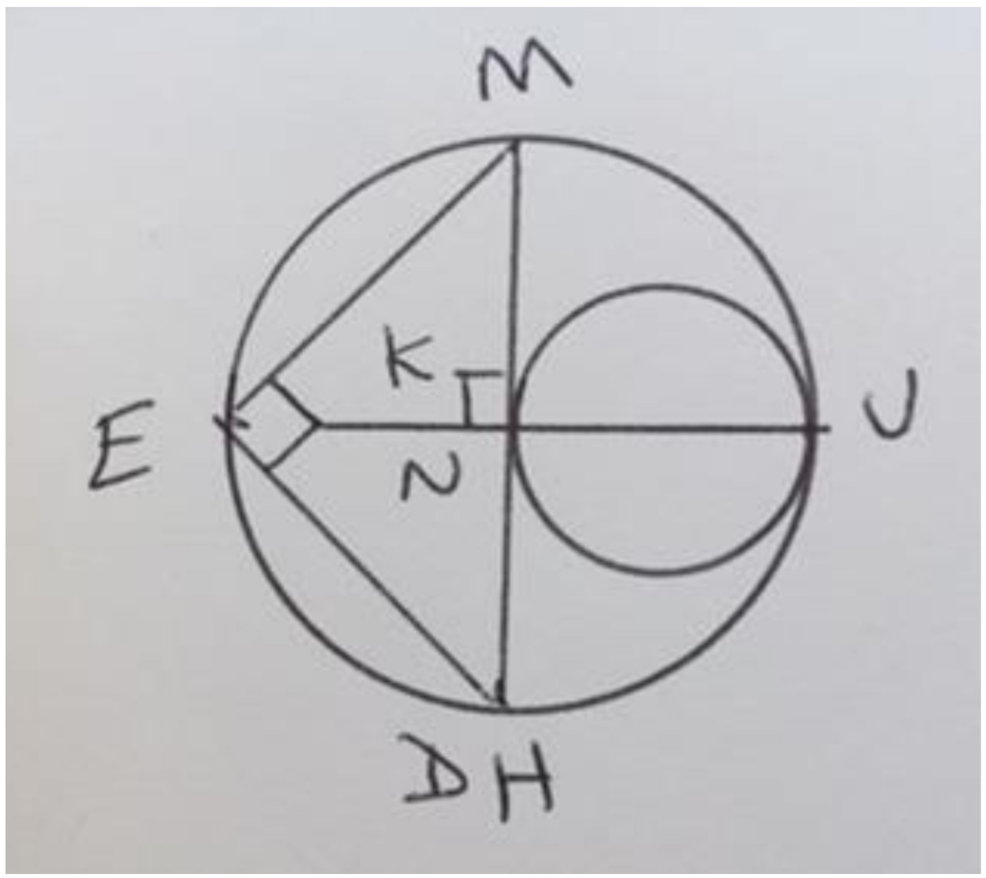
$$= \frac{2(\sim UM)}{UE} = \frac{2(\sim UM)}{2(UN)}$$

$$\mathbf{\angle KNU = 2\angle MEU}$$

$$\mathbf{\sim UK = \sim UM}$$

Figure 8:

Let K ⇒ N and D ⇒ H:



$$\frac{\sim UK}{UN} = \frac{\sim MH}{MD} = \frac{\sim MH}{UE} = \angle MEH$$

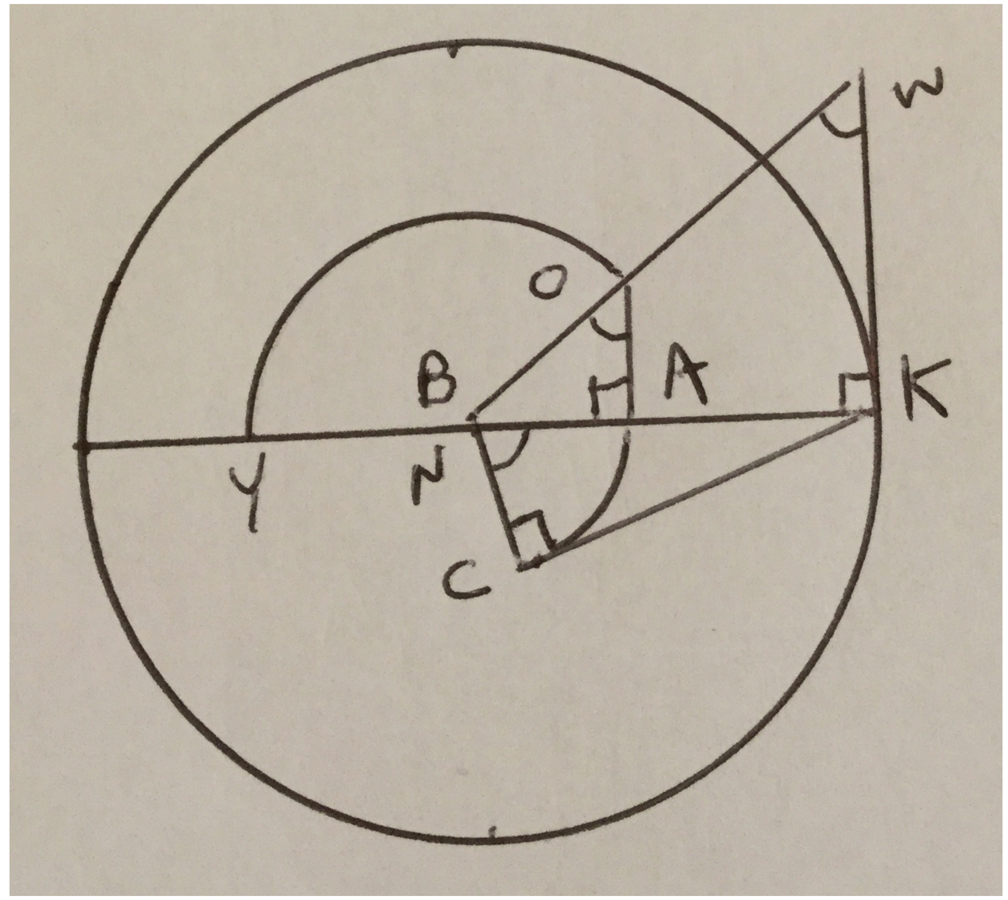
$$\frac{\sim UK}{UN} = \angle MNU$$

$$\frac{\mathbf{2(\sim UK)}}{\mathbf{UN}} = \angle \mathbf{MNH} = \boldsymbol{\pi}$$

Section 2

Refraction Along a Line

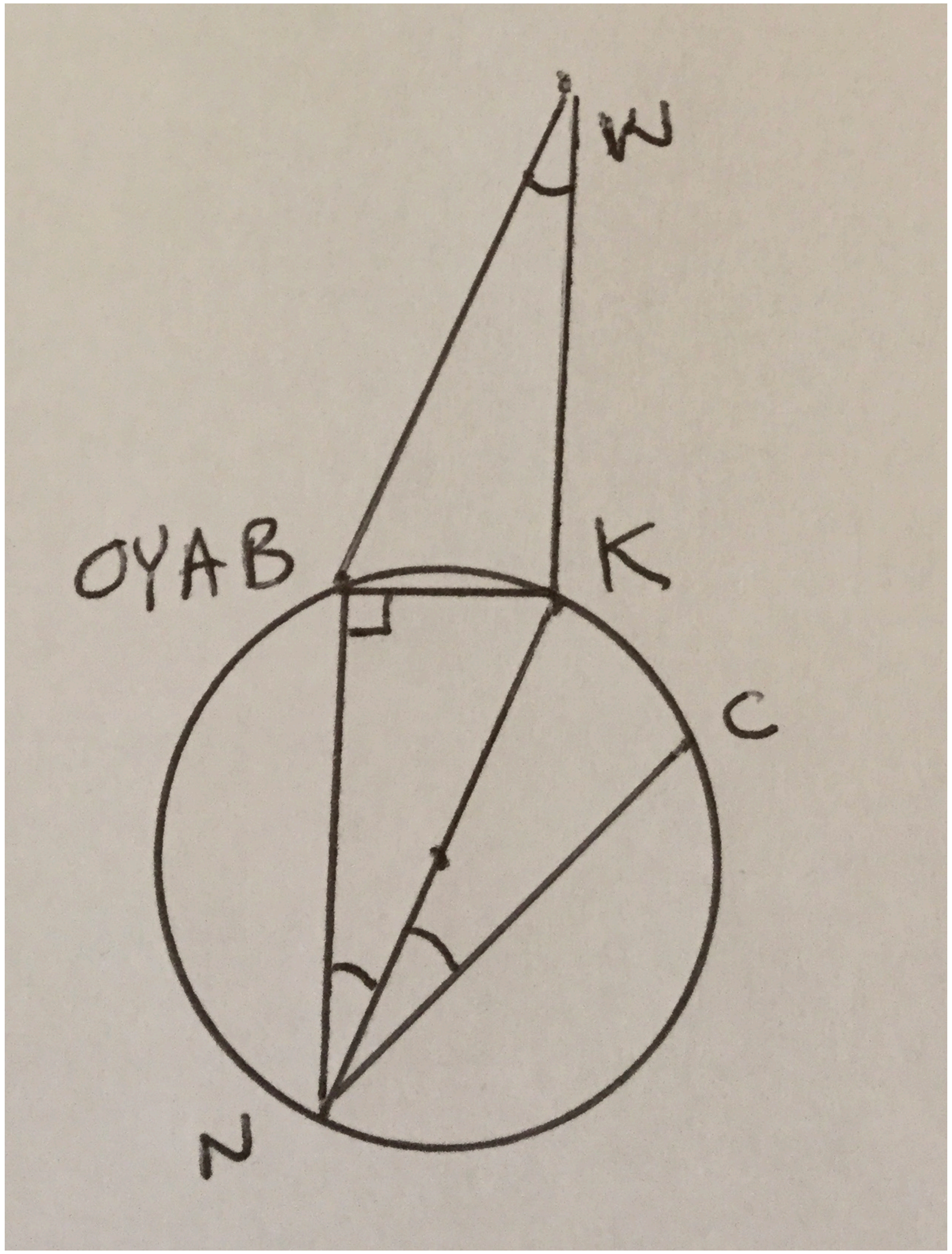
Figure 9:



$$\frac{(KW)}{(OA)} = \frac{NK}{NA} = \frac{NK}{NC} = \frac{OB}{OA} = \frac{WB}{WK}$$

KW (=OB) = YN

Figure 10:



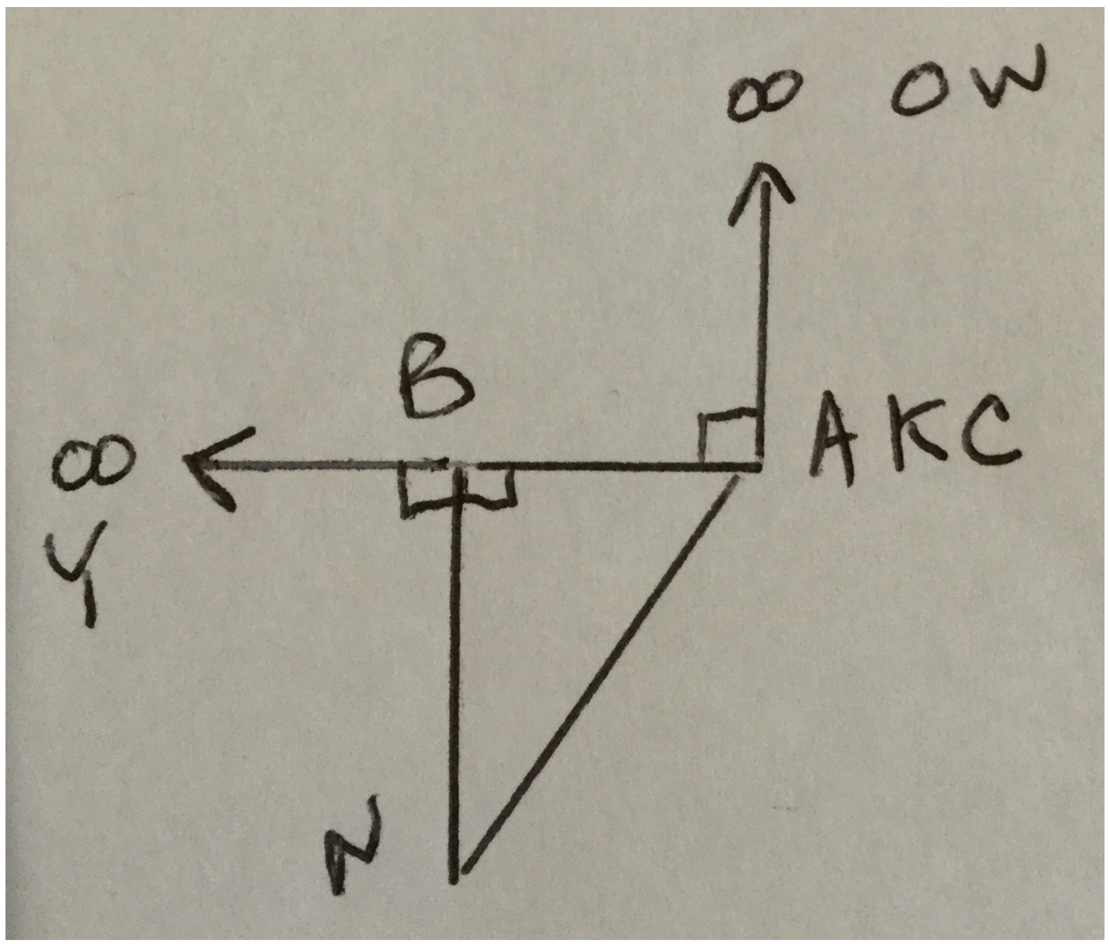
Create right triangle NBK.

When A lies at B:

$$\frac{NK}{NA} = \frac{NK}{NC} = \frac{(OB)}{(OA)} = \frac{WB}{WK}$$

KW = YN

Figure 11:

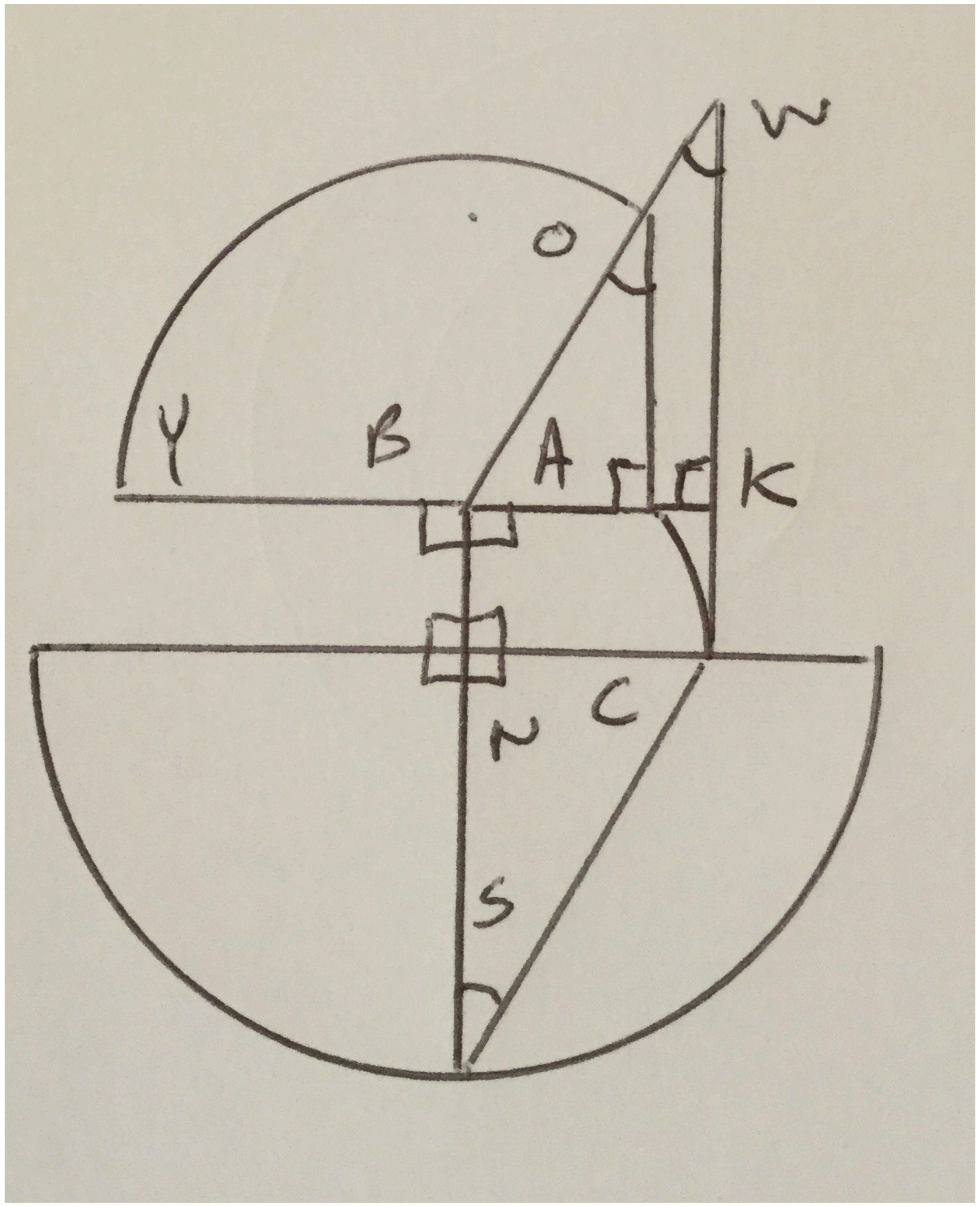


When A lies at K:

$$\frac{NK}{NA} = \frac{NK}{NC} = \frac{(OB)}{(OA)} = \frac{WB}{WK}$$

KW = YN = infinity

Figure12:



when:

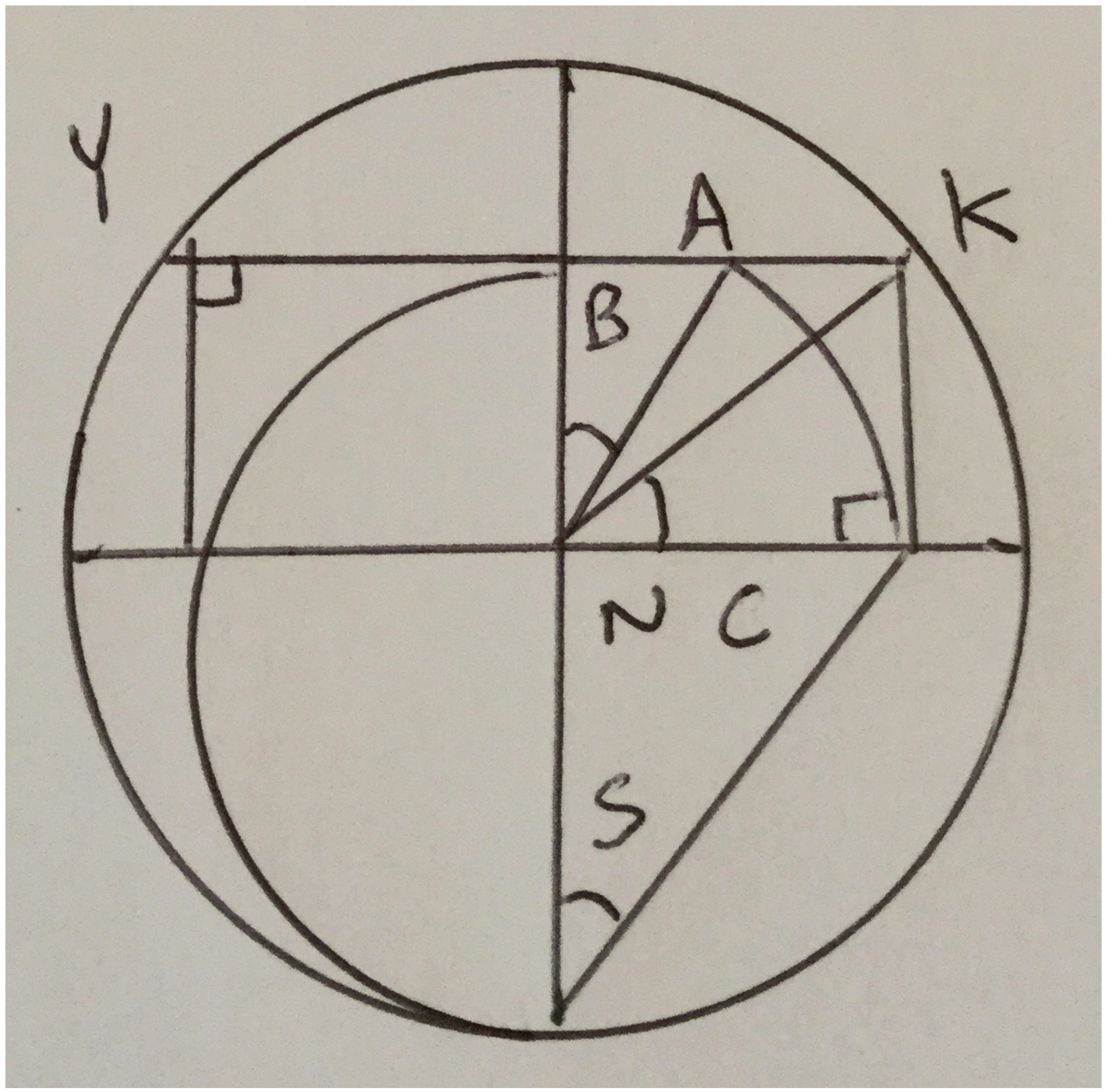
SC = BW II SC

KW = NS

$$\frac{NS}{NC} = \frac{NS}{NA}$$

$$\frac{NC}{NB} = \frac{NA}{NB}$$

Figure 13:



if: $\frac{NS}{NC} = \frac{NC}{NB}$

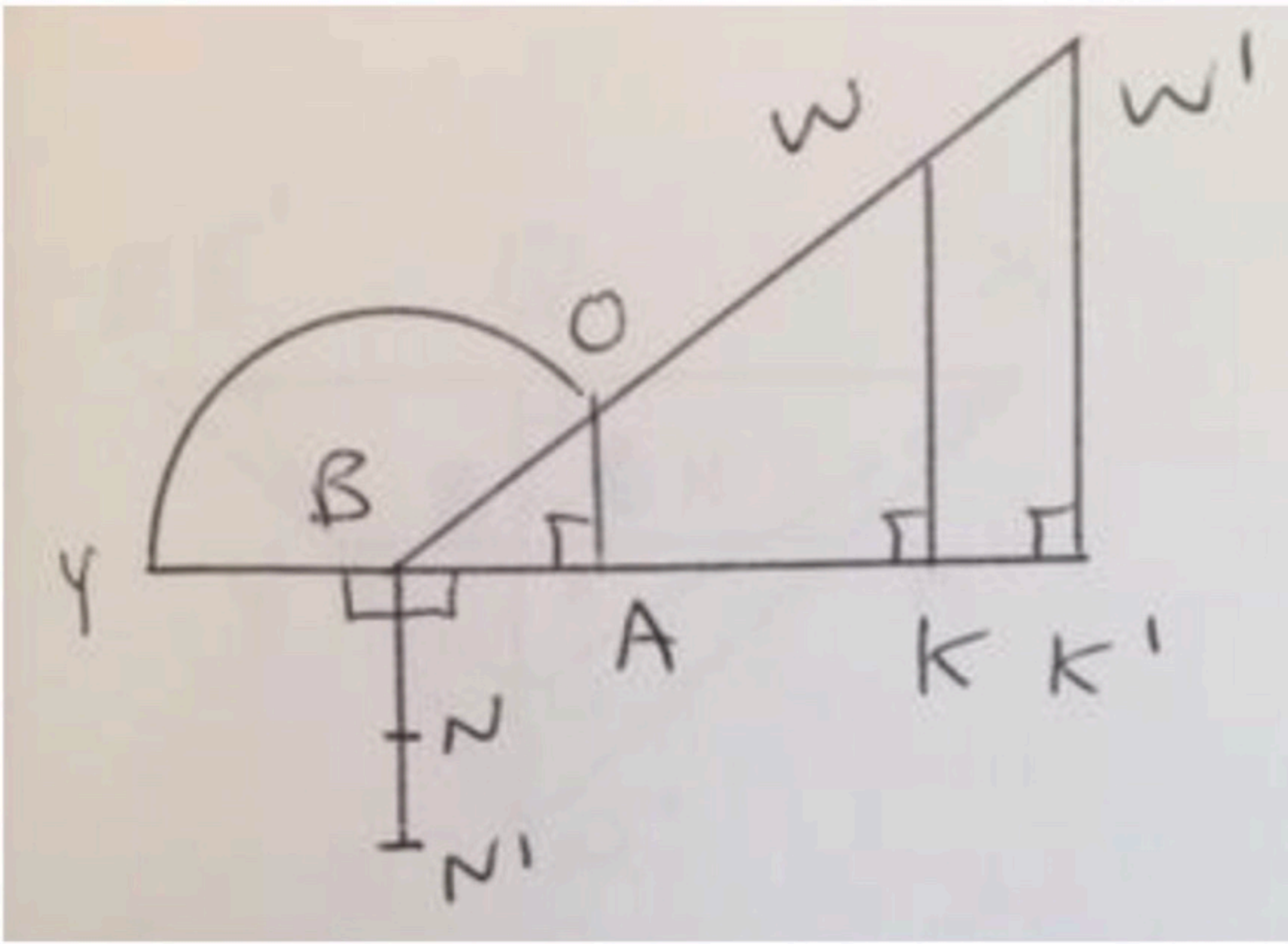
then: $\frac{NK}{NC} = \frac{NA}{NB}$

NA II SC

KW (= NS) = YN

It is obvious that as A approaches K from B, the relative rate that YN and KW approach infinity does not plateau, peak, or dip. Since we have shown that YN = KW when A lies at a point along BK other than B, as well as at B, we have shown that YN = KW for all points A along BK. (The Appendix provides the Law of Cosines approach to further illustrate this).

Figure 14:



$$\frac{OB}{OA} = \frac{NK}{NA} = \frac{N'K'}{N'A}$$

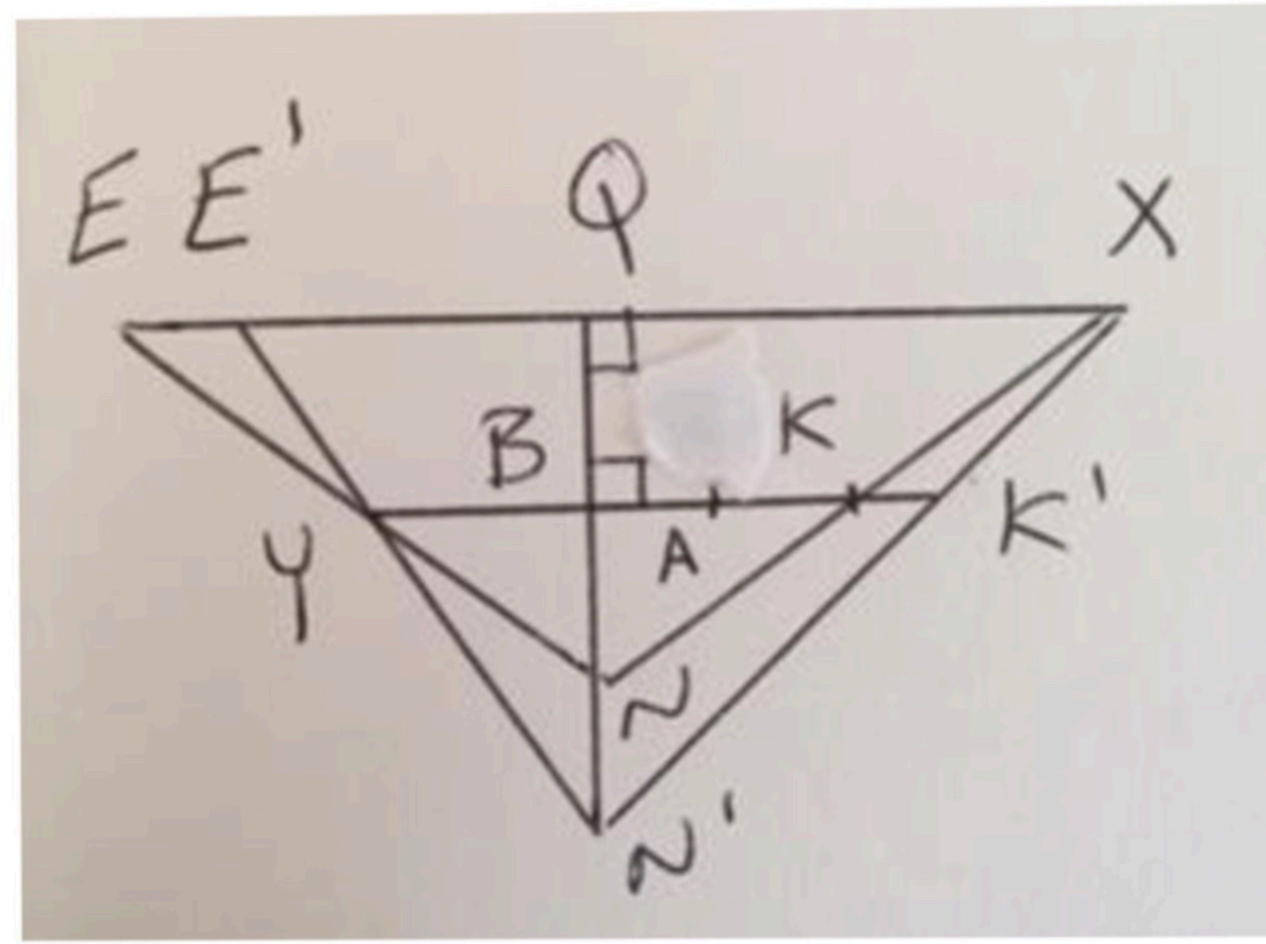
$$KW = YN$$

$$K'W' = YN'$$

$$\frac{KB}{YN} = \frac{K'B}{YN'}$$

Figure 15:

As the equal lengths of EN and E’N’ rotate about Y until they overlap, they approach their minimum which also occurs when N’K’X’ and NKX overlap.

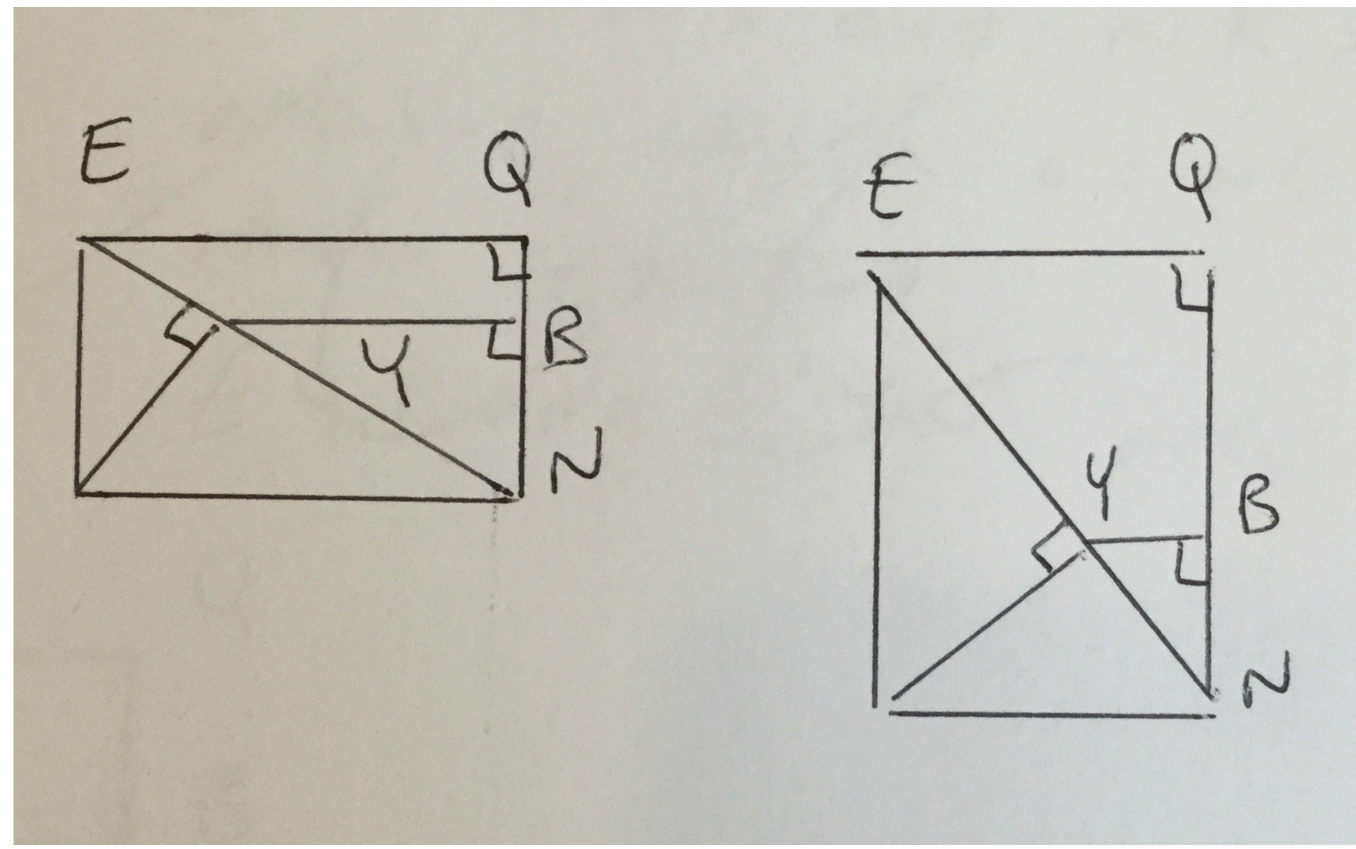


$$\frac{QX}{EN} = \frac{KB}{YN} = \frac{K'B}{YN'} = \frac{QX}{E'N'}$$

only one N’K’X exists for NKX because only one E’N’ equals EN

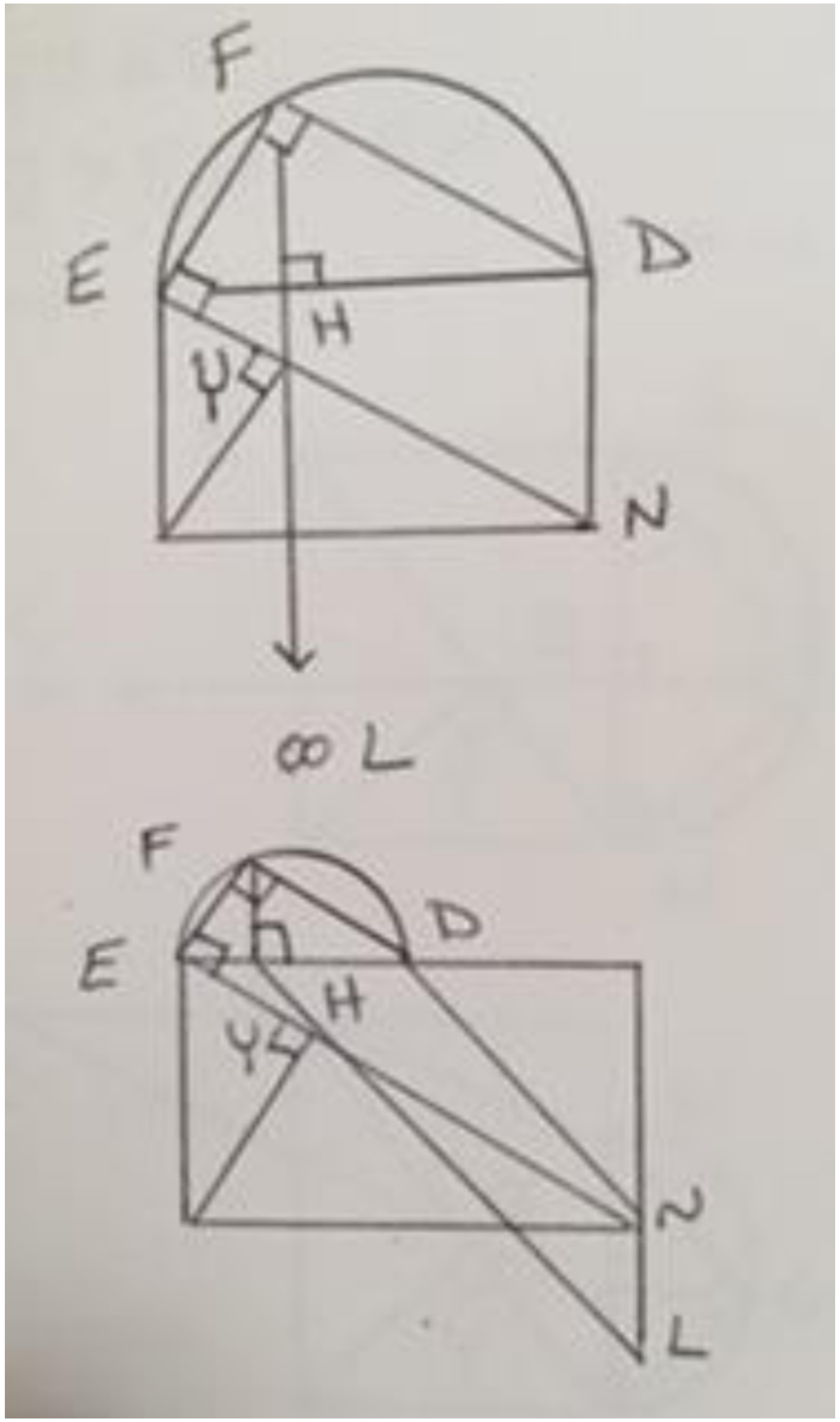
Figure 16:

Let X = Z when both NKX and N'K'X overlap, which occurs when EYN is the shortest line segment through Y connecting line QB to its perpendicular at Q. This occurs when:



because:

Figure 17:



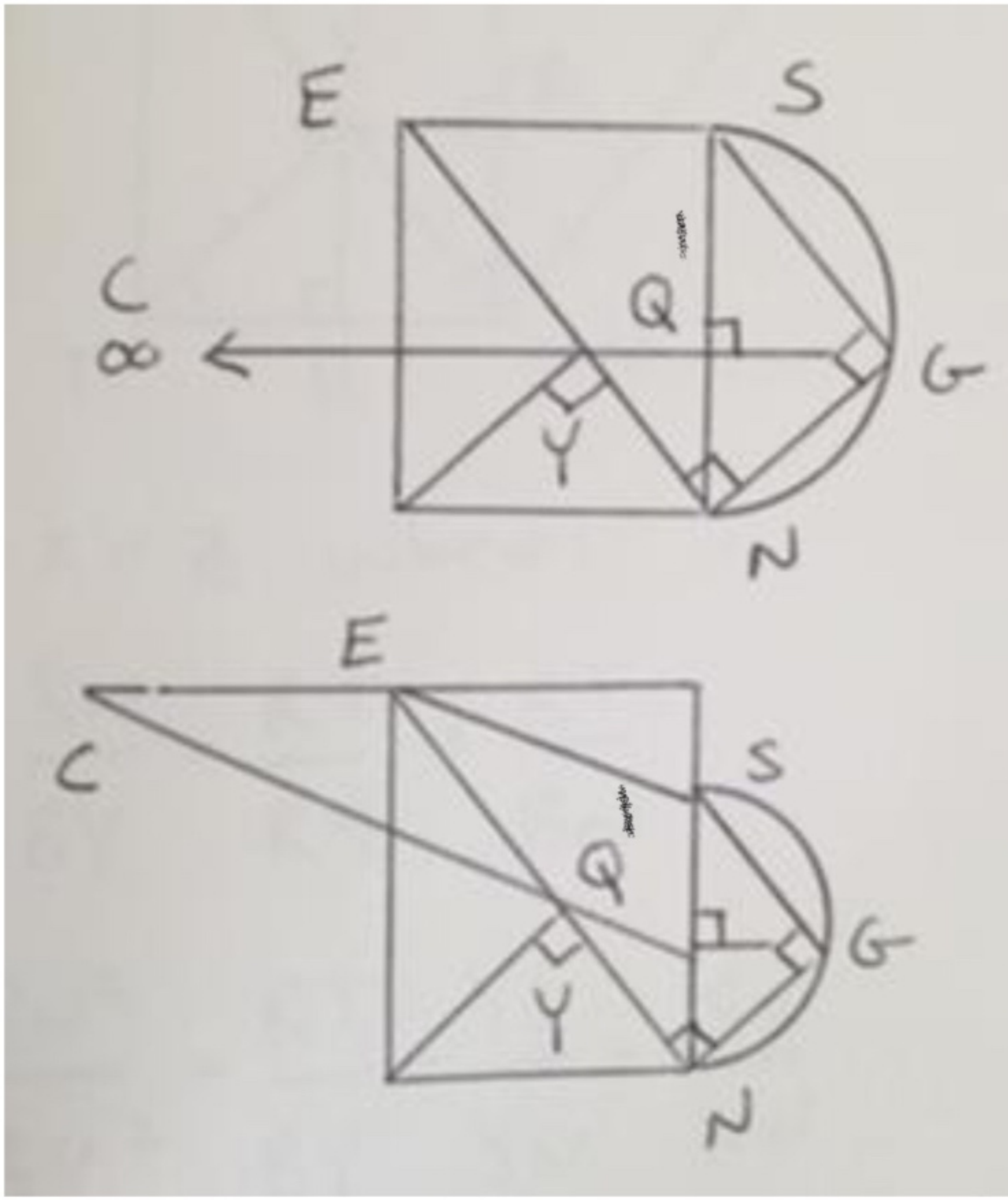
LH || ND

LH > NF > NE

holds true as:

H ⇒ E

Figure 18:



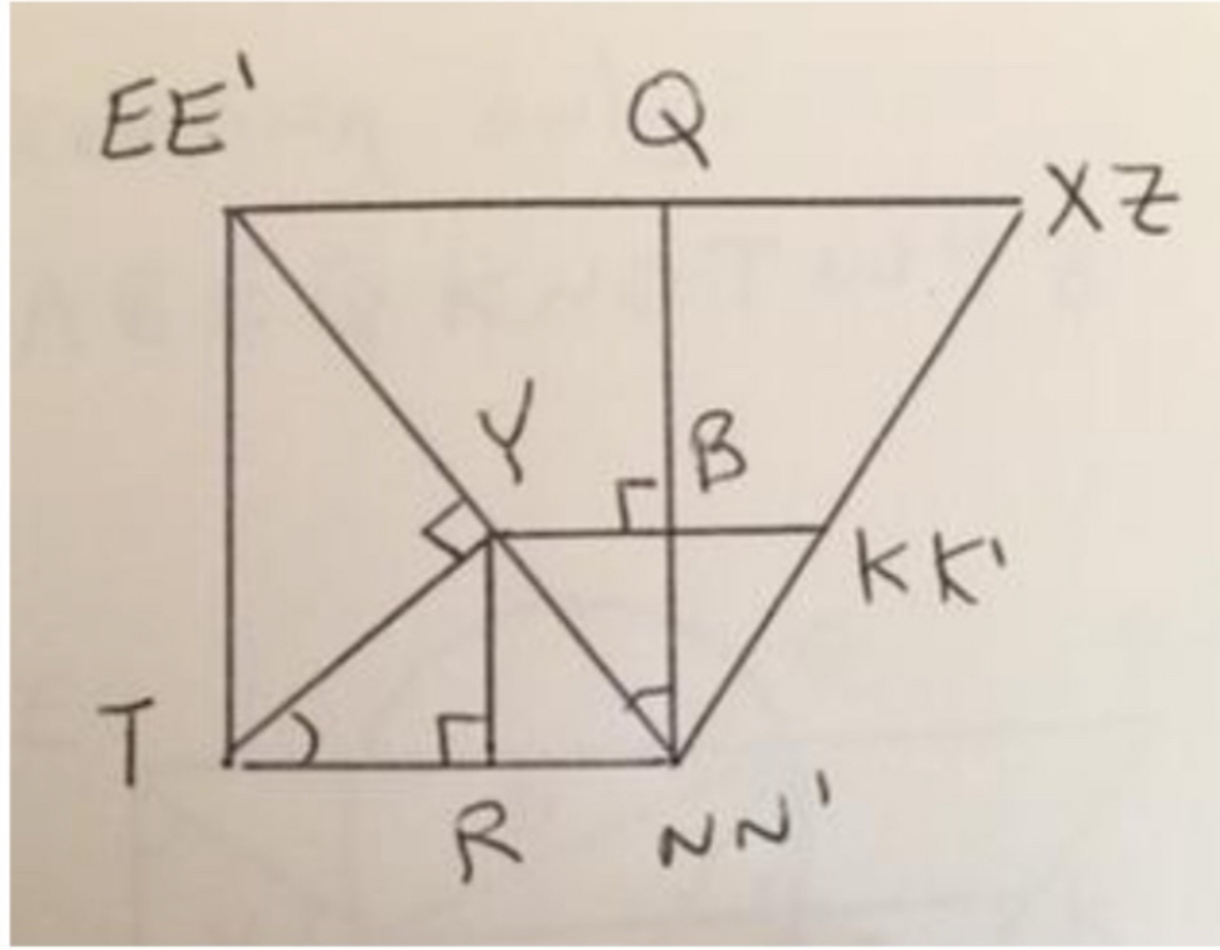
CQ' || ES

CQ' > EG > EN

holds true as

Q' ⇒ N

Figure 19:

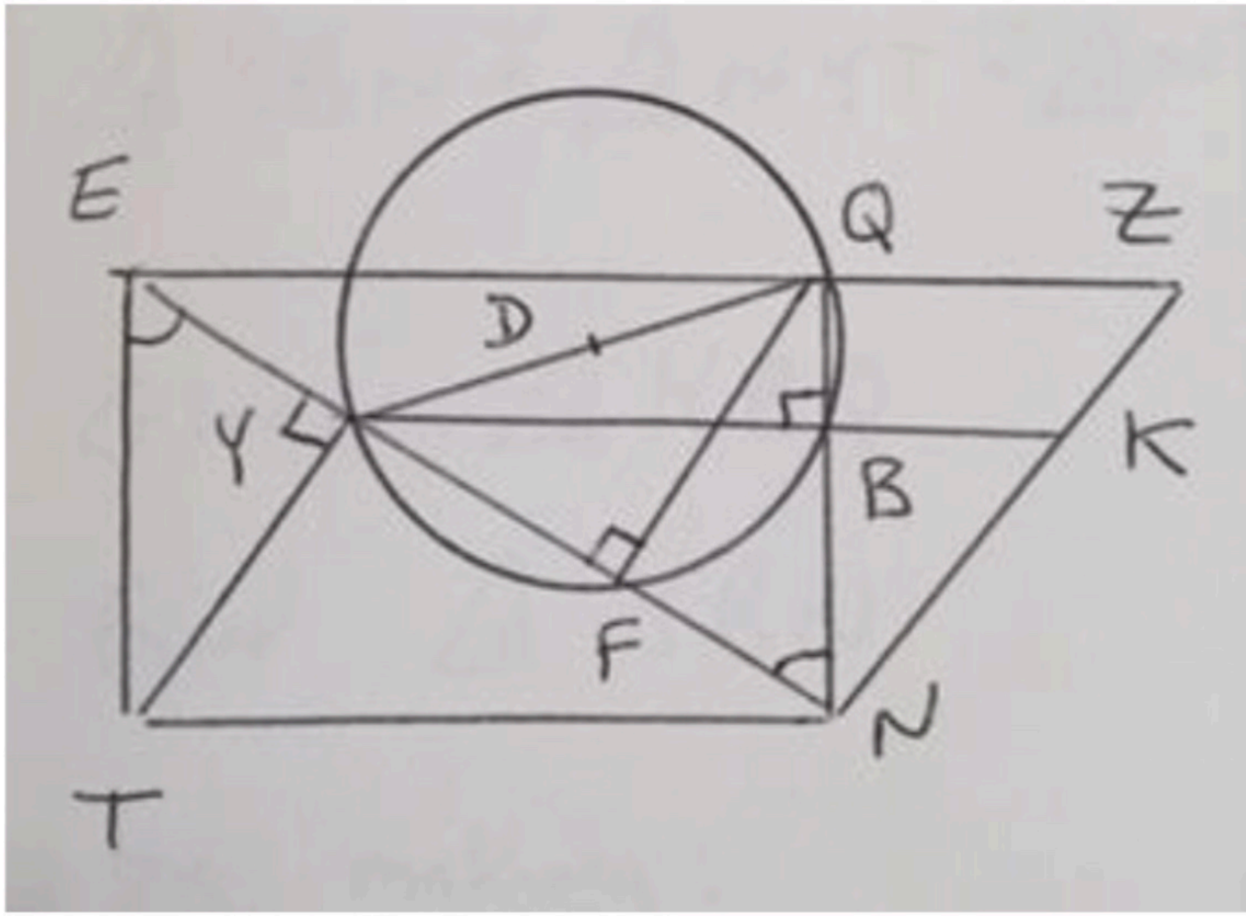


X = Z when:

$$\frac{BN}{BY} = \frac{RT}{RY} = \frac{RT}{BN}$$

$$\frac{BN^2}{BY^2} = \frac{RT}{BY} = \frac{YE}{YN} = \frac{KX}{KN}$$

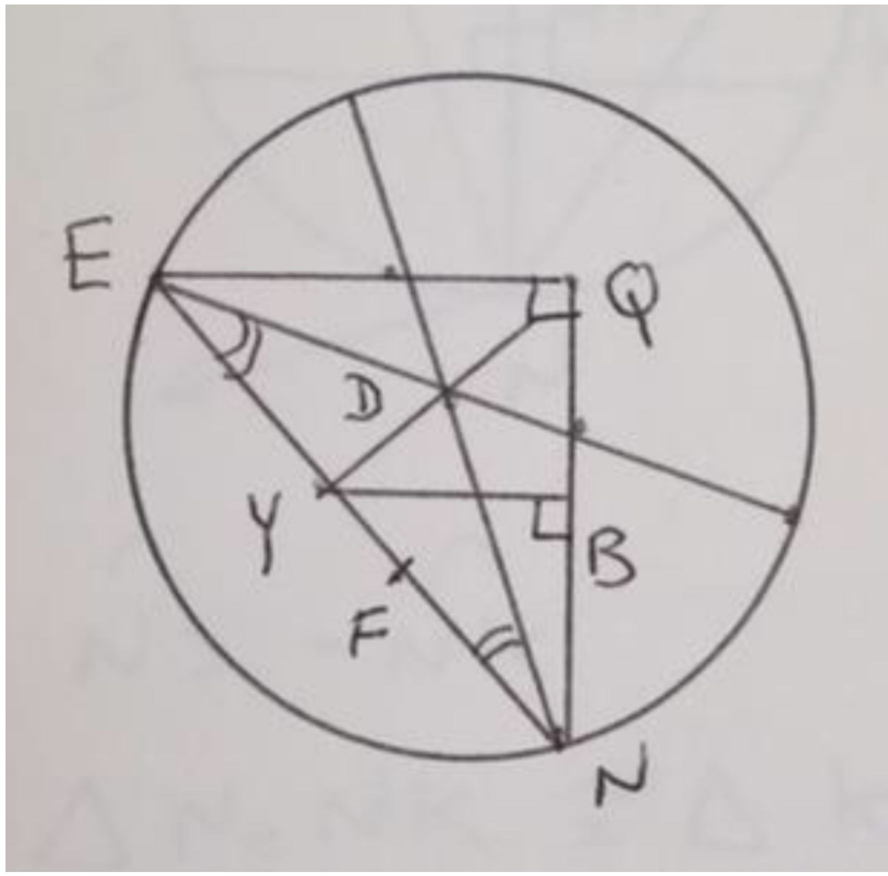
Figure 20:



given ΔYBN, find ΔYBQ using:

$\Delta YBN \cong \Delta NYT \cong \Delta NTE$

Figure 21:



given ΔYBQ, find ΔYBN by making:

EY = NF

which occurs when ~EN lies on a circle concentric with circle YFBQ

because:

DY = DF

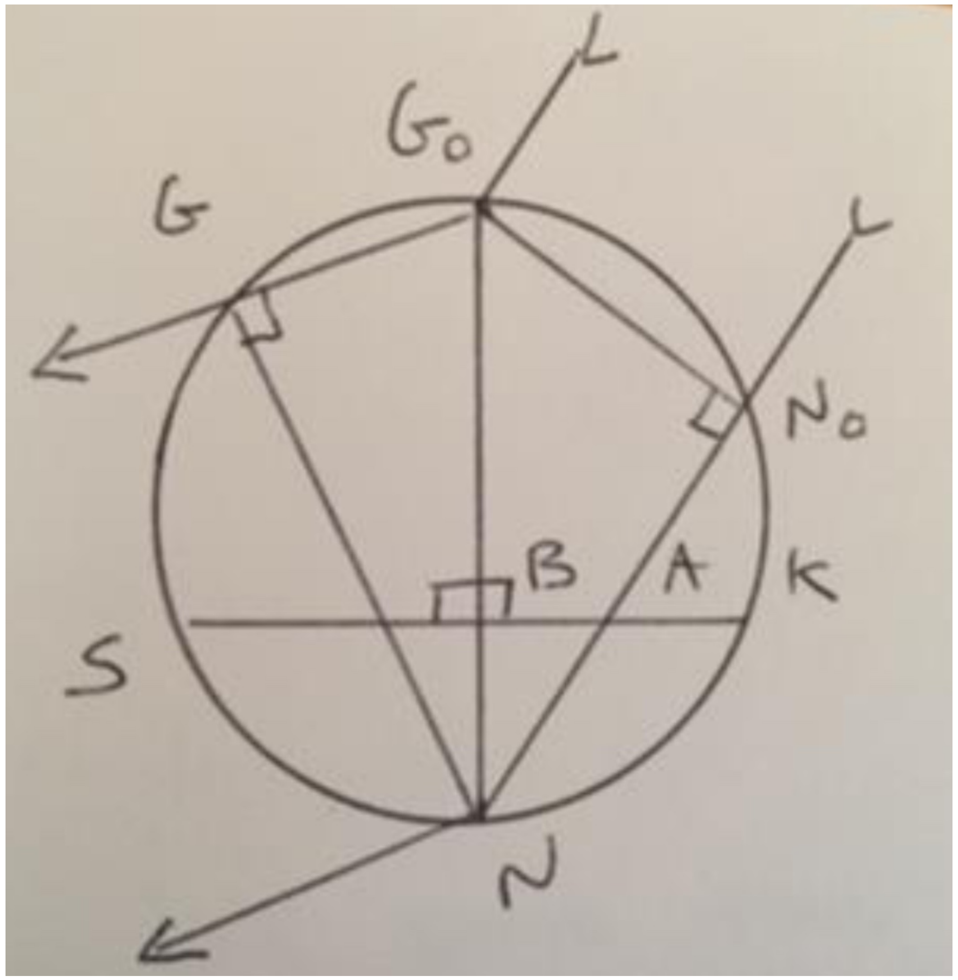
ΔEDY = ΔNDF

EY = NF

Before considering refraction along a line, picture yourself sitting on the beach watching waves roll in. Notice that even when wavefronts far out in the ocean are traveling perpendicular to the beach, they become closer to parallel with the beach as they crash. On beaches that are long and sloped, or have many sandbars, these wavefronts all crash parallel to the beach, regardless of their orientation in the open ocean.

Now picture yourself in a car applying brakes while driving. If the brakes on the front right wheel grip harder, the car will turn to the right. This is intuitive. For the same reason, when a wavefront hits a sandbar at an angle, one side of the wavefront will slow before the other, and this will tend to turn the wavefront parallel to the beach. This essentially represents refraction along a line.

Figure 22:



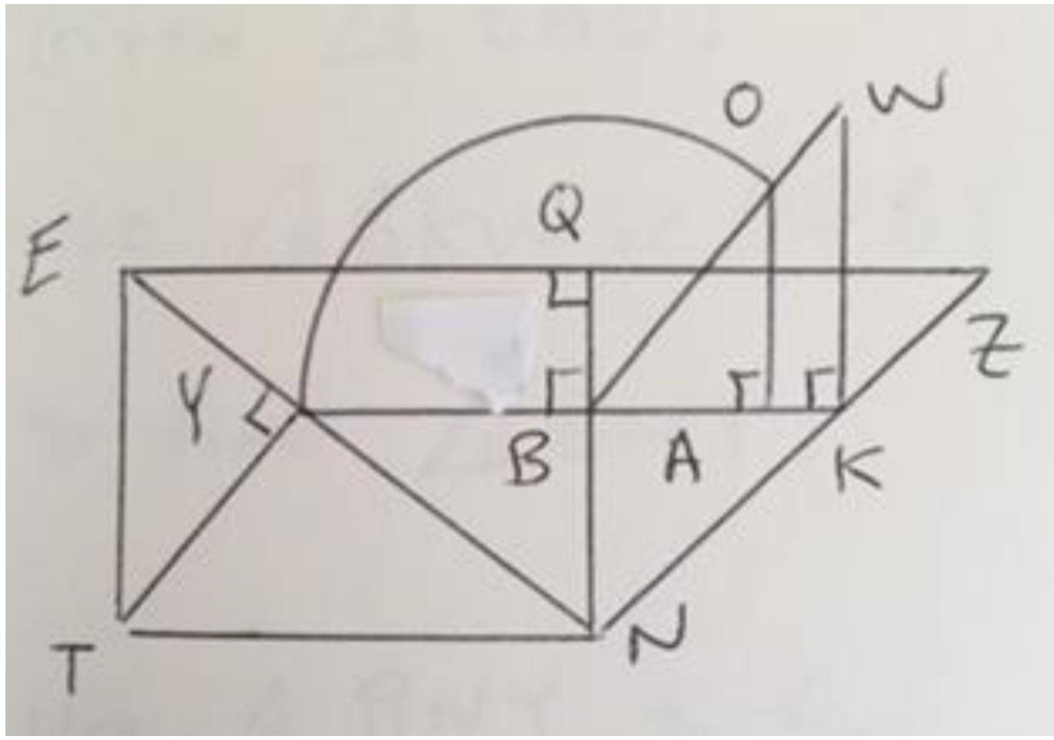
$\sim NS \;=\; \sim NK$

$\Delta N_\circ NK \;\cong\; \Delta KNA$

$$\mathbb{R} = \frac{NN_\circ}{GG_\circ} = \frac{NN_\circ}{NK} = \frac{NK}{NA}$$

wavefront $G_\circ N_\circ$ refracts into
wavefront GN along $G_\circ N$,
because it travels $G_\circ G$
in the same time it travels $N_\circ N$

Figure 23:



If $\mathbb{R} = \frac{OB}{OA}$ and $KW = YN$:

$\mathbb{R} = \frac{NK}{NA}$

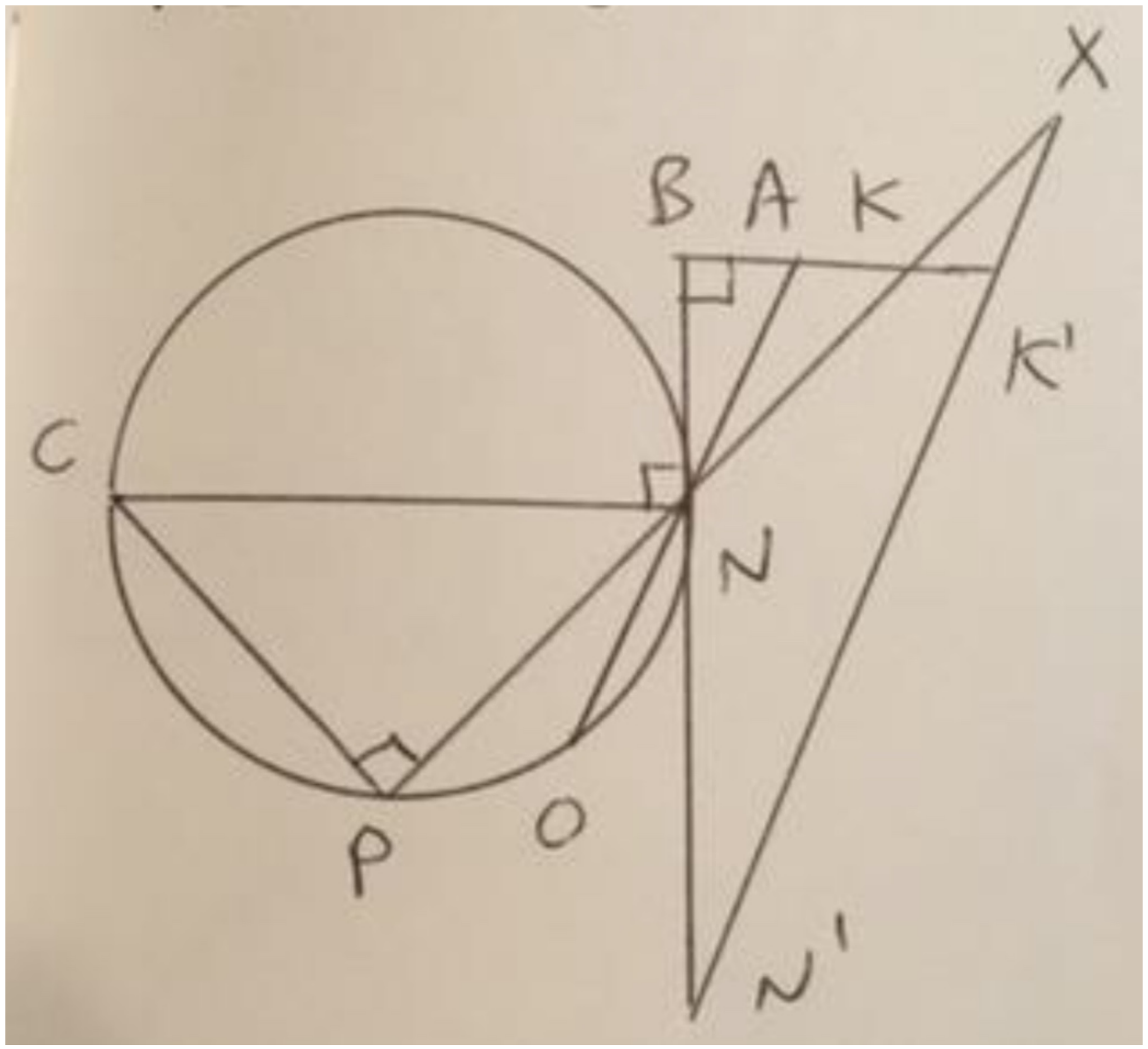
and Z is the clear image of object A refracted at N along BN.

given ΔBAO:
use ΔBKW or ΔQBY to find ΔBNY.
use ΔBNY to find ΔBKW or ΔQBY.

Section 3

Refraction Along a Circle

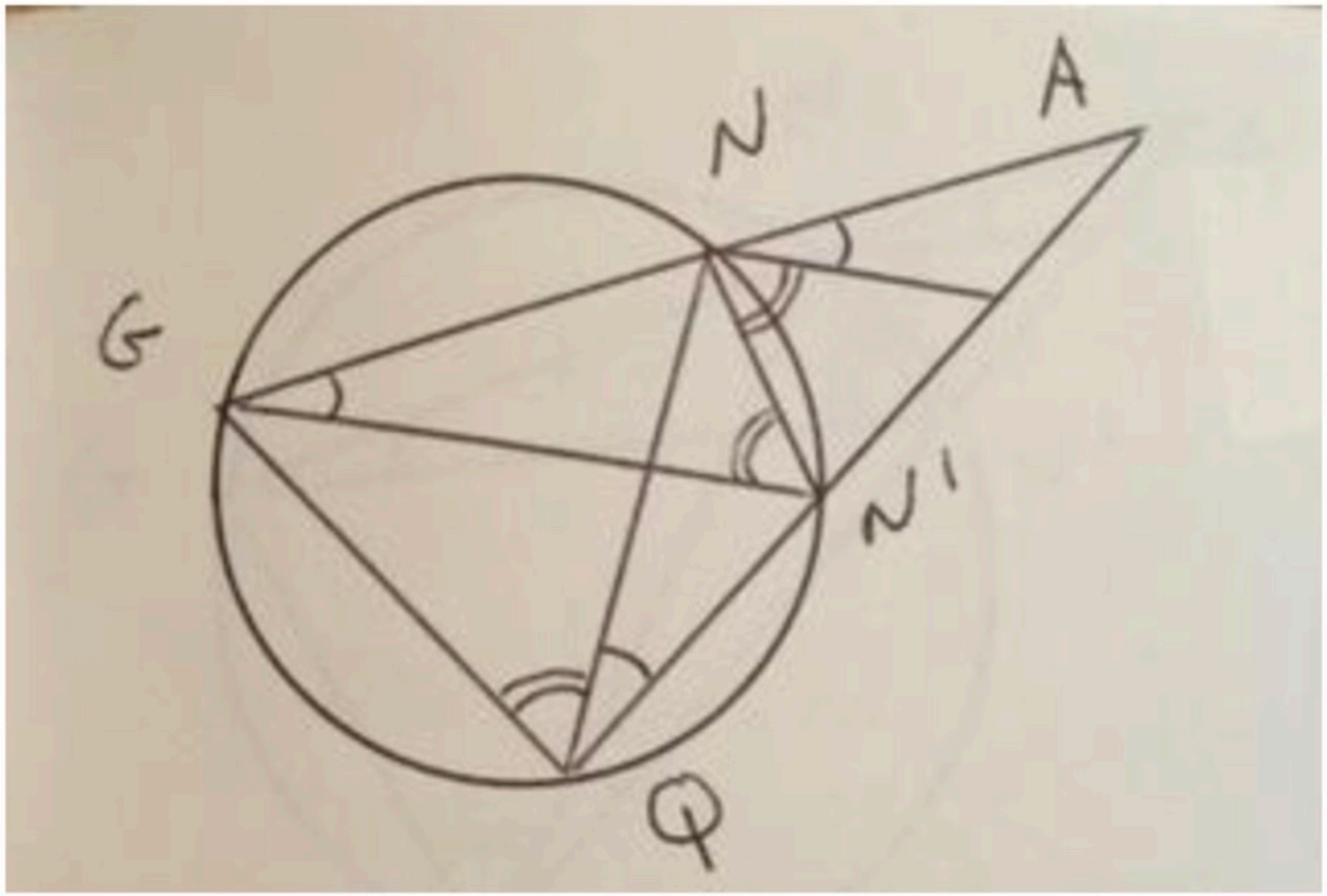
Figure 24:



$$\Delta KNA \cong \Delta OCP$$

$$\mathbb{R} = \frac{NK}{NA} = \frac{N'K'}{N'A} = \frac{CO}{CP}$$

Figure 25:



ΔANN’ ≅ ΔAQG

Figure 26:

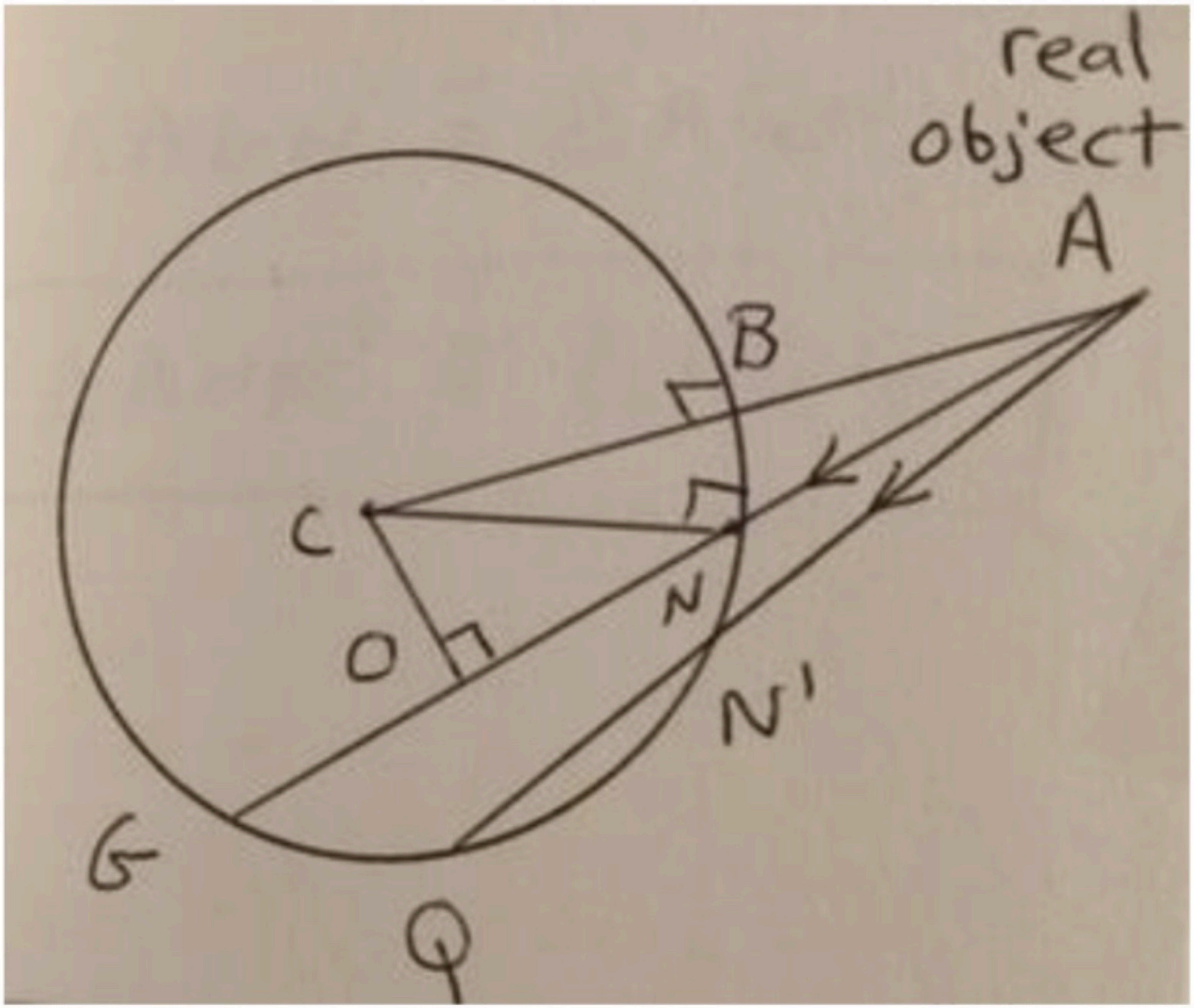
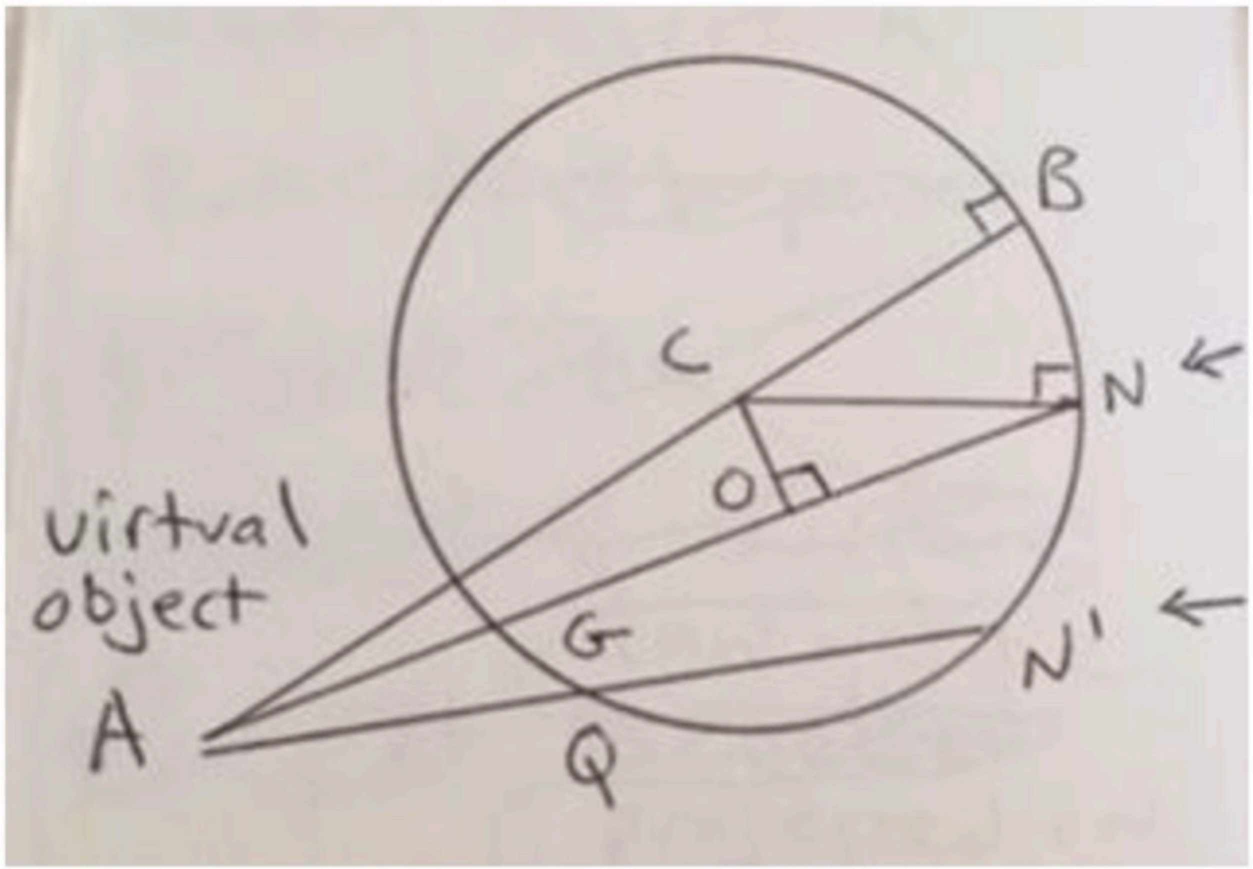
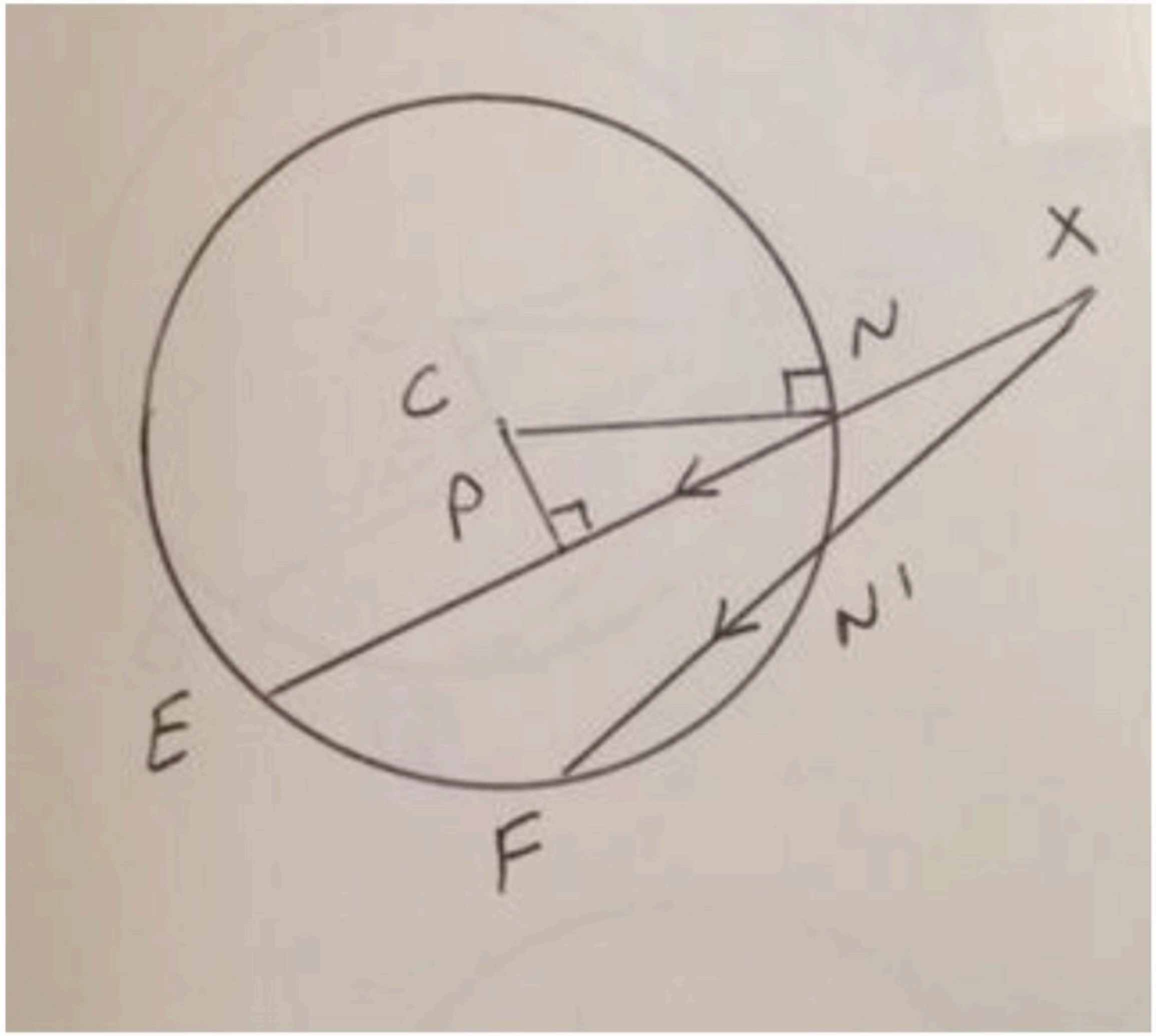


Figure 27:



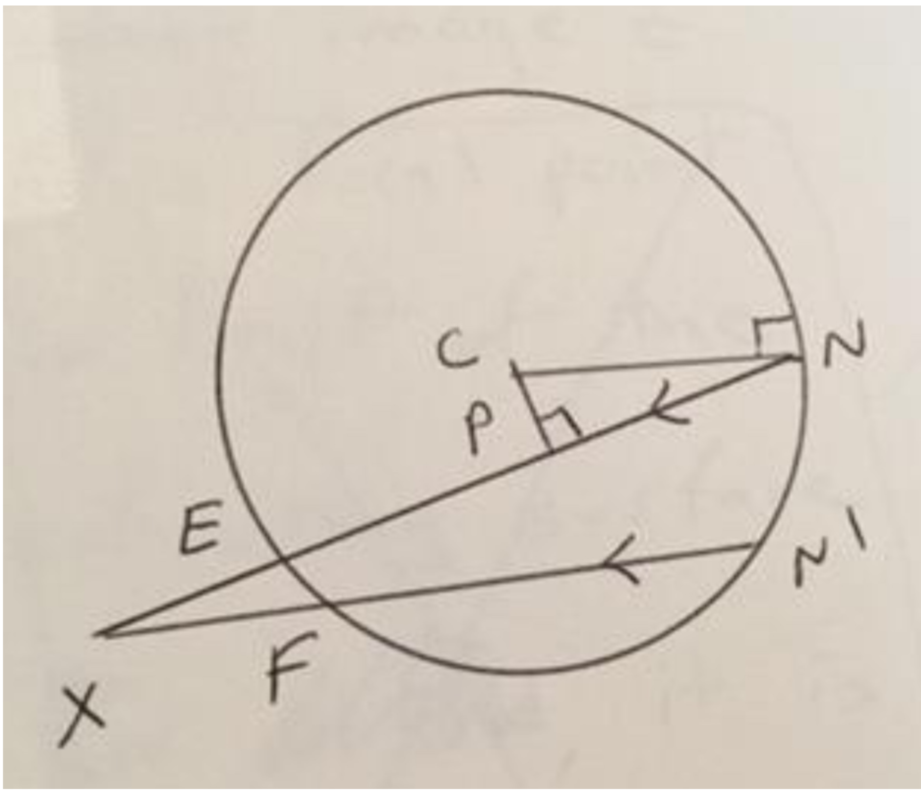
the virtual object A can not be projected on a screen
due to refraction at BN

Figure 28:



ΔXNN' ≅ ΔXFE
the virtual image (Z) can not be projected on a screen

Figure 29:



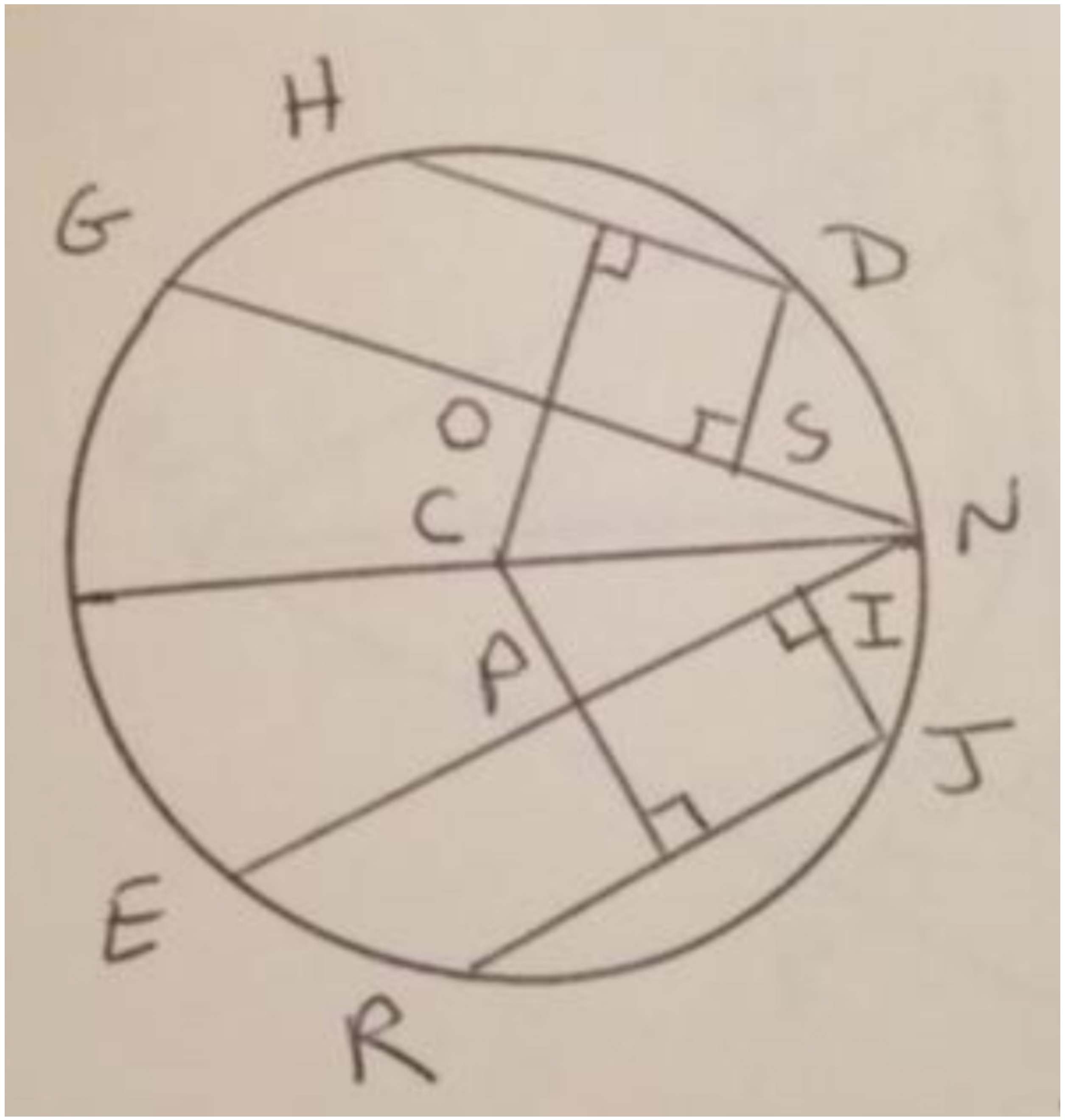
the real image (Z) can be projected on a screen

$$\frac{AG + AN'}{2AN'} = \frac{QG + NN'}{2NN'}$$

$$\frac{XE + XN'}{2XN'} = \frac{EF + NN'}{2NN'}$$

$$\frac{QG + NN'}{EF + NN'} = \frac{(AG + AN')}{2AN'} \frac{2XN'}{(XE + XN')}$$

Figure 30:



HD = QN'

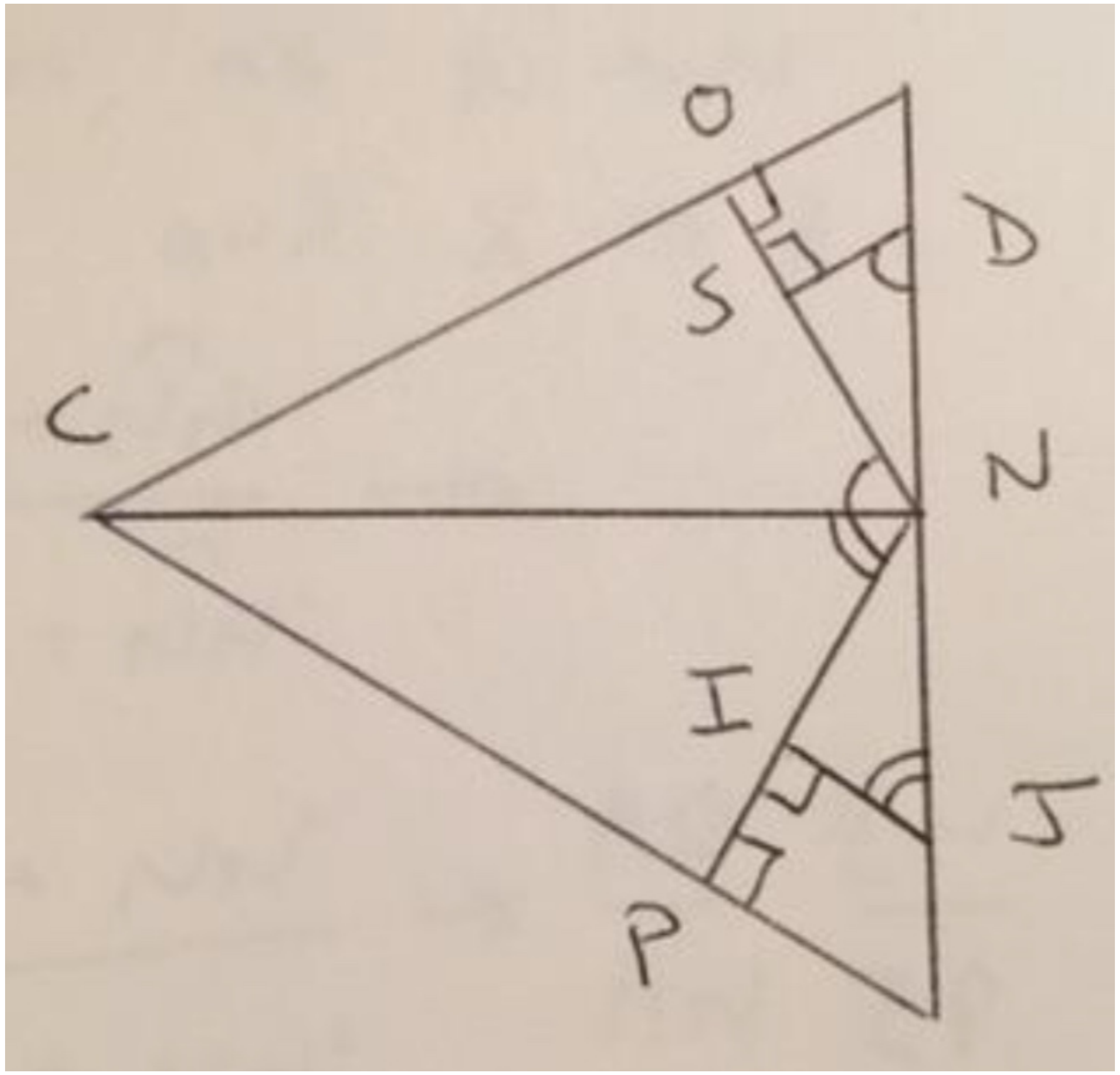
RJ = FN'

as N'⇒ N:

X⇒ Z, and ~DJ⇒ DJ

so that:

Figure 31:



$$\frac{DS}{JI} \Rightarrow \frac{CO}{CP}$$

$$\frac{JI}{JN} \Rightarrow \frac{NP}{NC}$$

$$\frac{DN}{DS} \Rightarrow \frac{NC}{NO}$$

$$\frac{ND}{NJ} \Rightarrow \frac{NP}{NO} \quad \frac{CO}{CP}$$

thus, as N’ $\Rightarrow$ N and X $\Rightarrow$ Z:

$$\frac{\sim QG + \sim NN'}{\sim EF + \sim NN'} \Rightarrow \frac{QG + NN'}{EF + NN'} \Rightarrow$$

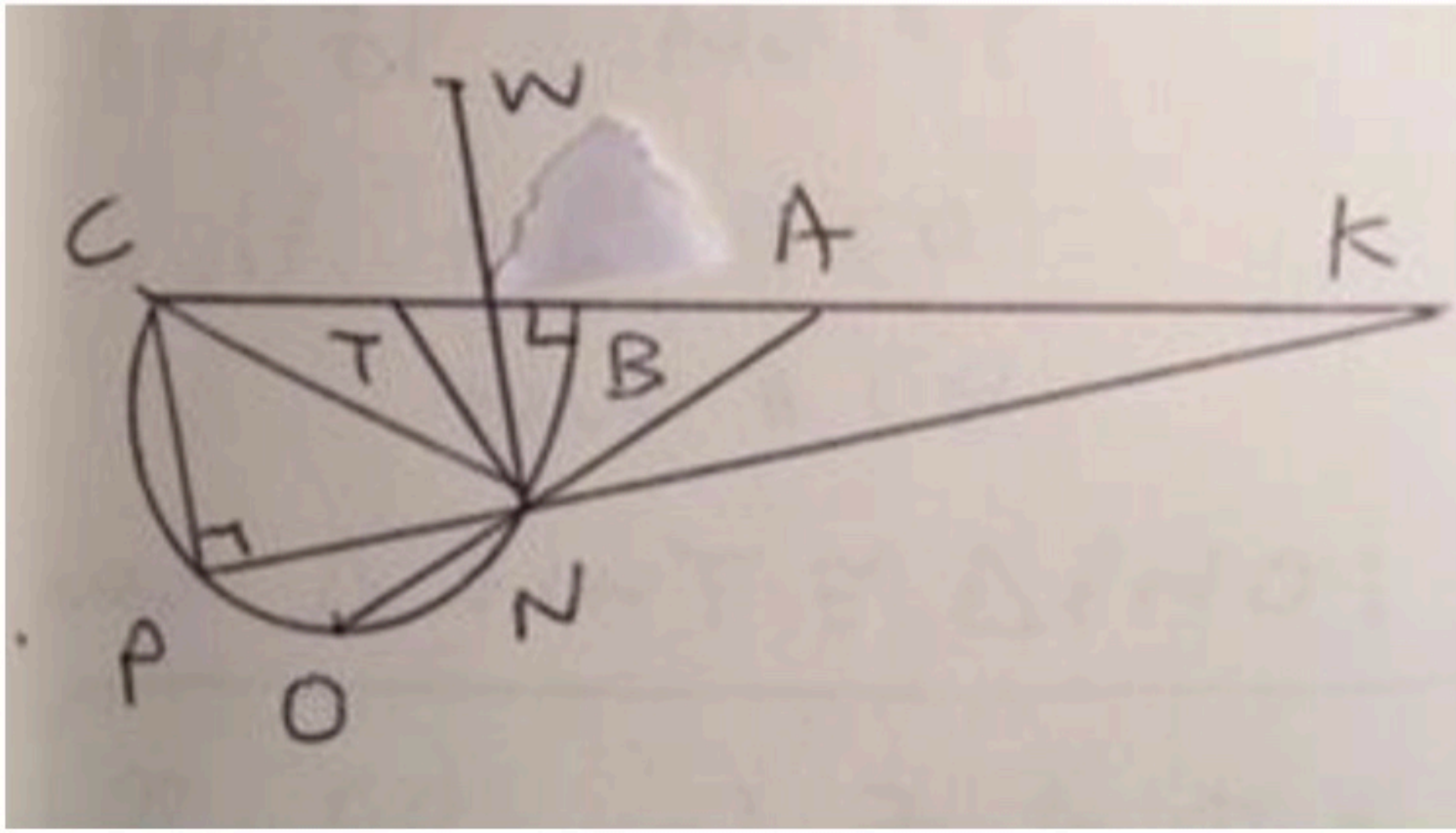
$$\frac{AO}{AN}\ \frac{ZN}{ZP}$$

and:

$$\frac{\sim QG + \sim NN'}{\sim EF + \sim NN'} = \frac{2\,(\sim ND)}{2\,(\sim NJ)} \Rightarrow$$

$$\frac{ND}{NJ} \Rightarrow \frac{NP}{NO}\ \frac{CO}{CP}$$

Figure 32:



NT || CO
NW || CP
when X = Z lies along
both NP and CW:

$$\frac{AO}{AN}\,\frac{ZN}{ZP} = \frac{CO}{NT}\,\frac{NW}{CP}$$

when $\Delta WNT \cong \Delta PNO$, $NW > NT$

and

$$\frac{AO}{AN}\frac{ZN}{ZP} = \frac{NP}{NO}\frac{CO}{CP}$$

so if:

$NT \parallel CO$

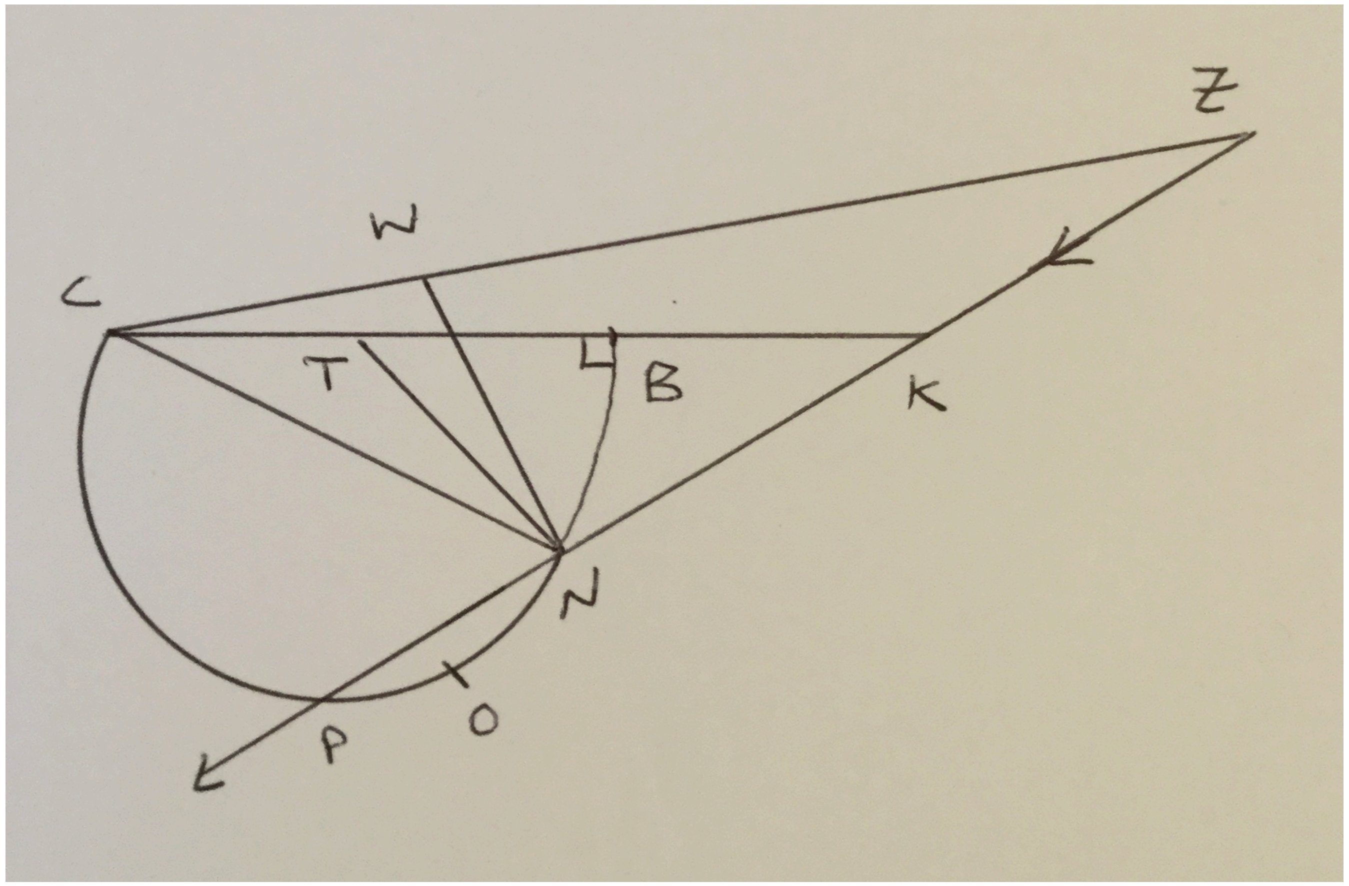
$NW \parallel CP$

and $\Delta WNT \cong \Delta PNO$:

$$\mathbb{R} = \frac{CO}{CP}$$

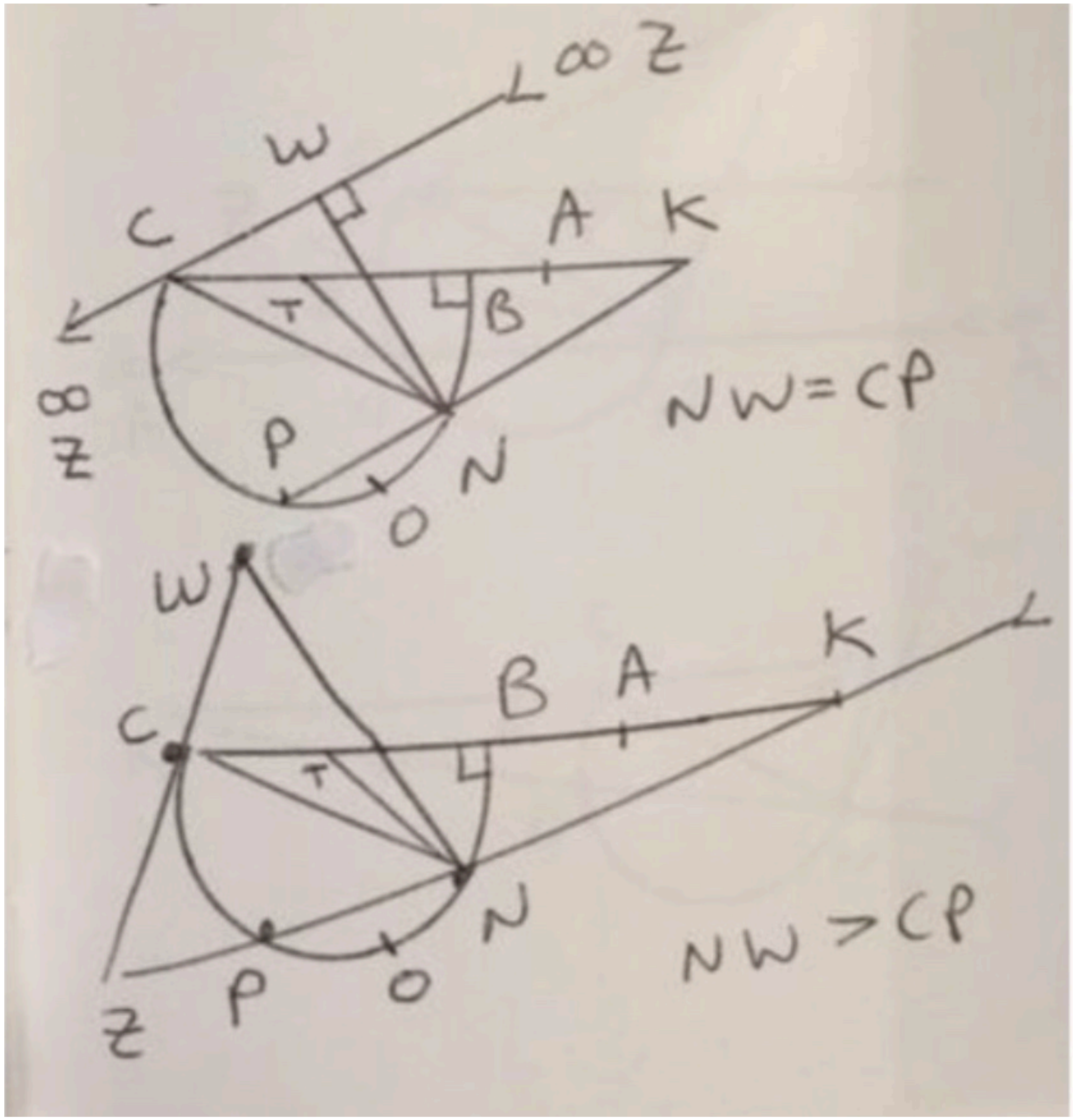
and Z is the clear image of object A refracted at N along ~BN

Figure 33:



Off-axis rays from any on-axis object A, (real or virtual), can not form a virtual on-axis image Z because NW must be less than CP for Z to be virtual; but NW must also be greater than NT.

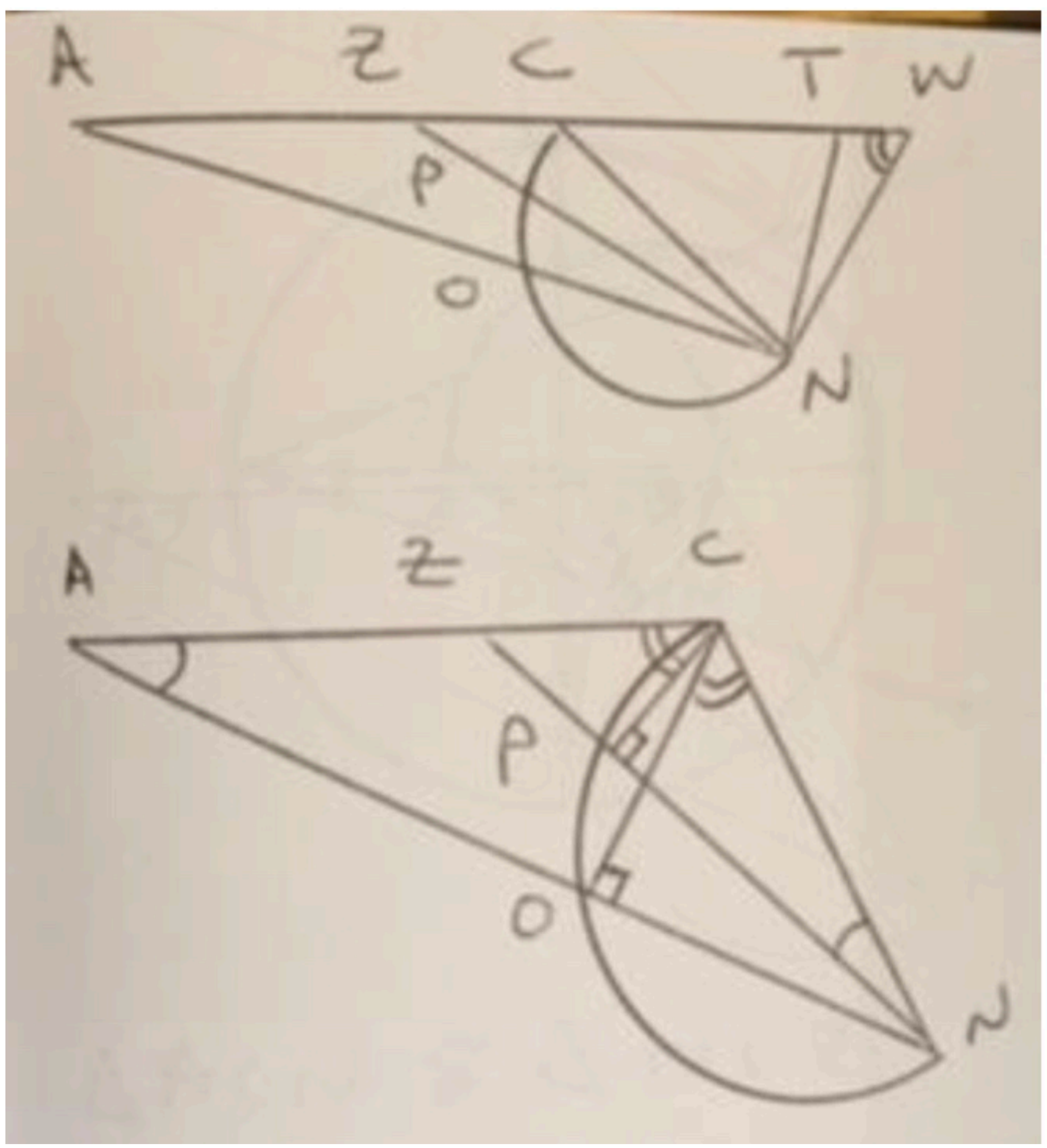
Figure 34:



Off-axis rays from any real on-axis object A can not form a real on-axis image Z because NW must be greater than (or equal to) CP for Z to be real; but NW must also be greater than NT.

Figure 35:

Off-axis rays from a virtual on-axis object A ***can*** form a real on-axis image Z because NW must be greater than or equal to CP for Z to be real; and NW must also be greater than NT. When WT lies along the axis, so does Z. This occurs when:



$$NT \parallel CO$$

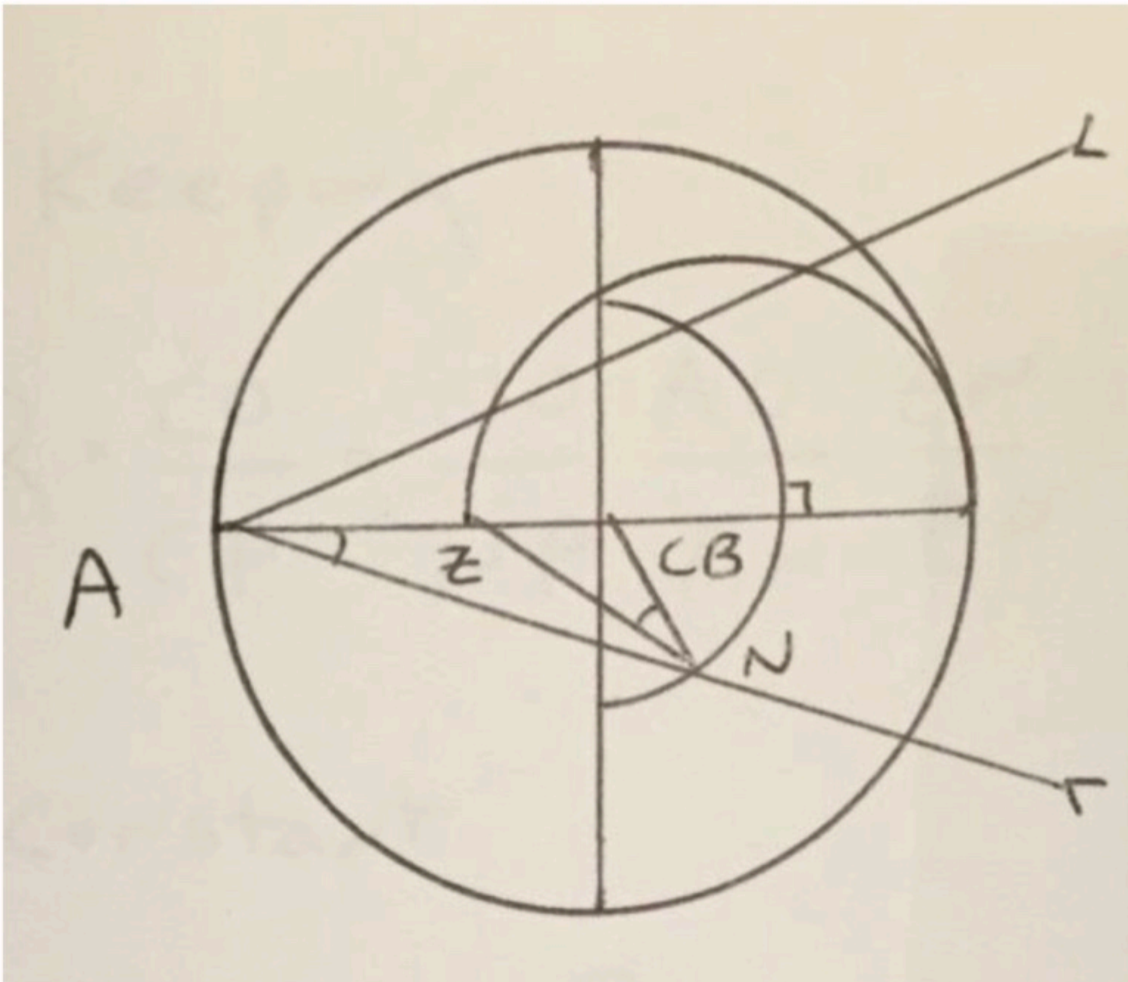
$$NW \parallel CP$$

$$\Delta WNT \cong \Delta PNO$$

$$\angle NWT = \angle NPO = \angle NCO$$

$$\Delta CPN \cong \Delta COA$$

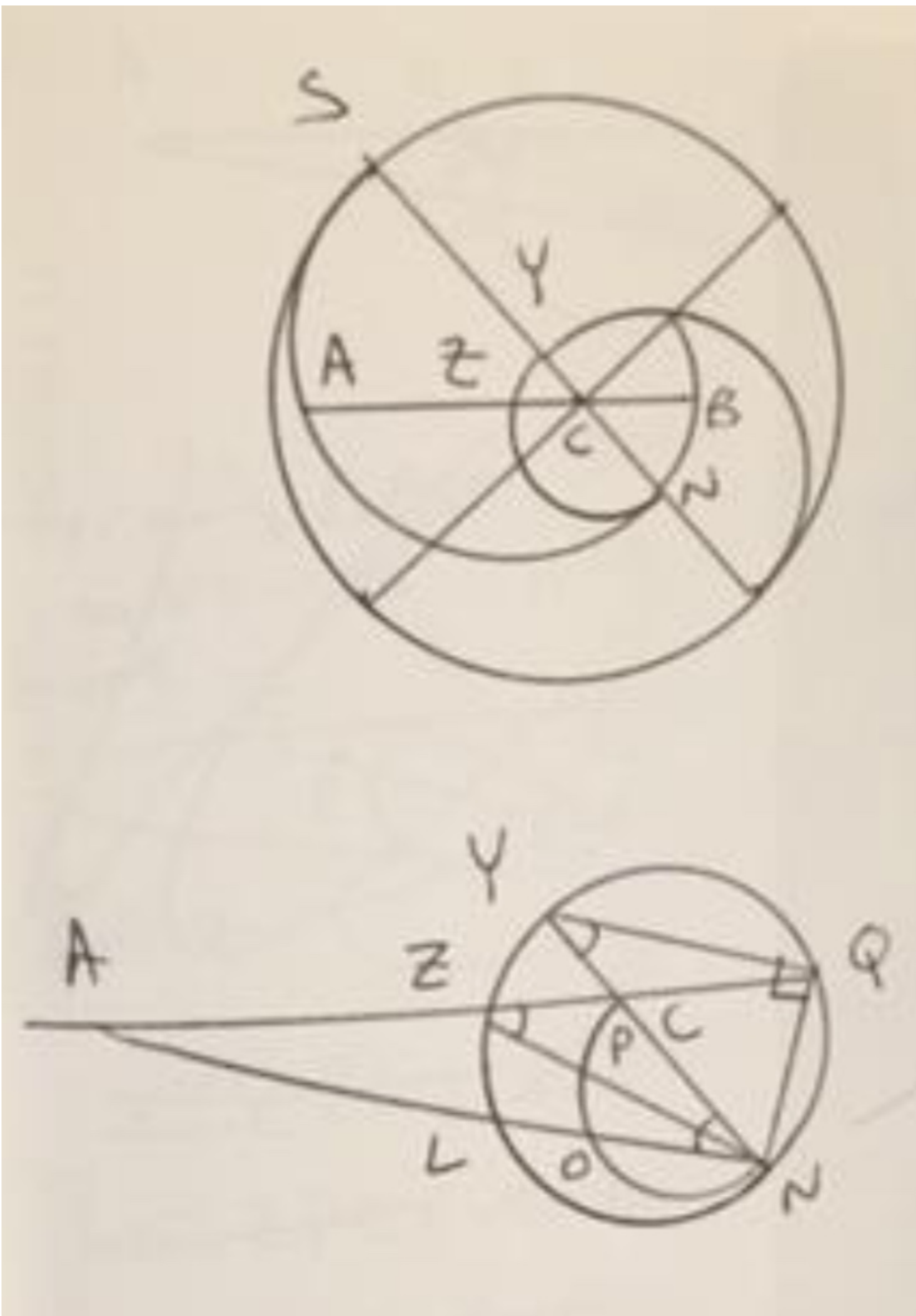
Figure 36:



When off-axis rays from a virtual on-axis object A form a real on-axis image Z, this is the on-axis real image of the on-axis virtual object A at all points N because:

$$\Delta ACN \cong \Delta NCZ \quad \text{for all } N$$

Figure 37:



This can also be demonstrated by constructing:

SC/CN = CN/CY

$$\frac{CY}{CN} = \frac{CN}{CS} = \frac{CY + CN}{CN + CS} = \frac{NY}{NS}$$

$$\frac{AO}{AN}\ \frac{ZN}{ZP} = \frac{SC}{SN}\ \frac{ZN}{ZP} = \frac{NC}{NY}\ \frac{ZN}{ZP} =$$

$$\frac{NC}{NY}\ \frac{YN}{YC} = \frac{CN}{CY}$$

$$\frac{CO}{CP}\ \frac{NP}{NO} = \frac{LY}{LN}\ \frac{PN}{PC} = \frac{QN}{QY}\ \frac{PN}{PC} =$$

$$\frac{QN\ (ZN)}{QY\ (ZY)} = \frac{CN}{CY}$$

Section 4

Axial Refraction at a Circle

keeping:

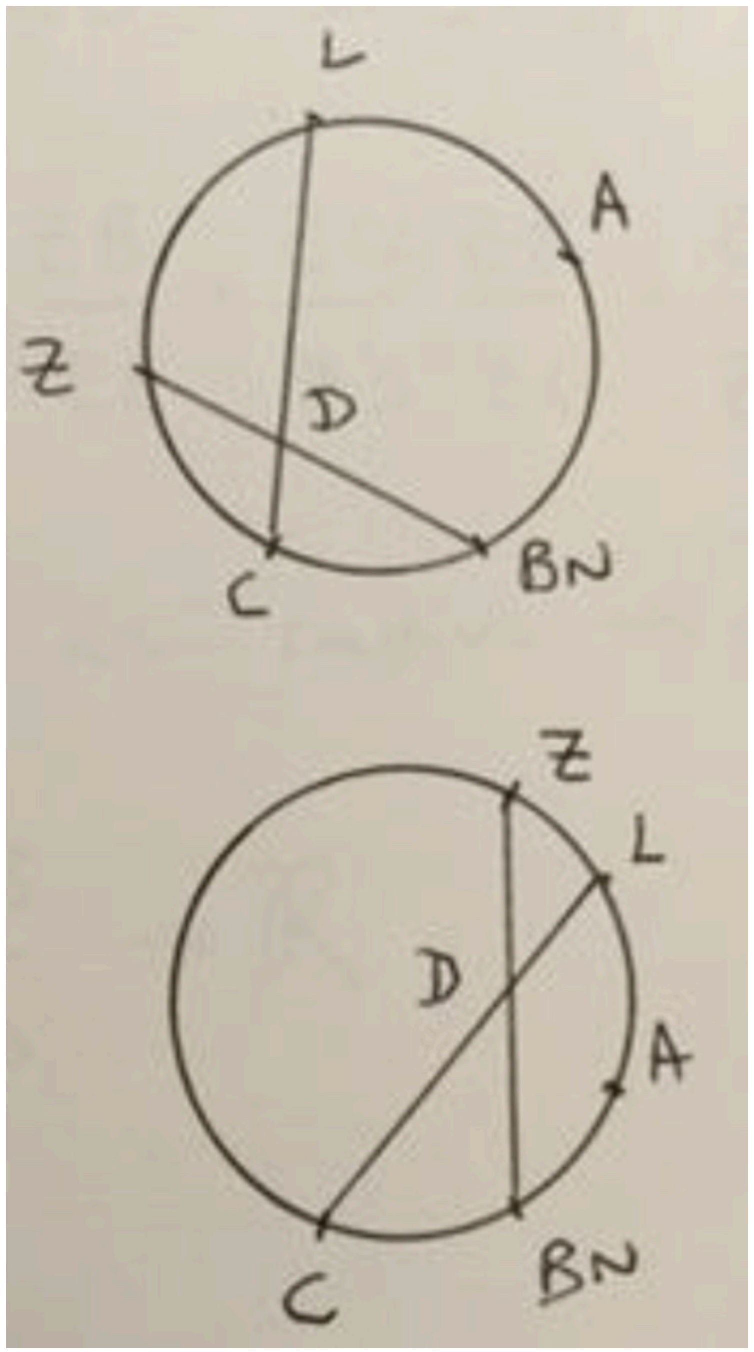
$$\mathbb{R} \quad = \quad \frac{CO}{CP} \quad = \quad \frac{NO}{NP}\ \frac{AO}{AN}\ \frac{ZN}{ZP}$$

constant as N $\Rightarrow$ B:

$$\frac{BC}{BC}\ \frac{AC}{AB}\ \frac{ZB}{ZC} \quad \Rightarrow \quad \mathbb{R}$$

Figure 38:

"axial" refraction can be described along a circle of infinite radius



draw CDL so:

AL || ZB so:

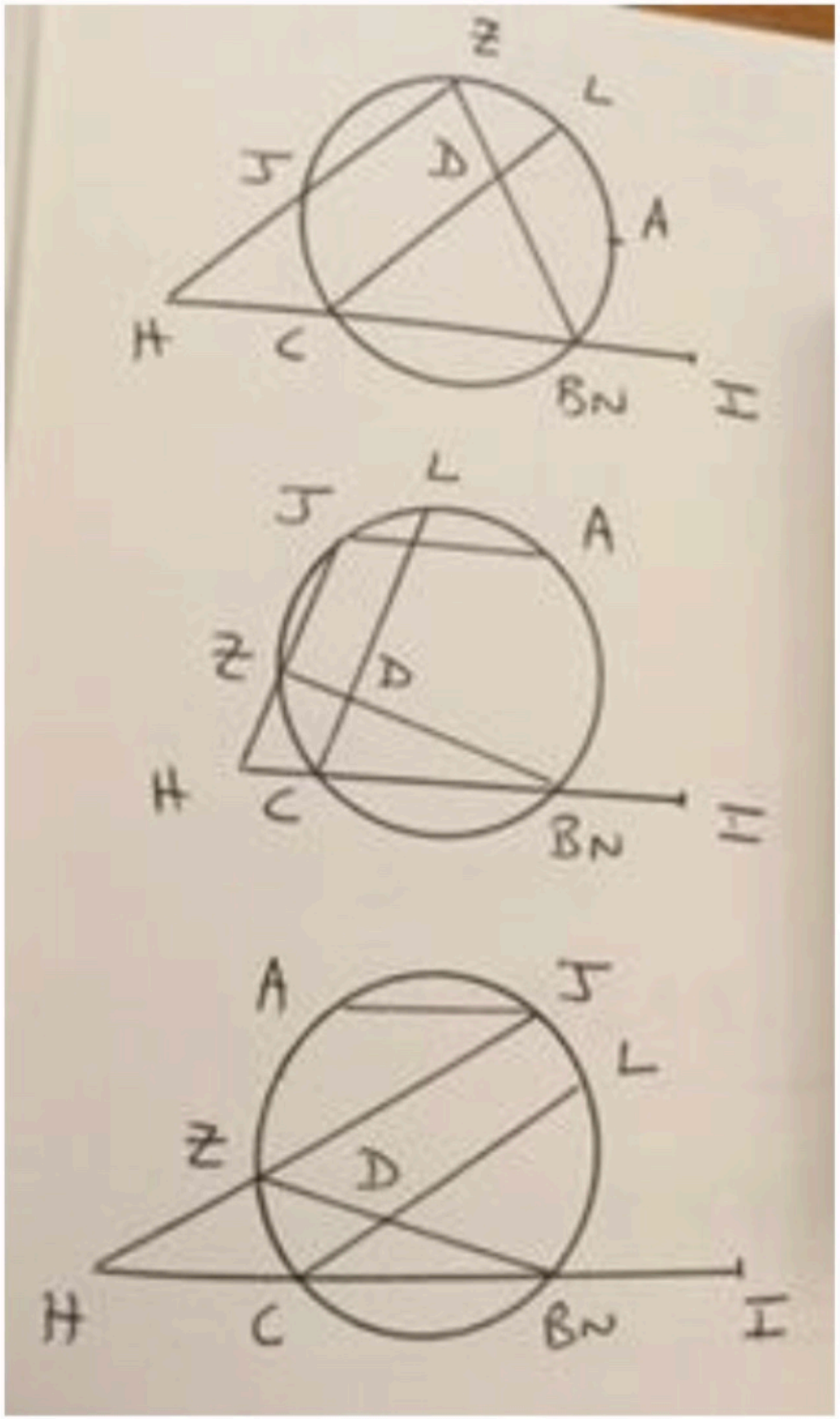
$\Delta ACB \cong \Delta ZCD$ and:

$$\frac{AC}{AB}\frac{ZB}{ZC} = \frac{ZC}{ZD}\frac{ZB}{ZC} = \frac{ZB}{ZD}$$

so as the radius $\Rightarrow \infty$

$$\frac{ZB}{ZD} \Rightarrow \mathbb{R}$$

Figure 39:



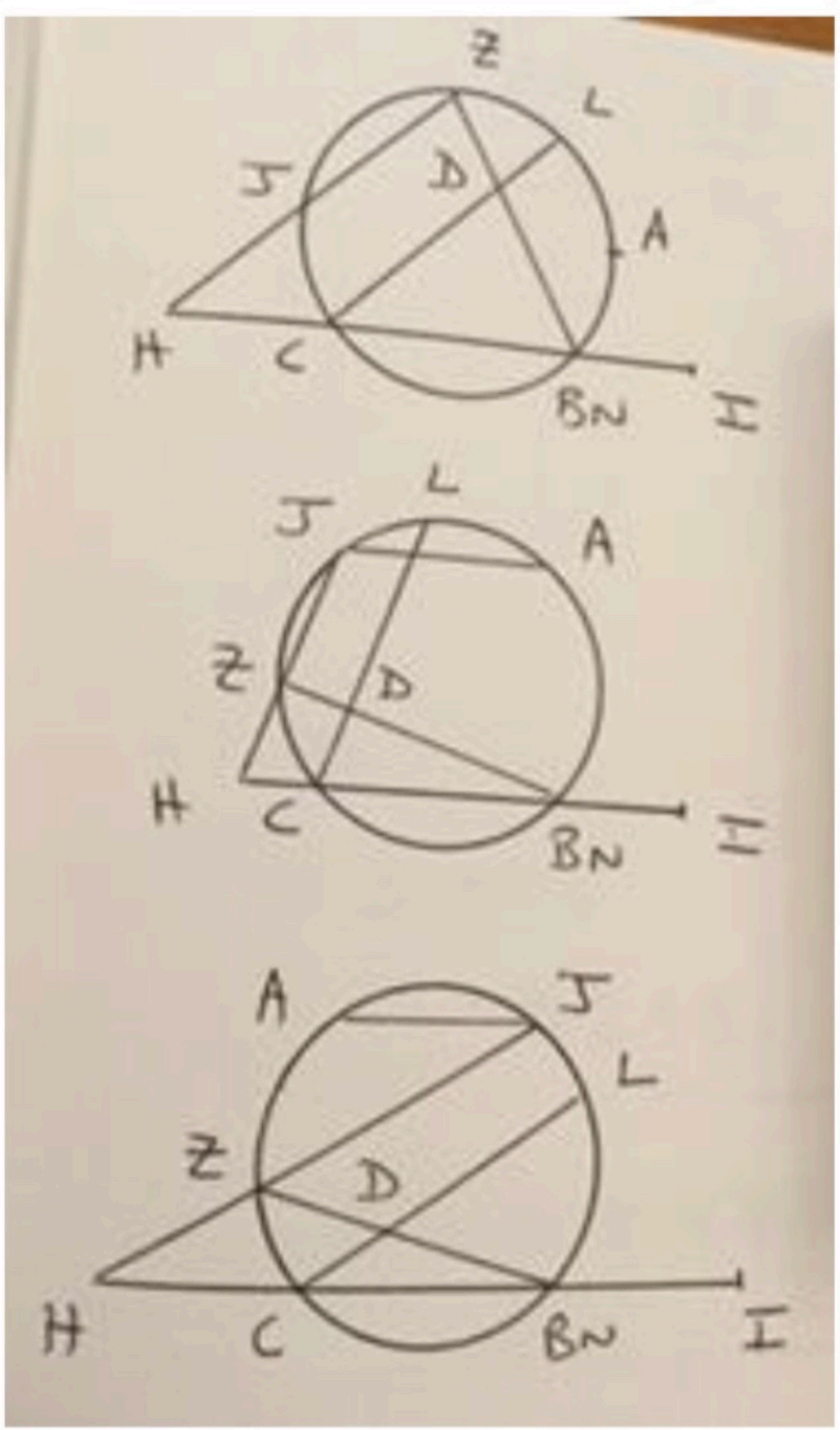
AL ll ZB
AZ = BL
~AZ = ~BL

HZ ll CL
ZC = LJ
~ZC = ~LJ

~AZ + ~ZC = ~AZC
~BL + ~LJ = ~BLJ

~AZC = ~BLJ
AJ ll CB

Figure 40:



HZ II CL

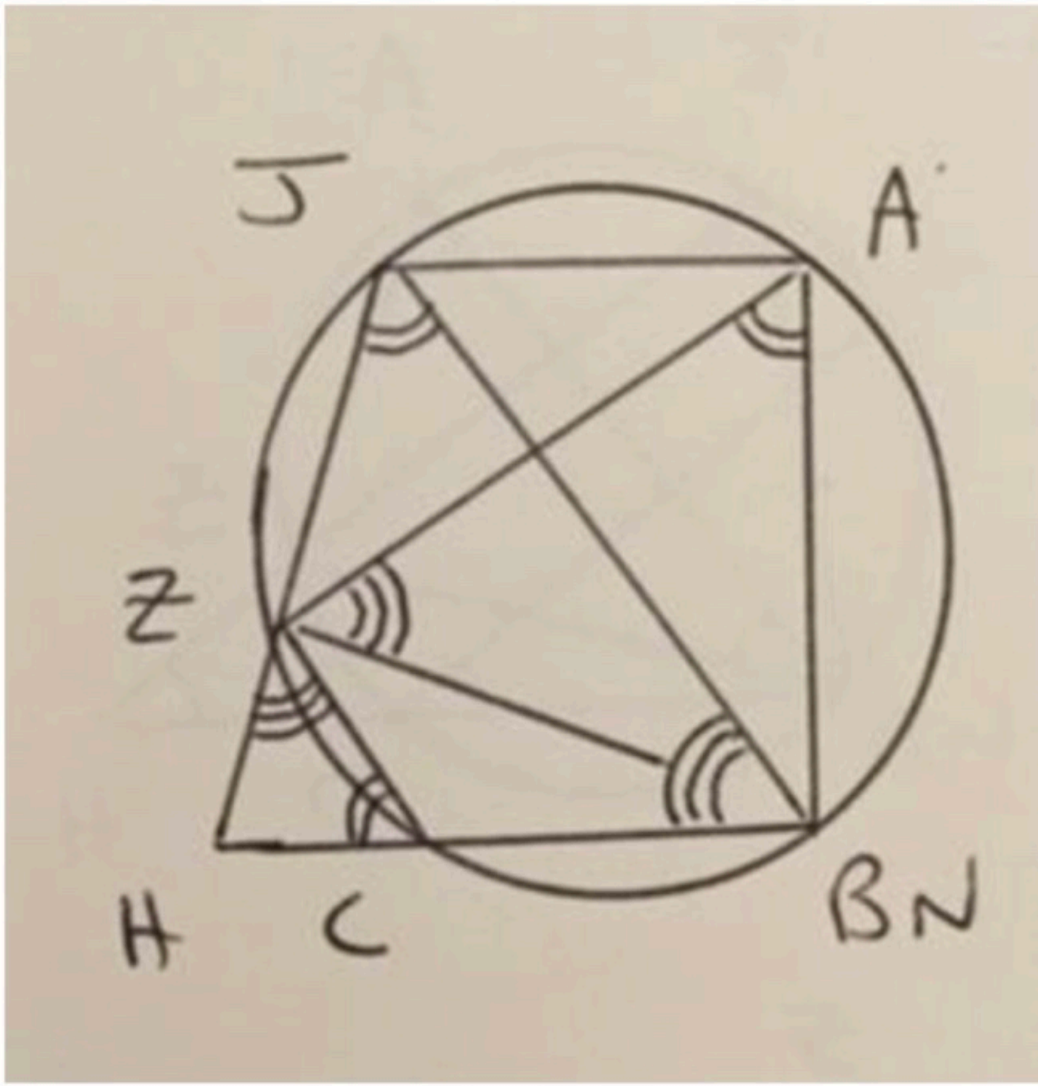
$$\frac{ZB}{ZD} = \frac{HB}{HC}$$

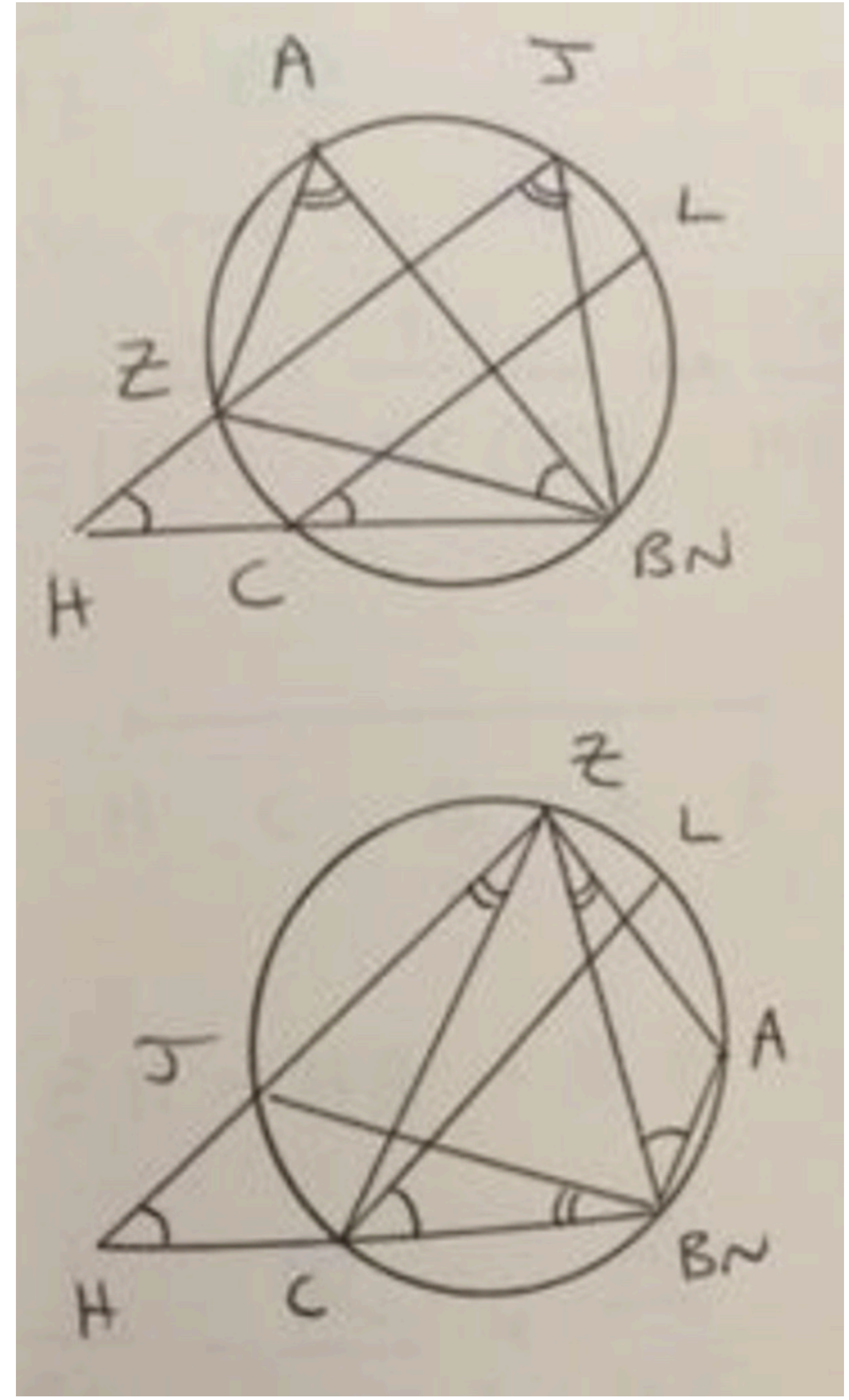
ΔHBZ ≅ ΔHJC

when ΔHJC = ΔIAB:

$$\frac{IB}{IA} = \frac{HZ}{HB}$$

Figure 41:





ΔHCZ ≅ ΔHJB ≅ ΔBAZ

$$\Delta HCZ \cong \Delta HJB \cong \Delta BAZ$$

$$\frac{HC}{HZ} = \frac{BA}{BZ}$$

as the radius $\Rightarrow \infty$

$$\frac{1}{HZ\,(BA)} = \frac{1}{HC\,(BZ)} \Rightarrow \frac{\mathbb{R}}{HB\,(BZ)}$$

These equalities are used with the following possible sums resulting from the circle with infinite radius, to produce the conjugate foci equations:

HZ= HB + BZ or
HB = HZ + BZ or
BZ = HZ + HB

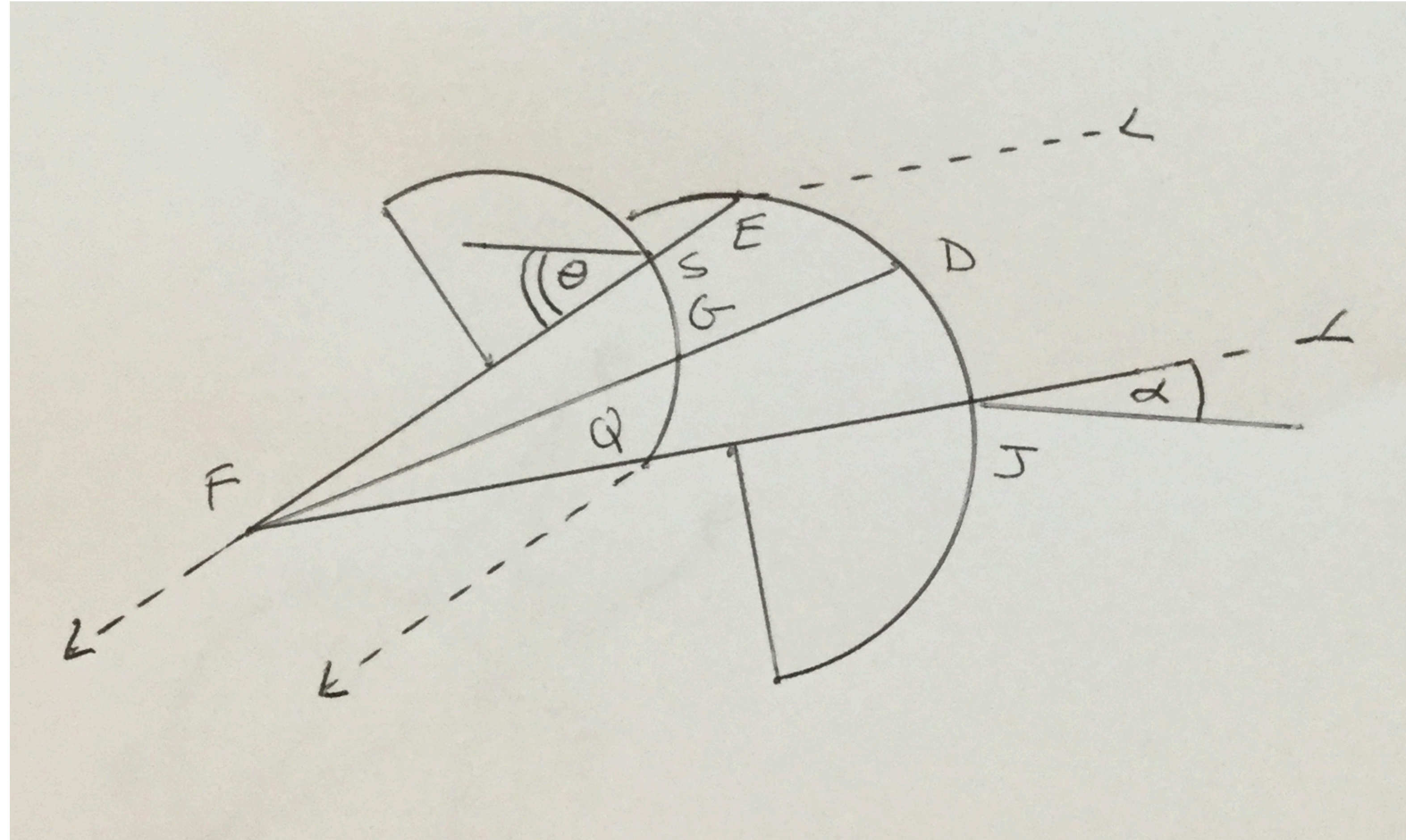
Section 5

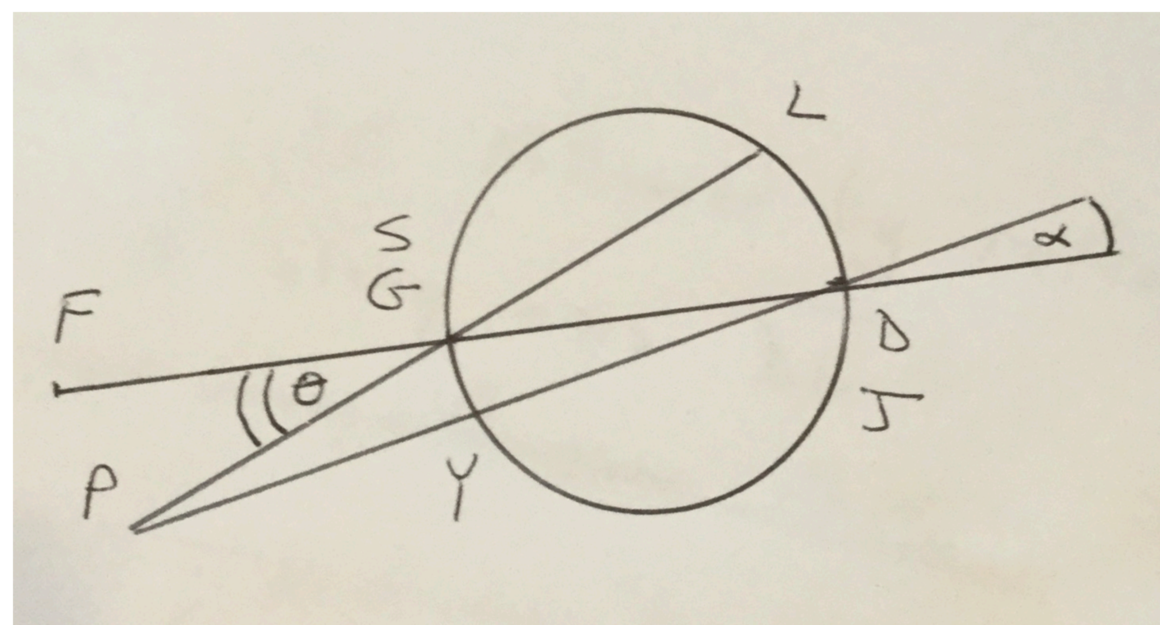
Afocal Axial Angular Magnification

Before considering afocal axial angular magnification, imagine two cars driving down the same street. When one car passes a sign post, it speeds up until it reaches the next sign post, then slows back down to its original speed, which is the same speed of the other car. Not only will the car that sped up be further down the road, it will also have had a greater average speed during the trip. This effect depends on two factors. The first is the degree to which the car speeds up between the sign posts, and the second is the distance between those sign posts.

This metaphor can be used to illustrate afocal axial angular magnification, which simply depends on two factors. The first is the degree to which light rays change between two lenses or refracting surfaces. The second is the separation of those two lenses or refracting surfaces. This is why a collapsible telescope no longer magnifies a distant object when it is “collapsed,” and its lenses are no longer separated.

Figure 42:





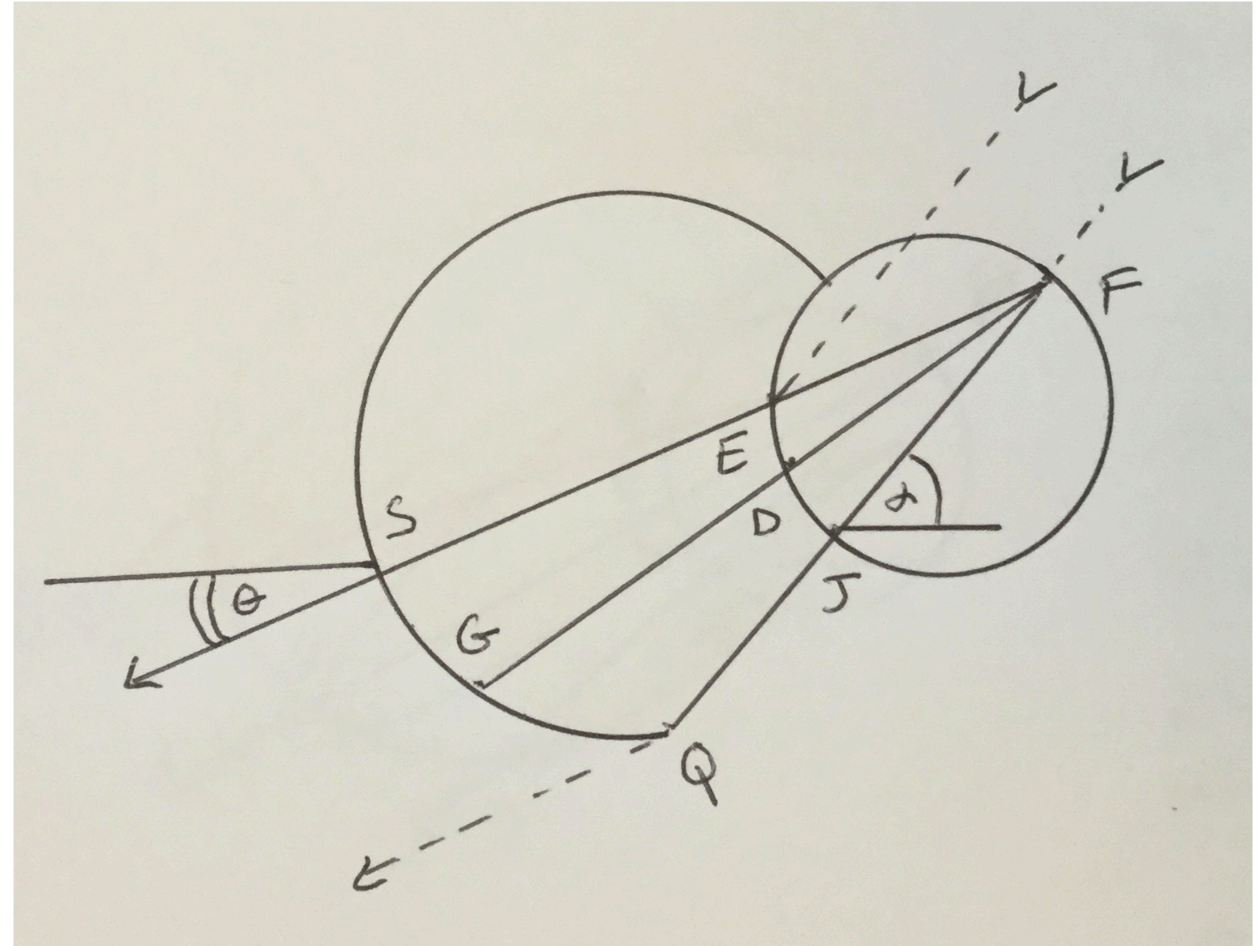
In figure 41, given distance refraction at ~JDE followed by refraction into distance at ~QGS along axis DGF:

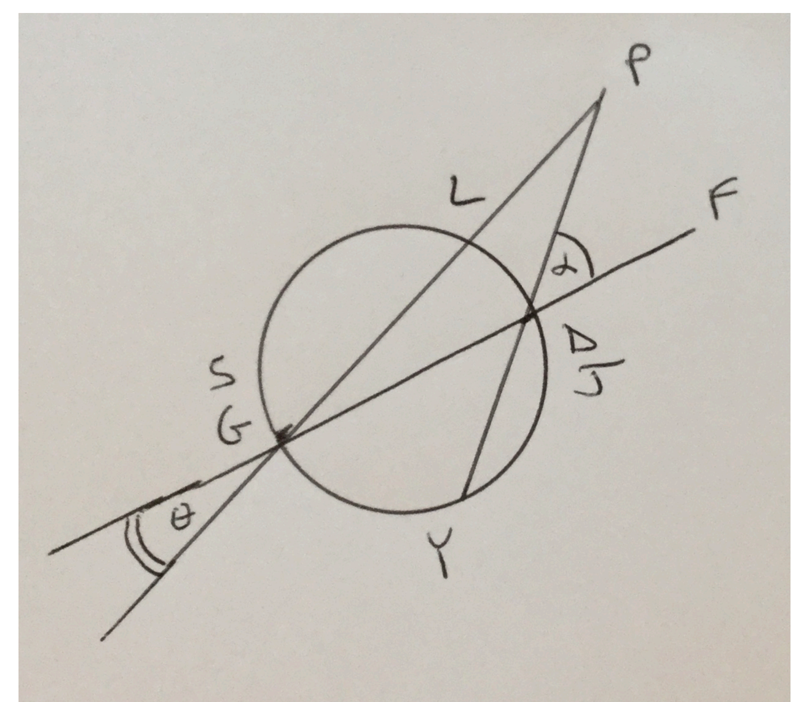
as angle JFD = angle SFG, and both approach zero,

$$\frac{\theta}{\alpha} \Rightarrow \frac{\sim LD/GD}{\sim YG/GD} \quad \text{as} \quad P \Rightarrow F$$

$$\frac{\theta}{\alpha} \Rightarrow \frac{FD}{FG} \quad \text{as} \quad P \Rightarrow F$$

Figure 43:





In figure 42, given distance refraction at ~JDE followed by refraction into distance at ~QGS along axis FDG:

as angle JFD = angle SFG, and both approach zero,

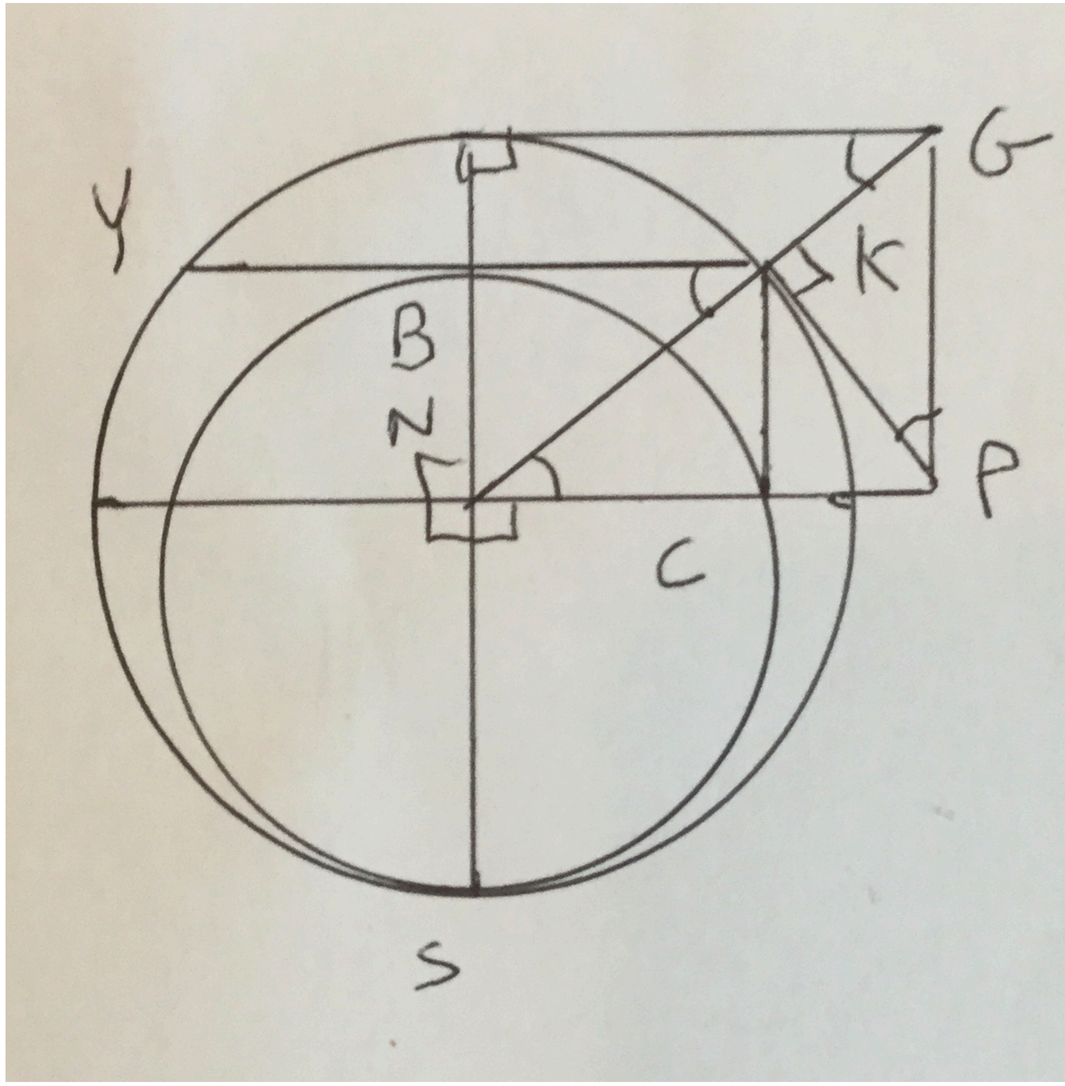
$$\frac{\theta}{\alpha} \Rightarrow \frac{\sim LD/GD}{\sim YG/GD} \quad \text{as} \quad P \Rightarrow F$$

$$\frac{\theta}{\alpha} \Rightarrow \frac{FD}{FG} \quad \text{as} \quad P \Rightarrow F$$

Section 6

Clinical Determination of Axial Retinal Image Size Magnification

Figure 44:



$\Delta NBK = \Delta GKP$

From figure 13, recall the "continued proportion"

$$\frac{NS}{NC} = \frac{NC}{NB}$$

and notice that:

$$\frac{NK}{NB} = \frac{KN + KG}{GP}$$

which equals:

$$\frac{NK + NB}{NK}$$

We have just shown that:

$$\frac{NK}{NB} = 1 + \frac{NB}{NK} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

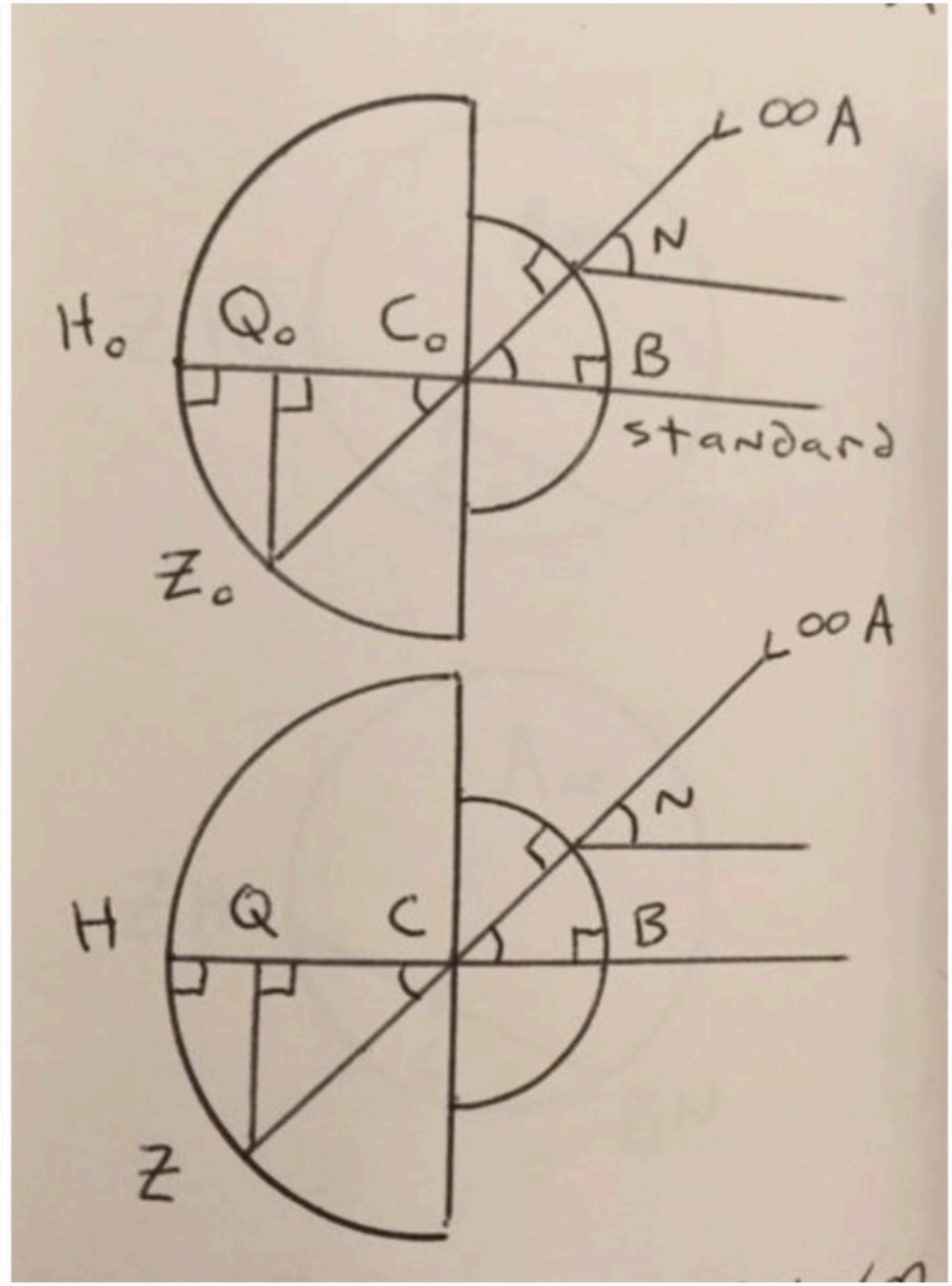
Since we have shown that neither NK or NB can measure the other length, we have shown that there is no length relative to itself, ("unit length"), that will measure all finite lengths.

This is relevant in any discussion of magnification. We can either consider such non-measurable distances to be irrational numbers, which are continuing fractions, or we can consider “number theory” itself to be irrational, along with the presumption that anything, even a unit measurement, can be real defined by itself.

Axial retinal image size magnification is not a number, but rather a ratio. It therefore requires a standard retinal image size for comparison. It is fair to call any such magnification using a standard, which is by definition arbitrary, meaningless in and of itself. However, it is simply a tool to use for ***comparing*** magnifications. Such comparisons are meaningful and not arbitrary, because arbitrary standards factor out when comparing ratios.

Figure 45:

The top diagram references the standard eye. The bottom diagram references any eye used for comparison, with the retinal image size designated as HZ.

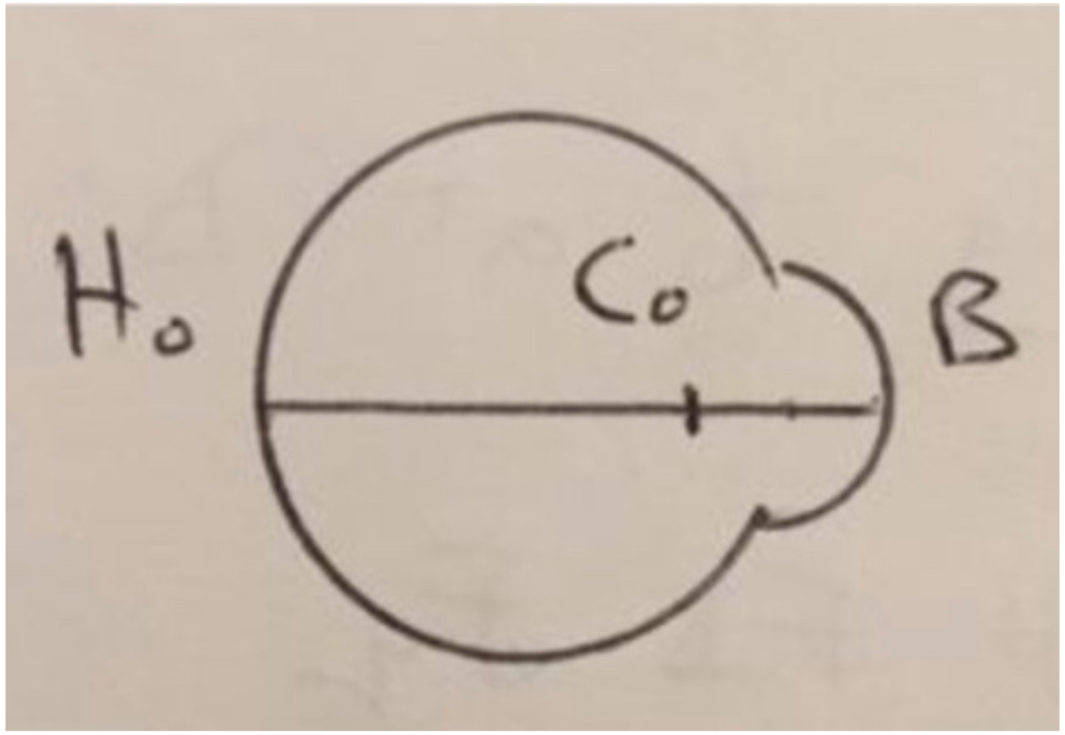


$$\frac{ZQ}{Z_\circ Q_\circ} = \frac{ZC}{Z_\circ C_\circ} = \frac{HC}{H_\circ C_\circ} = \frac{BH/\mathbb{R}}{BH_\circ/\mathbb{R}}$$

as N $\Rightarrow$ B:

$$\boldsymbol{M} \Rightarrow \frac{ZQ}{Z_\circ Q_\circ} = \frac{BH}{BH_\circ}$$

Figure 46:



In order to find the magnification ***M***, (in this case that of retinal image size magnification), we need to know both the standard BH_o, as well as BH for the eye in question. When a distant object is focused at Z, and a distance refractive error exists, Z lies at E rather than at H.

using $BH_\circ$ as the chosen ocular standard where:

$$\mathbb{R} = \frac{H_\circ B}{H_\circ C_\circ} = \frac{HB}{HC} = \frac{EB}{EL} = \frac{4}{3}$$

and $$\frac{\mathbb{R}}{BH_\circ} = 60 \text{ diopters}$$

(where a diopter is a unit of inverse meter length)

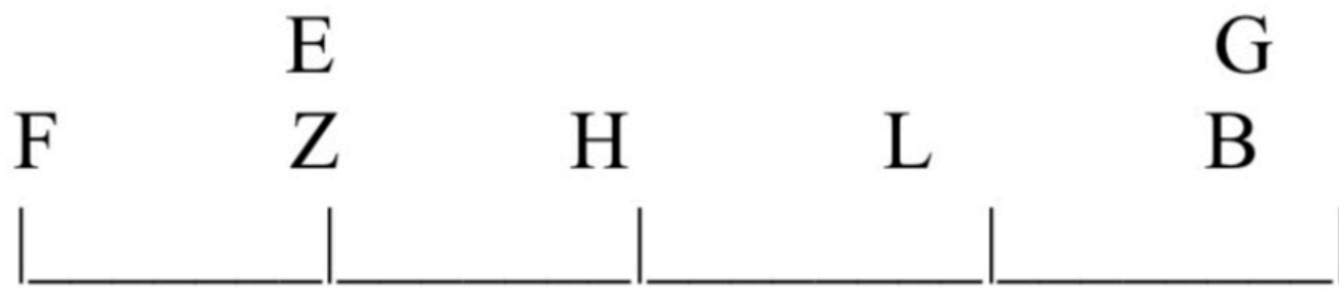
Measure BL to find:

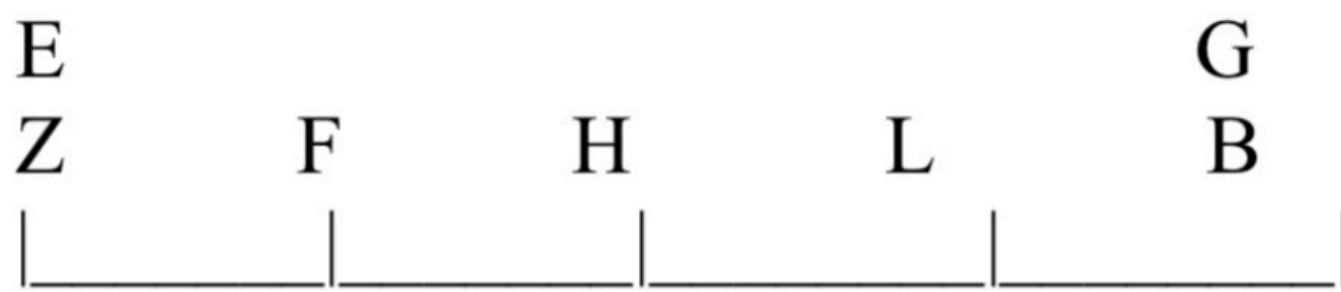
$$\frac{\mathbb{R}}{BE} = \frac{1}{EL} = \frac{\mathbb{R} - 1}{BL}$$

in order to calculate BH using:

$$\frac{\mathbb{R}}{BE} = \frac{1}{BF} + \frac{\mathbb{R}}{BH}$$

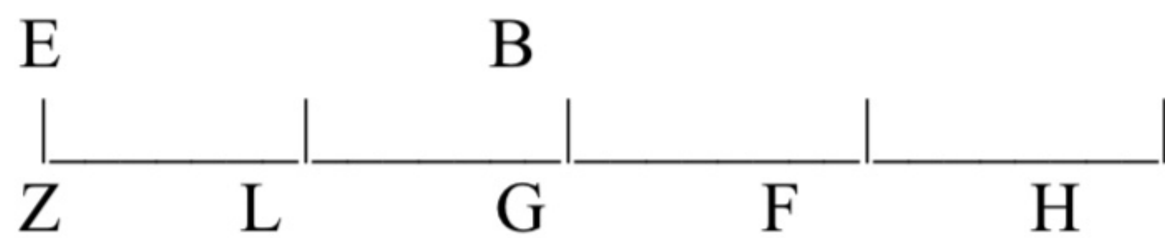
or





$$\frac{\mathbb{R}}{BH} = \frac{1}{BF} + \frac{\mathbb{R}}{BE}$$

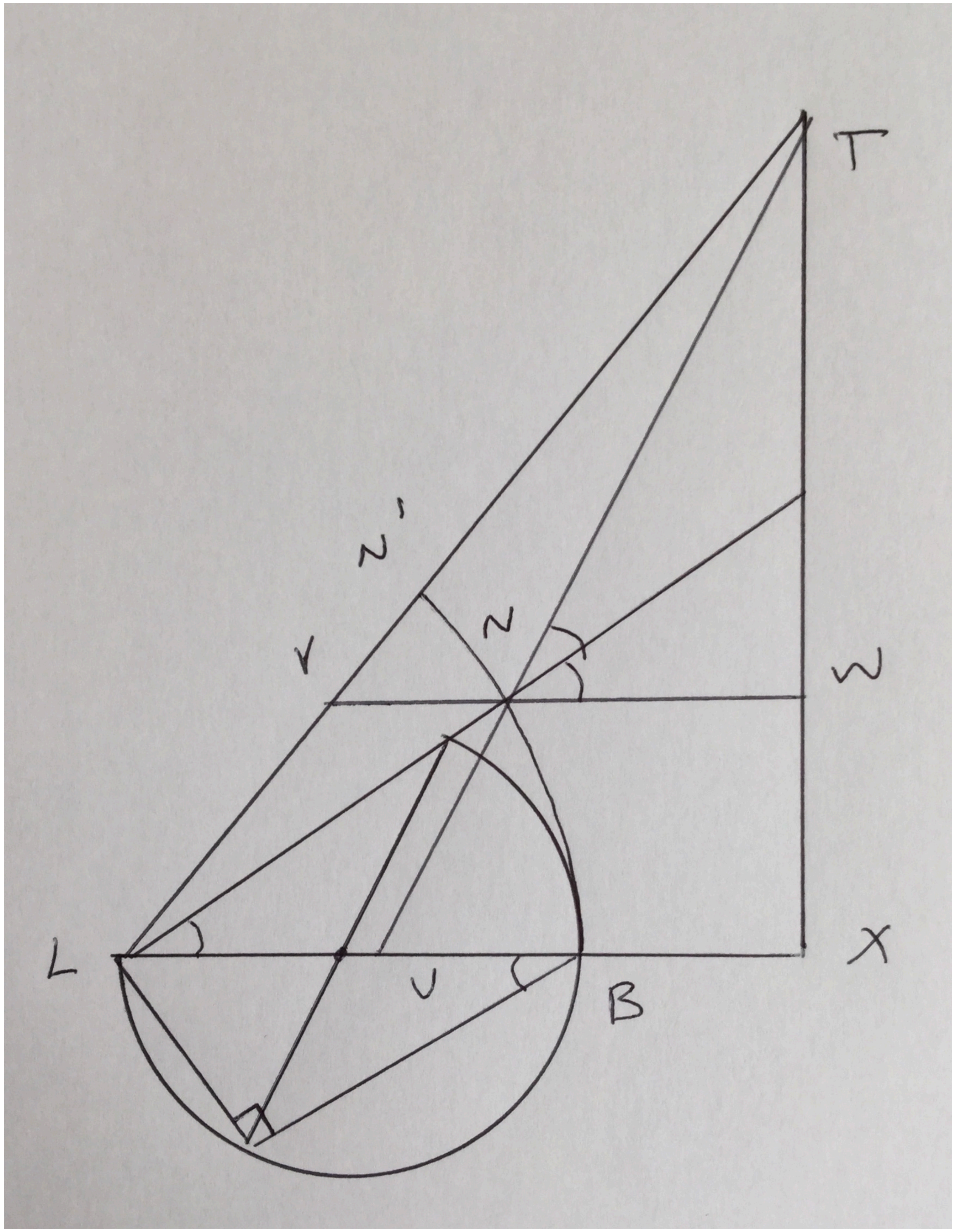
note that the condition producing a virtual image at H:



$$\frac{1}{BF} = \frac{\mathbb{R}}{BE} + \frac{\mathbb{R}}{BH}$$

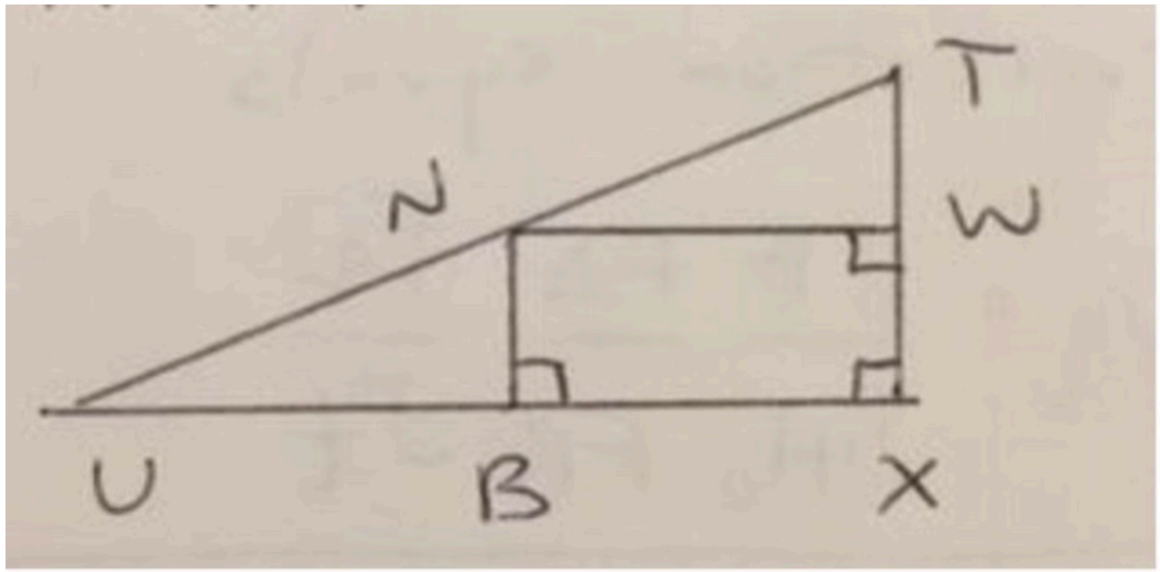
is meaningless when considering the focused axial image size magnification BH/BH_o when the standard image is real.

Figure 47:



BL Is found
by changing BX
to clearly focus
the reflected image V
of light source T

Figure 48:



make T ⇒ X
so that 2BU ⇒ BL
and ∠NBU ⇒ $\frac{\pi}{2}$

so that:

$$\frac{XT}{XW} \rightarrow \frac{UX}{UB} \rightarrow \frac{2UX}{BL} \leftarrow \frac{2VW}{BL}$$

with a very small XT
measure XW and VW
to approximate BL

only the corneal component $\boldsymbol{K}$ of $\frac{\mathbb{R}}{\text{BE}}$ can be approximated with BL from the reflection off B

when its deviation from the standard 42 is assumed to equal the deviation of the total $\frac{\mathbb{R}}{\text{BE}}$

from its standard of 60:

$$\boldsymbol{K} + (42 - \boldsymbol{K}) = 42$$

$$\frac{\mathbb{R}}{\text{BE}} + (42 - \boldsymbol{K}) = 60$$

$$\frac{\mathbb{R}}{\text{BE}} = \boldsymbol{K} + 18$$

and since:

$$\boldsymbol{M} = \frac{\mathbb{R}}{BH_{\circ}} \quad \frac{BH}{\mathbb{R}}$$

$$\boldsymbol{M} = \frac{60}{\dfrac{\mathbb{R}}{BE} \pm \dfrac{1}{BF}}$$

(Note that the traditional sign convention when considering the distance correction 1/BF allows for the +/- sign to be replaced by simply a + sign).

When the retinal image size magnification of two real eyes are compared, retinal image size magnification loses its arbitrary nature resulting from its presumed standard. However, that does not address the arbitrary assumption in this calculation that magnification differences between two eyes result solely from their front surfaces. This calculation is only as correct as that assumption.

Section 7

Axial Magnification of Distance Correction

Figure 49:

Standard emmetropic eye:

Non-standard emmetropic eye:

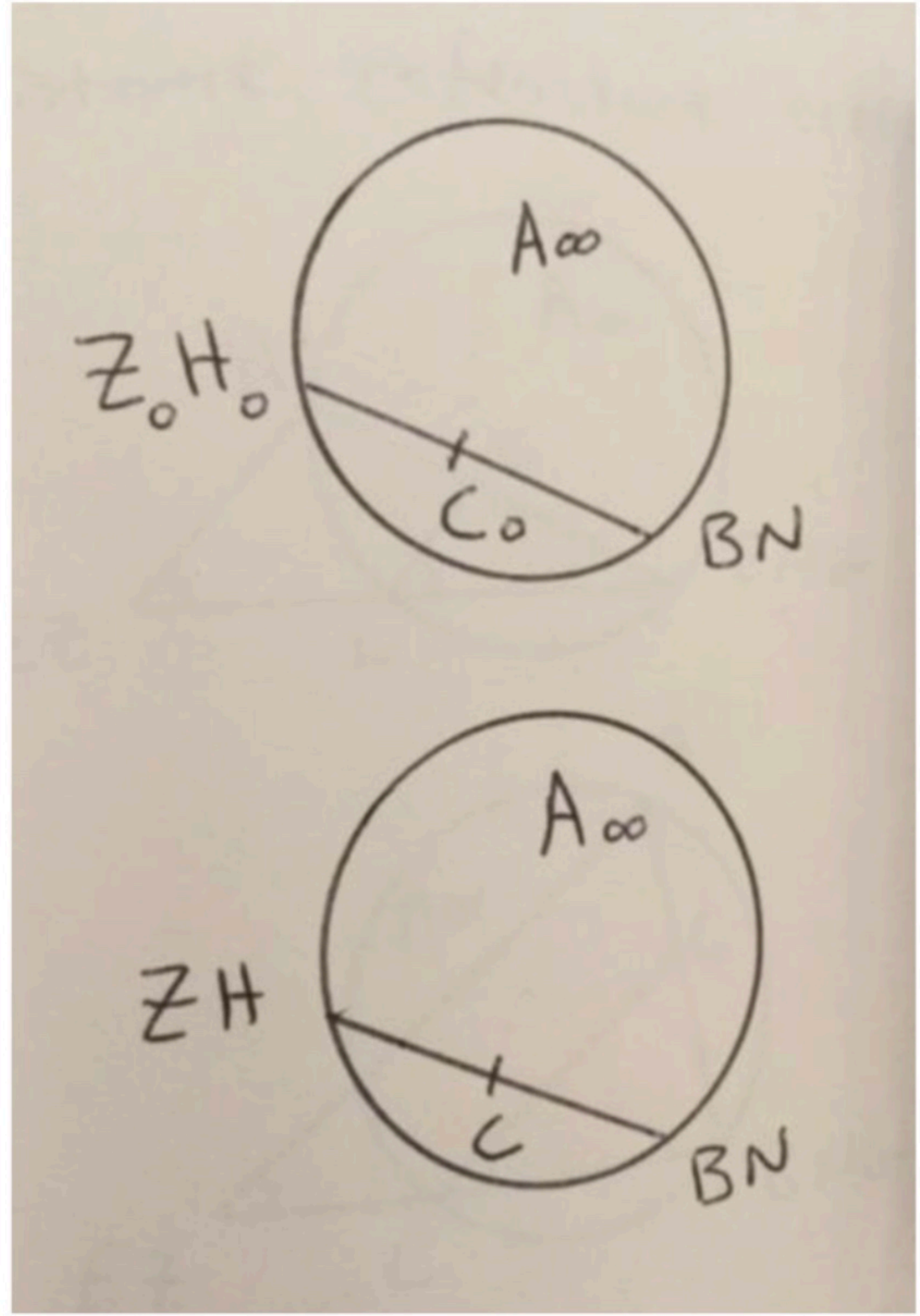


Figure 50:

Additional refraction at G (at B) creates distance refractive error with combined curvature of radius BL.

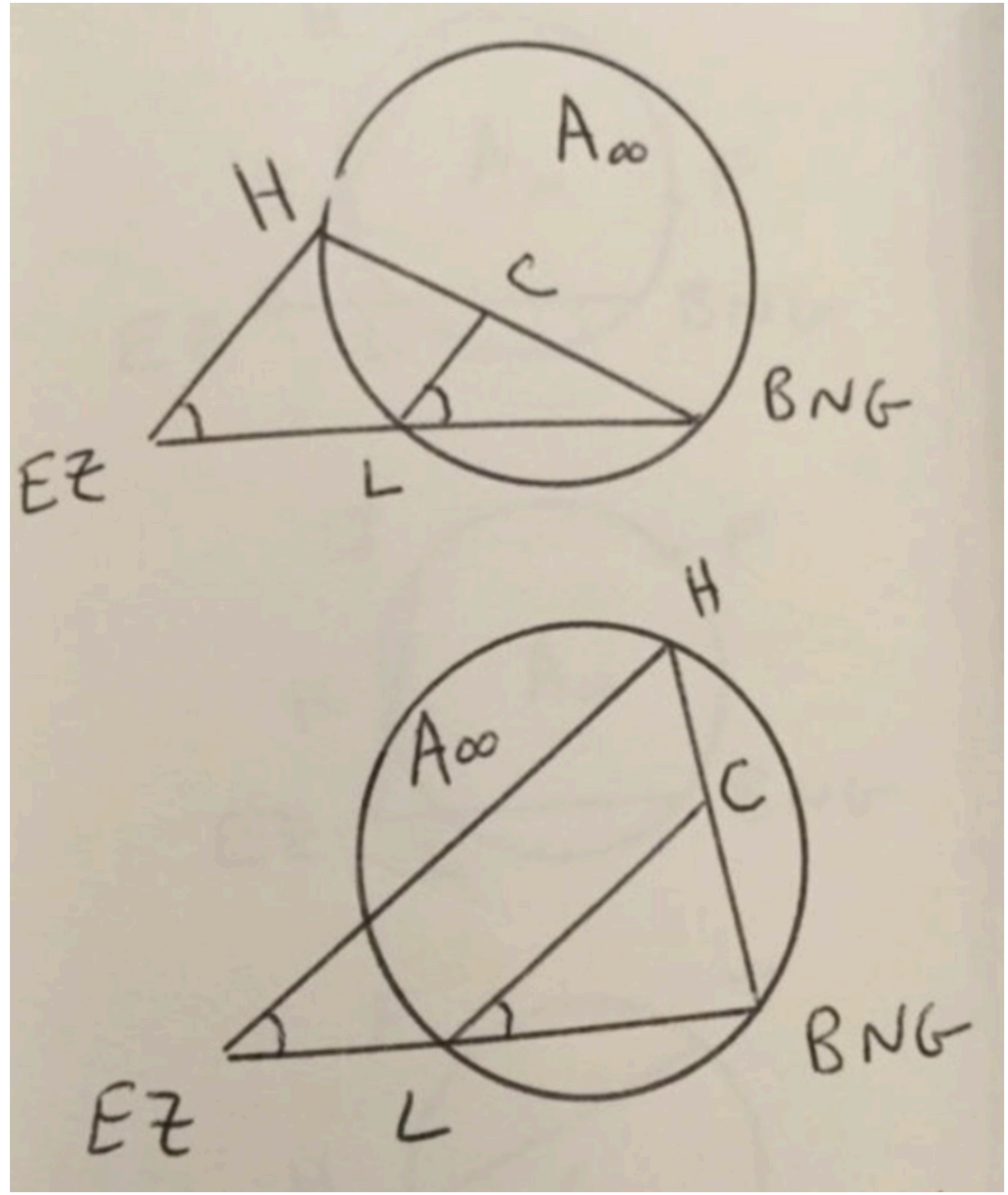


Figure 51:

The distance correction must focus infinity (A) at F so that:

JF ll BE

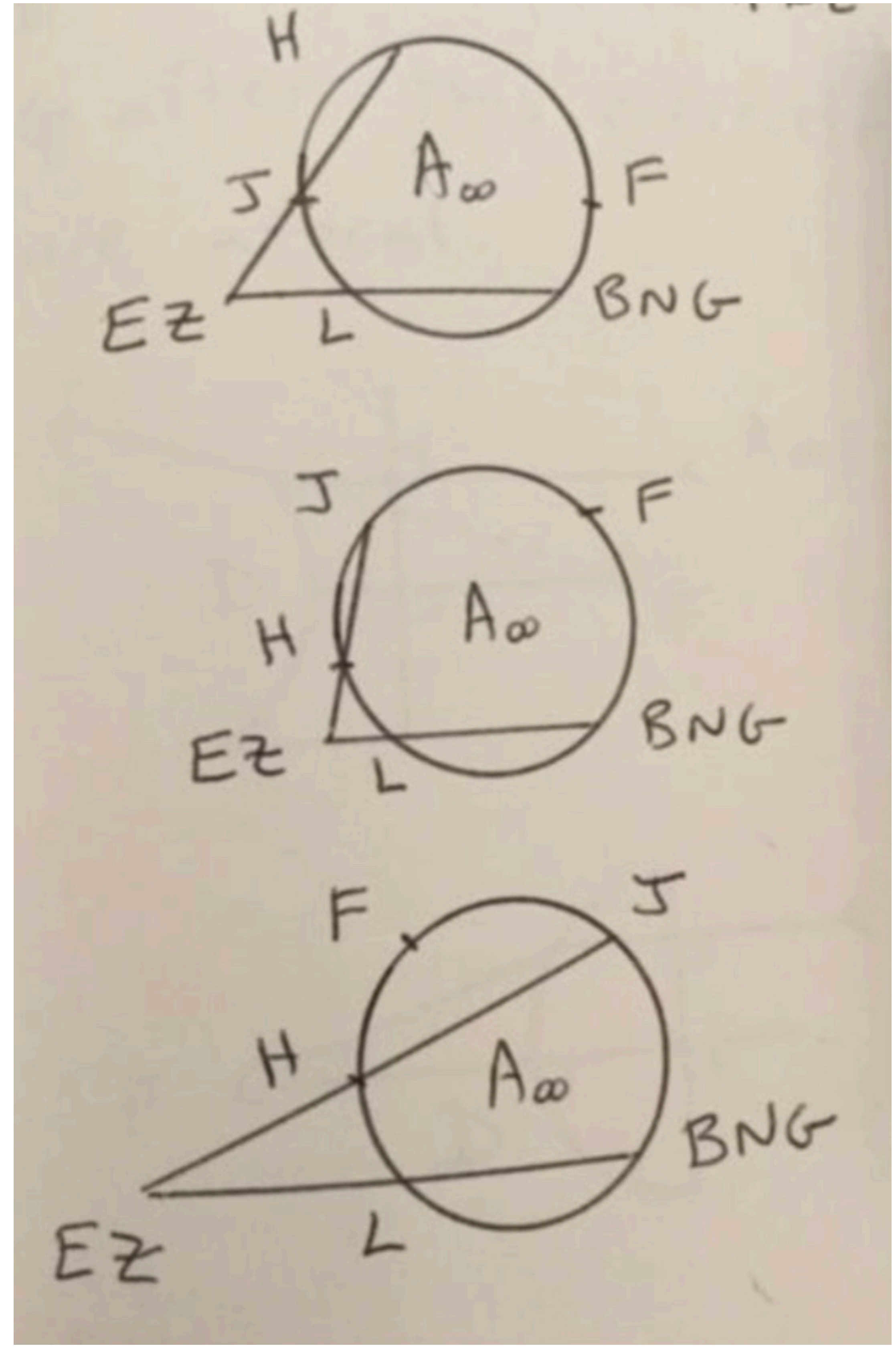
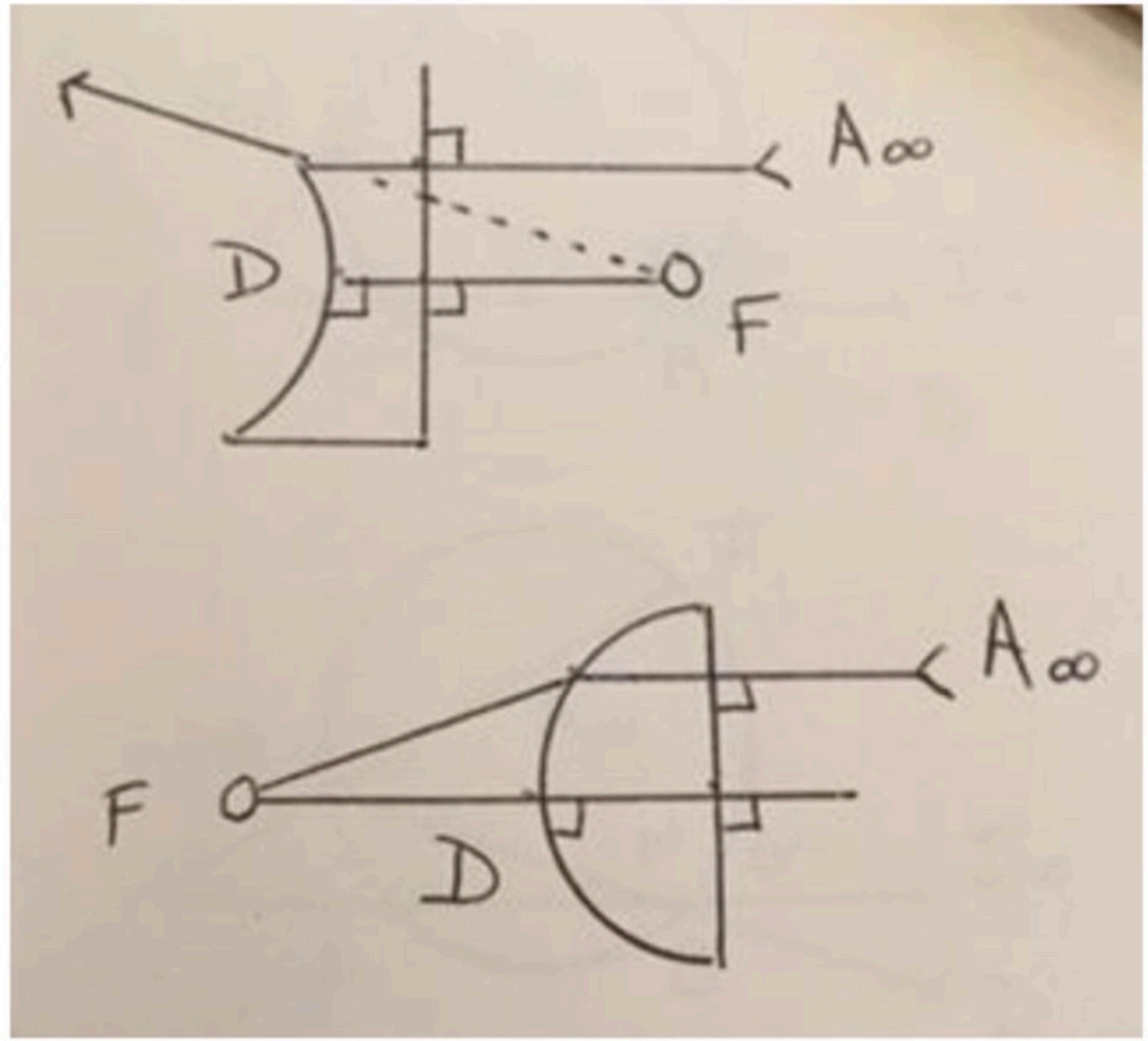
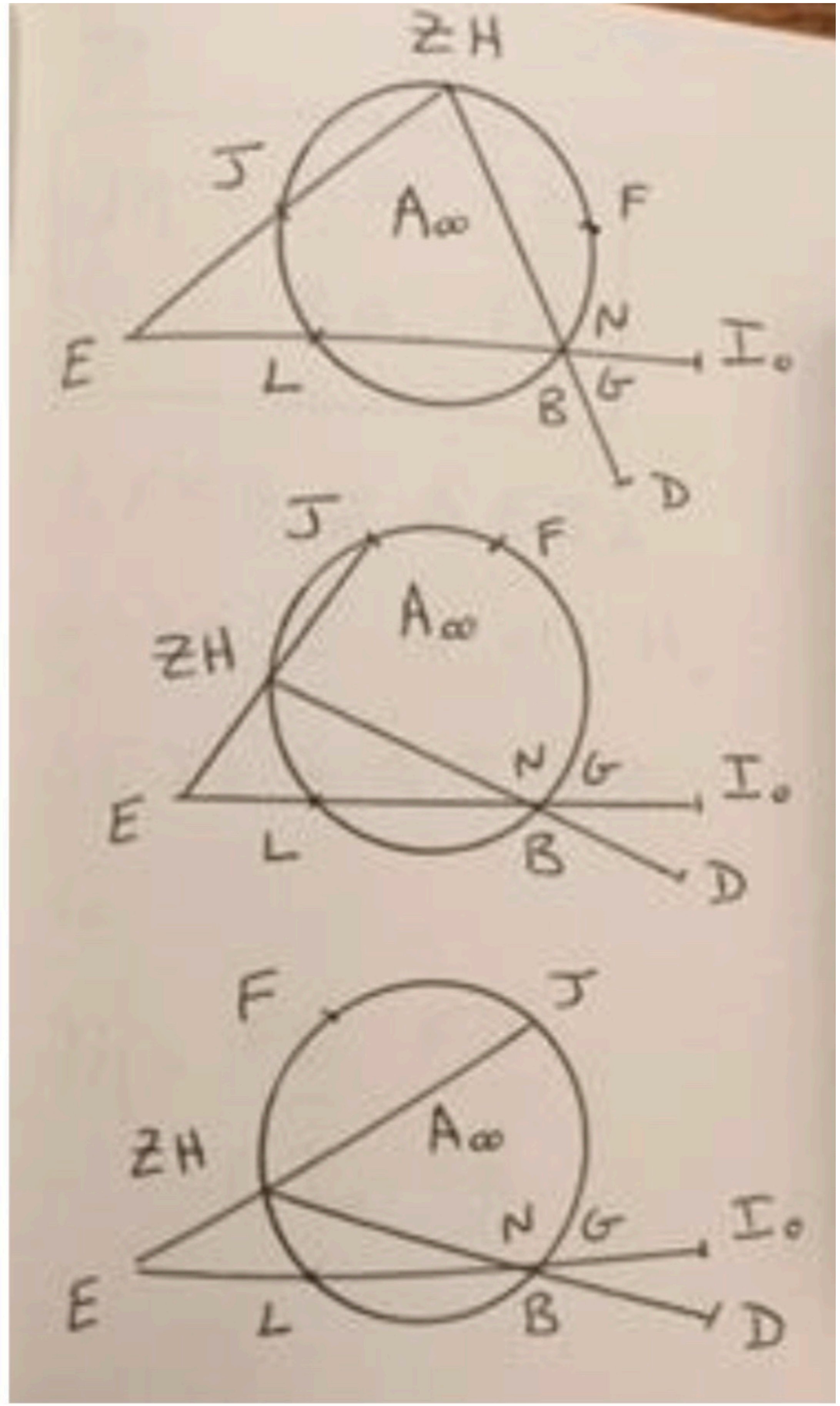


Figure 52:



since the distance correction
at D moves Z to H
rays leaving G after this correction are afocal

Figure 53:



$$\boldsymbol{M} = \frac{BH}{BH_{\circ}} \frac{FD}{FB}$$

$\Delta EBH \cong \Delta EJL$

when E is at $H_{\circ}$:

$\Delta EJL = \Delta I_{\circ}FB$ so:

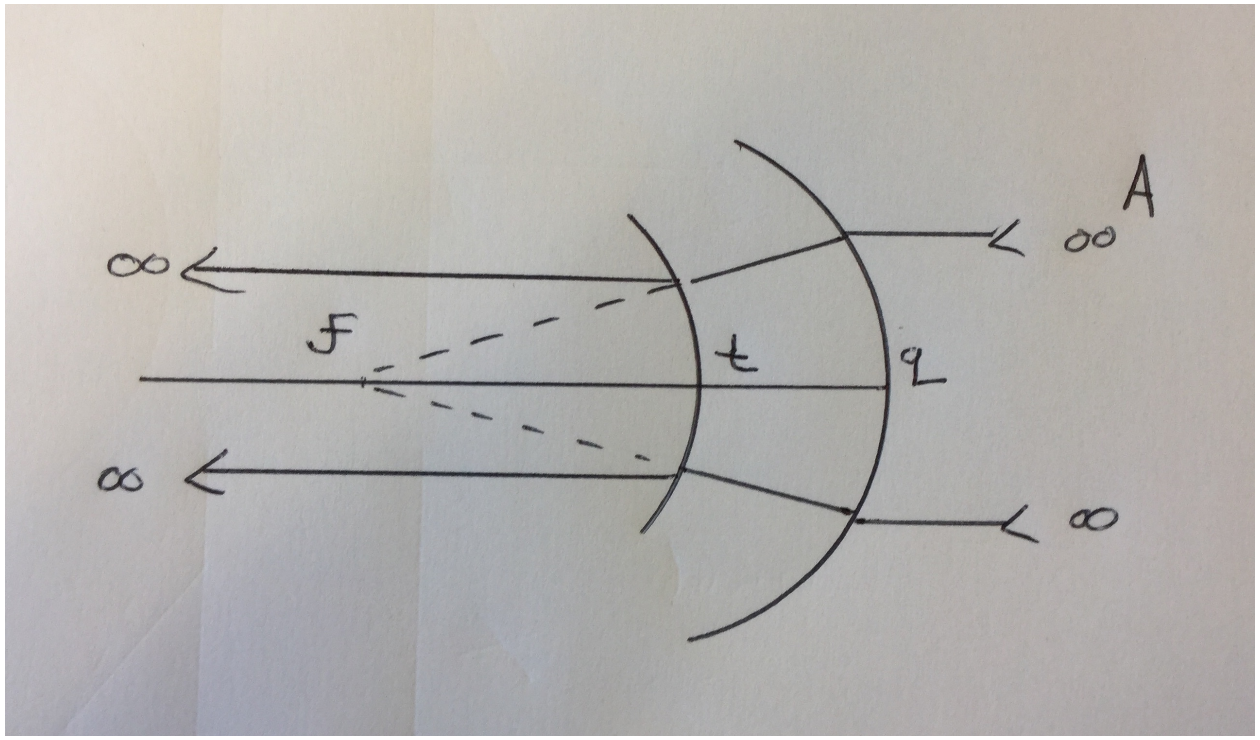
$$\boldsymbol{M} = \frac{FB}{FI_{\circ}} \frac{FD}{FB}$$

Note that when all the refractive error is due to the retina H lying at a position other than the standard, in other words, all the error is “axial” in nature, which occurs

when E is at H_o:

The magnification equals one when the distance correction at D lies at the standard eye’s front focal point.

Figure 54:



placing t at D:

$$\boldsymbol{M} = \frac{BH}{BH_o} \; \frac{FD}{FB} \; \frac{fq}{ft}$$

when the front surface of a spectacle lens that
corrects distance refractive error is not flat
it is convex
and produces additional
axial afocal angular magnification

In summary:

axial magnification of distance correction equals:

$$\boldsymbol{M} = \frac{BH}{BH_o} \frac{FD}{FB} \frac{fq}{ft}$$

where:

$\frac{BH}{BH_o}$ = axial corrected image size magnification

and:

$\frac{FD}{FB} \frac{fq}{ft}$ = axial afocal angular magnification of distance correction

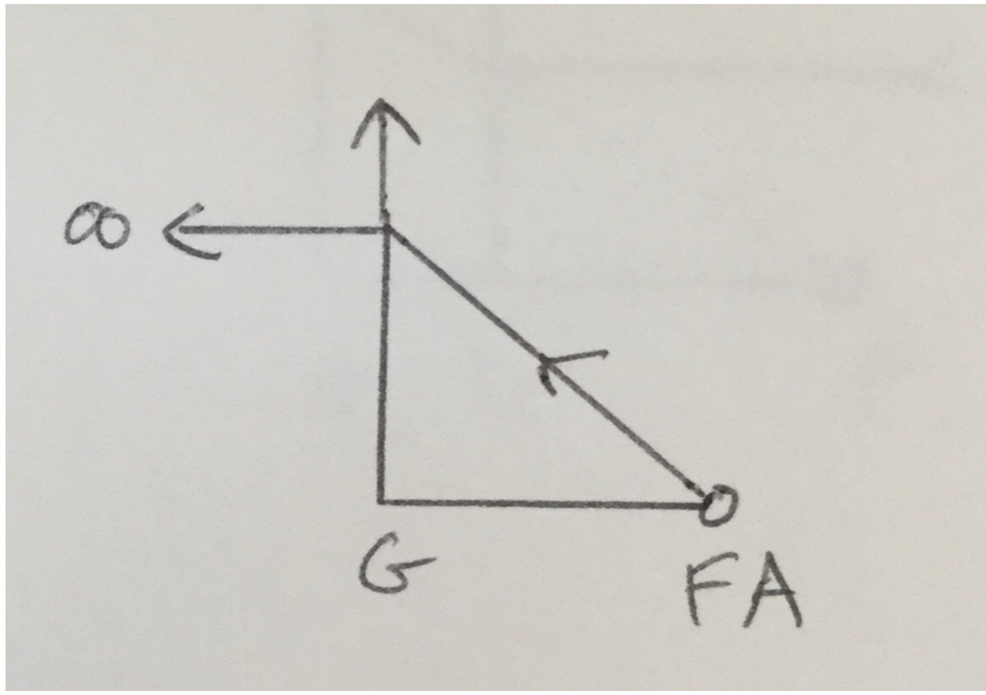
$\frac{FD}{FB}$ = "power factor"

$\frac{fq}{ft}$ = "shape factor"

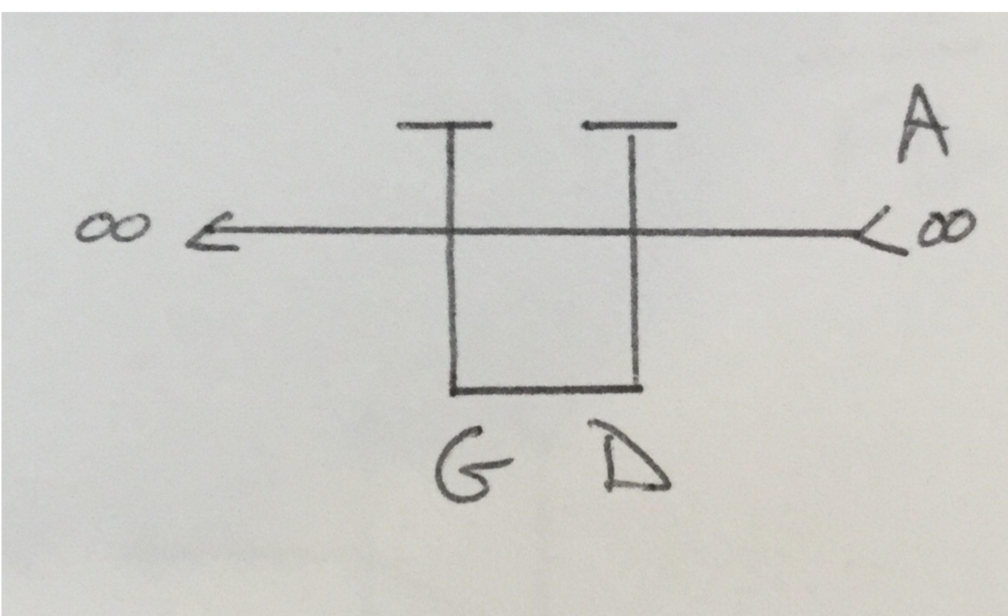
Section 8

Axial Magnification of Near Correction

Figure 55:

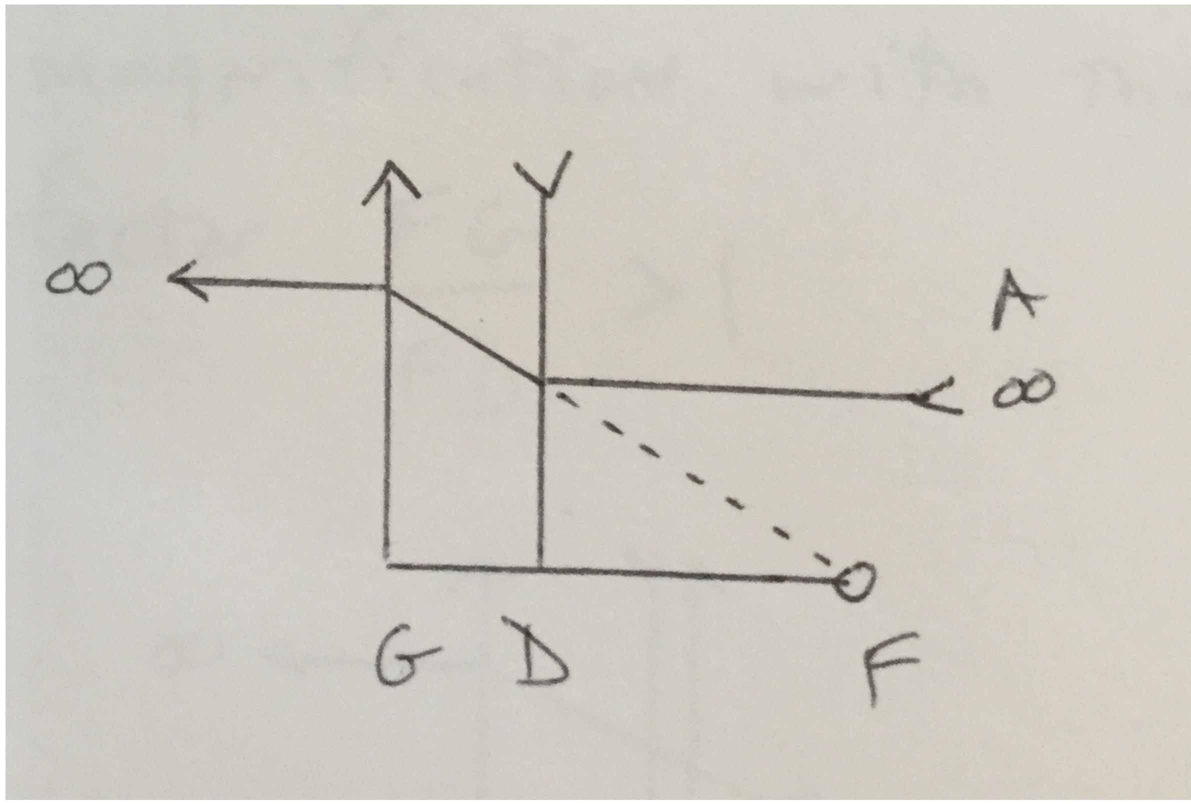


There is no afocal angular magnification when object A is at the front focal point of a myopic eye,



or at distance with an emmetropic eye.

Figure 56:

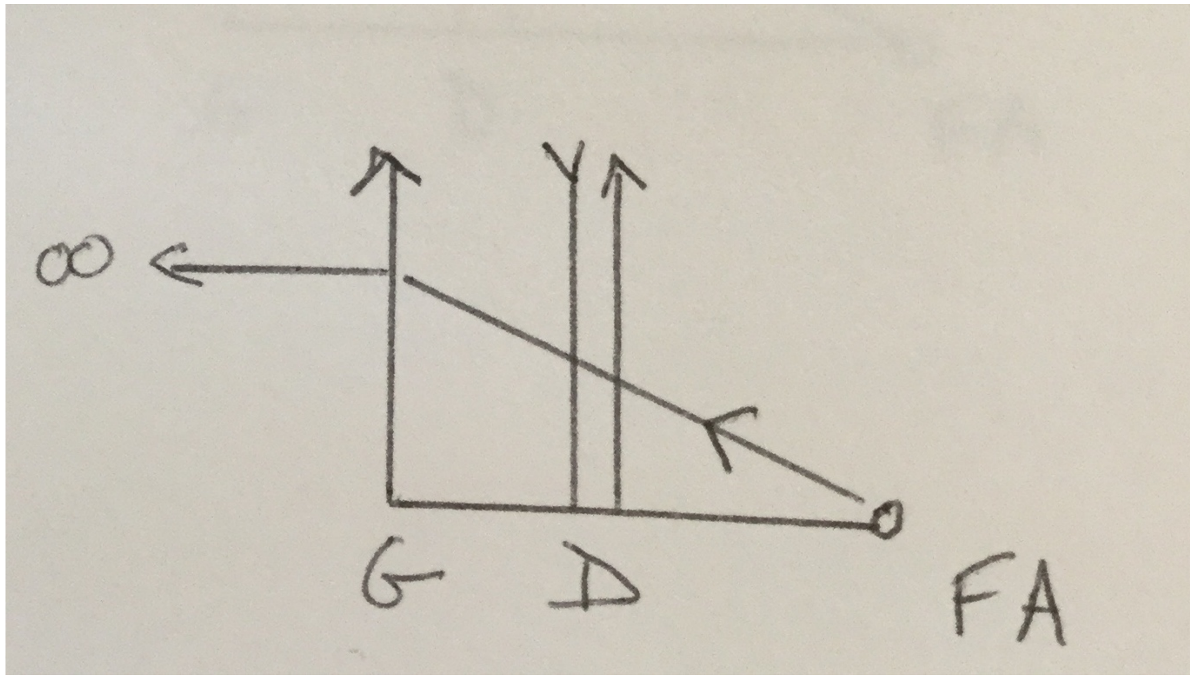


However, a distance myopic correction at D creates afocal angular magnification:

$$\frac{FD}{FG} < 1$$

and this is relative to both the myopic eye with object A at the myopic eye's front focal point F, as well as the emetropic eye with object A at distance.

Figure 57:

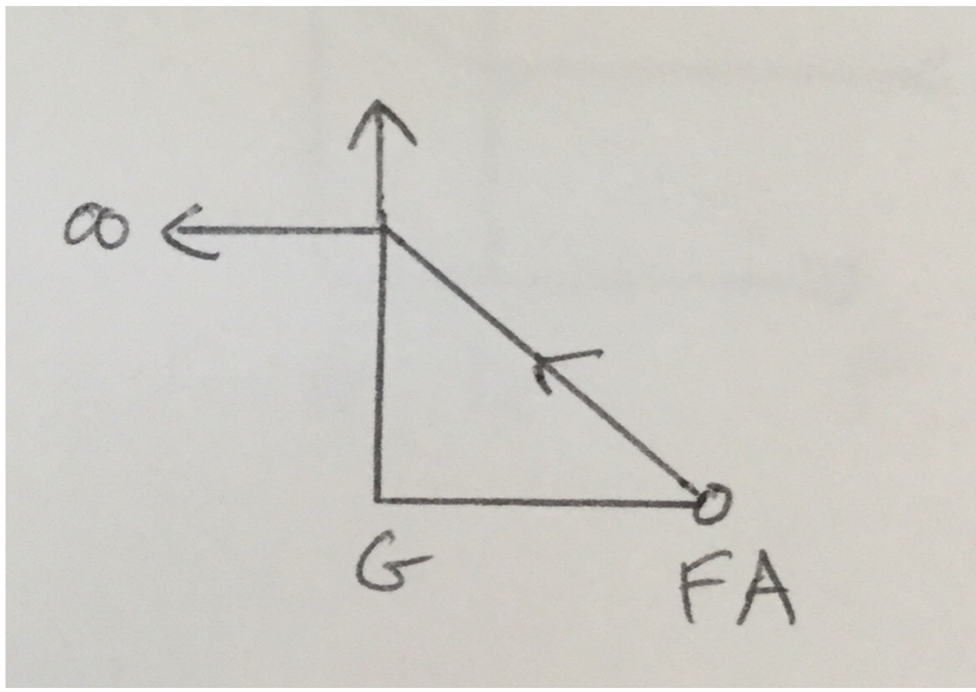


Removing the myopic distance correction at D with a converging lens at D removes this afocal angular magnification with the factor:

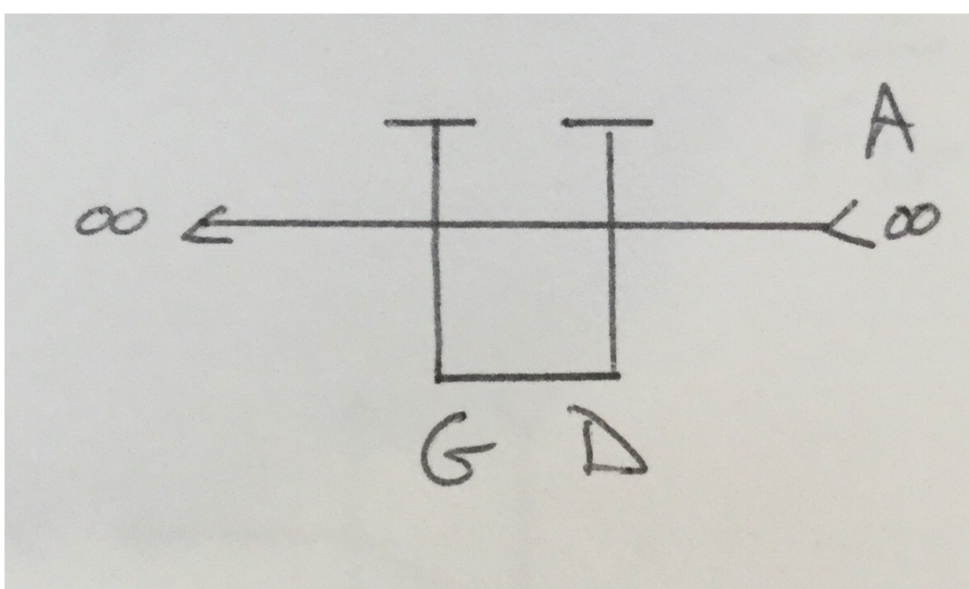
$$\frac{FG}{FD} > 1$$

and this magnification of near correction is relative to the distance corrected myope.

(Figure 55):



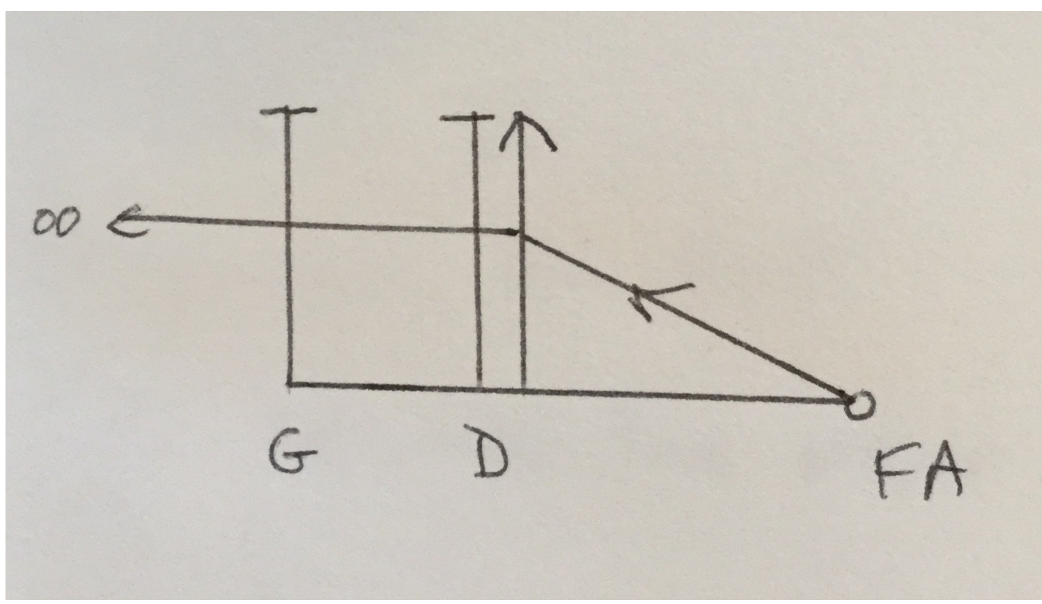
It is not relative to either the myope,



or an emmetrope.

Figure 58:

If additional converging power is added to the converging lens so that the near focal point is in focus for an emetropic eye, rather than the myopic eye, the afocal angular magnification removed with the factor:



$$\frac{FG}{FD} > 1$$

remains the same, and the reference eye is emetropic.

Figure 59:

When the converging lens at D is split into two converging lenses:

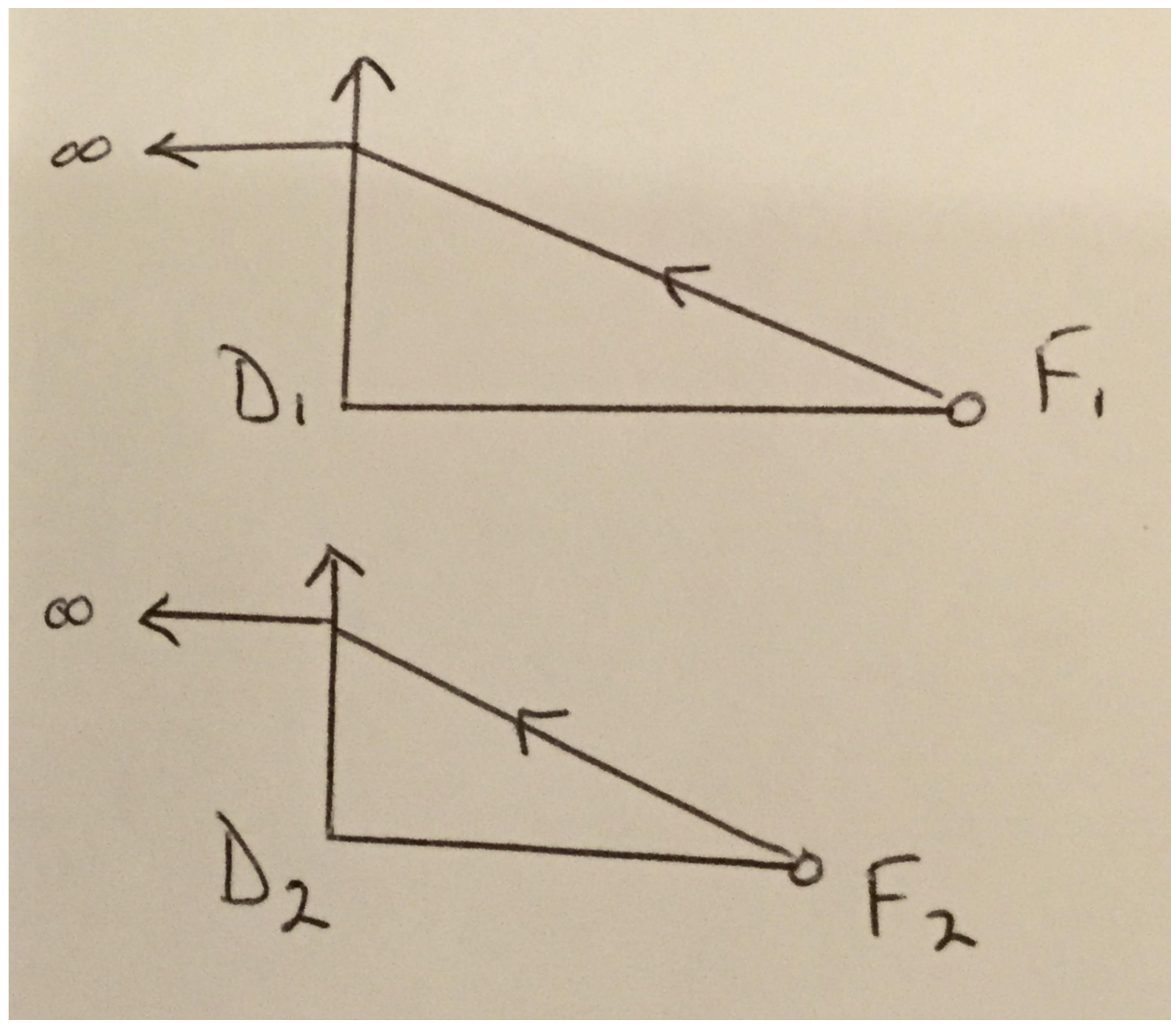


Figure 60:

With the same combined focus F:

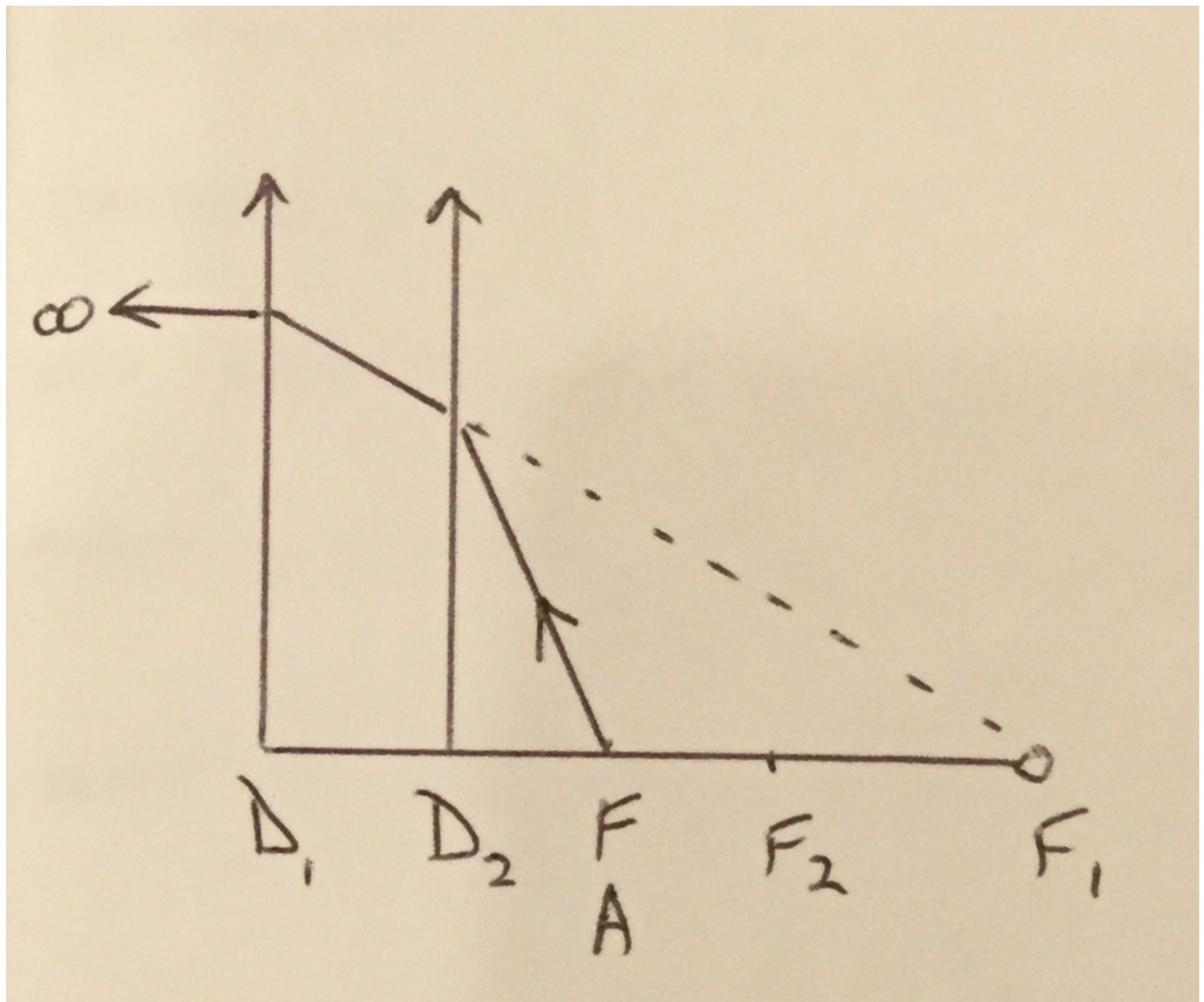
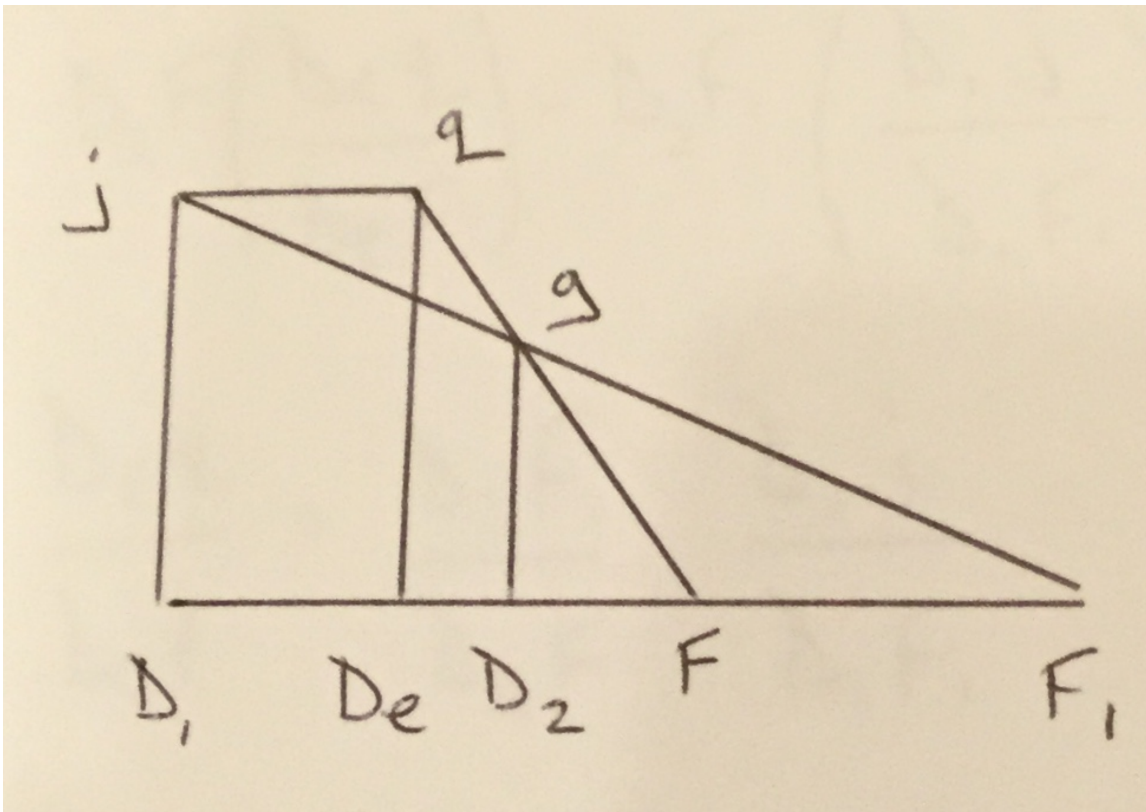


Figure 61:



The axial magnification of near correction can be specified as that produced

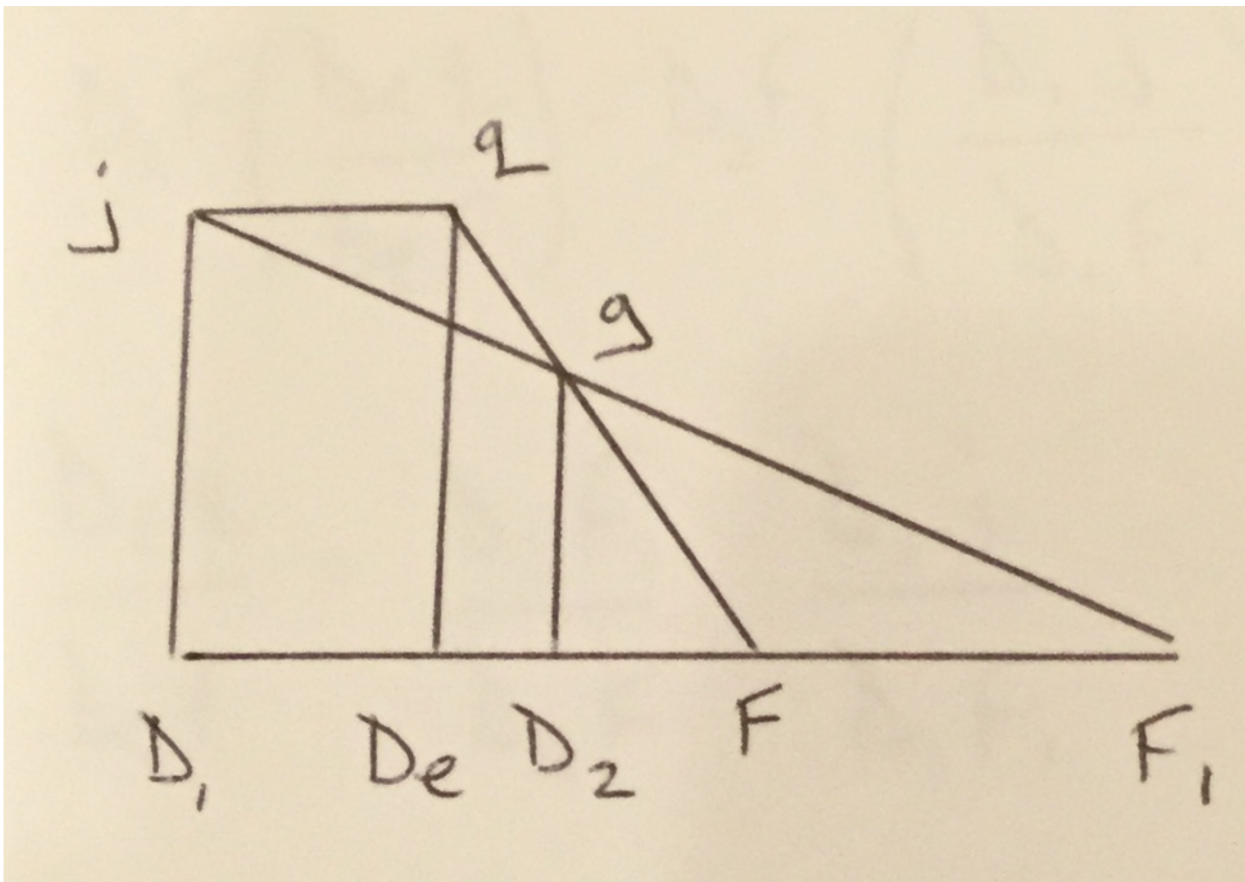
as if

all convergence occurs at a single unknown axial point De

and equals:

$$\frac{FG}{FDe} = \frac{FB}{FDe}$$

Figure 62:

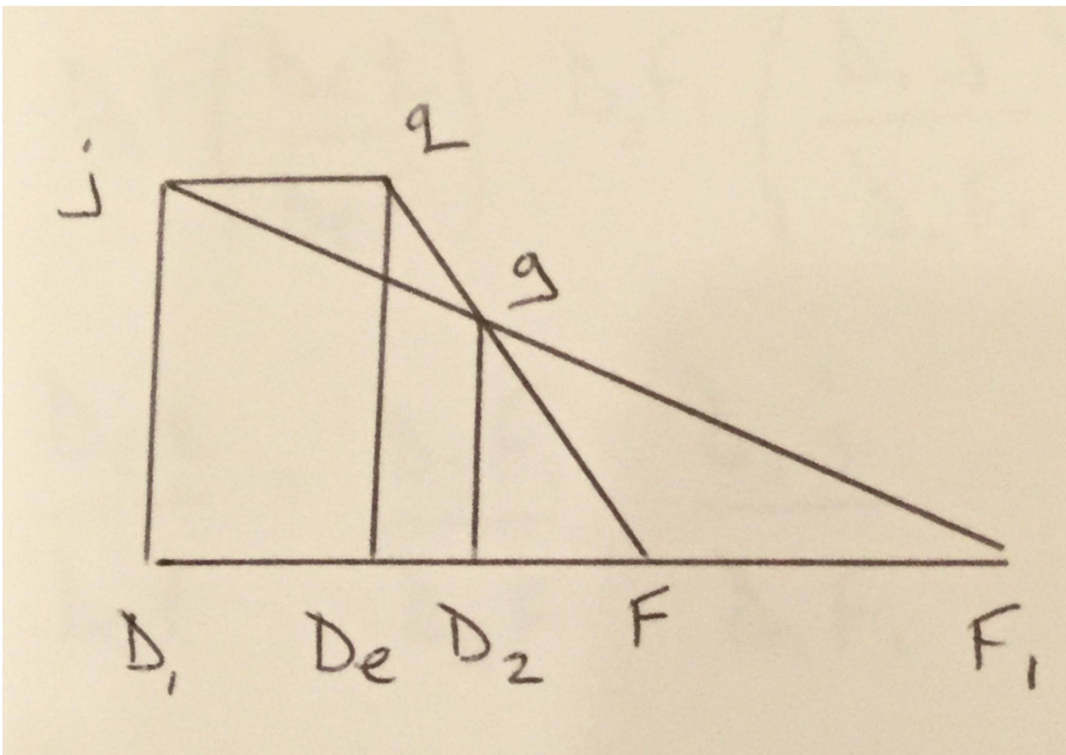


De can be located using triangles:

$$\frac{D_2g}{D_2F} = \frac{Deq}{DeF}$$

$$\frac{D_2g}{D_2F_1} = \frac{D_1j}{D_1F_1}$$

Figure 63:



$$D_2F \; \frac{Deq}{DeF} = D_2F_1 \; \frac{D_1j}{D_1F_1}$$

$$\frac{Deq}{DeF} = \frac{D_2F_1}{D_2F} \; \frac{D_1j}{D_1F_1}$$

$$\frac{1}{DeF} = \frac{D_2F_1}{D_2F} \; \frac{1}{D_1F_1}$$

$$\frac{FB}{FDe} = \frac{D_2F_1}{D_2F} \; \frac{FB}{D_1F_1}$$

Figure 64:

When an object at a standard distance Fs is moved to F:

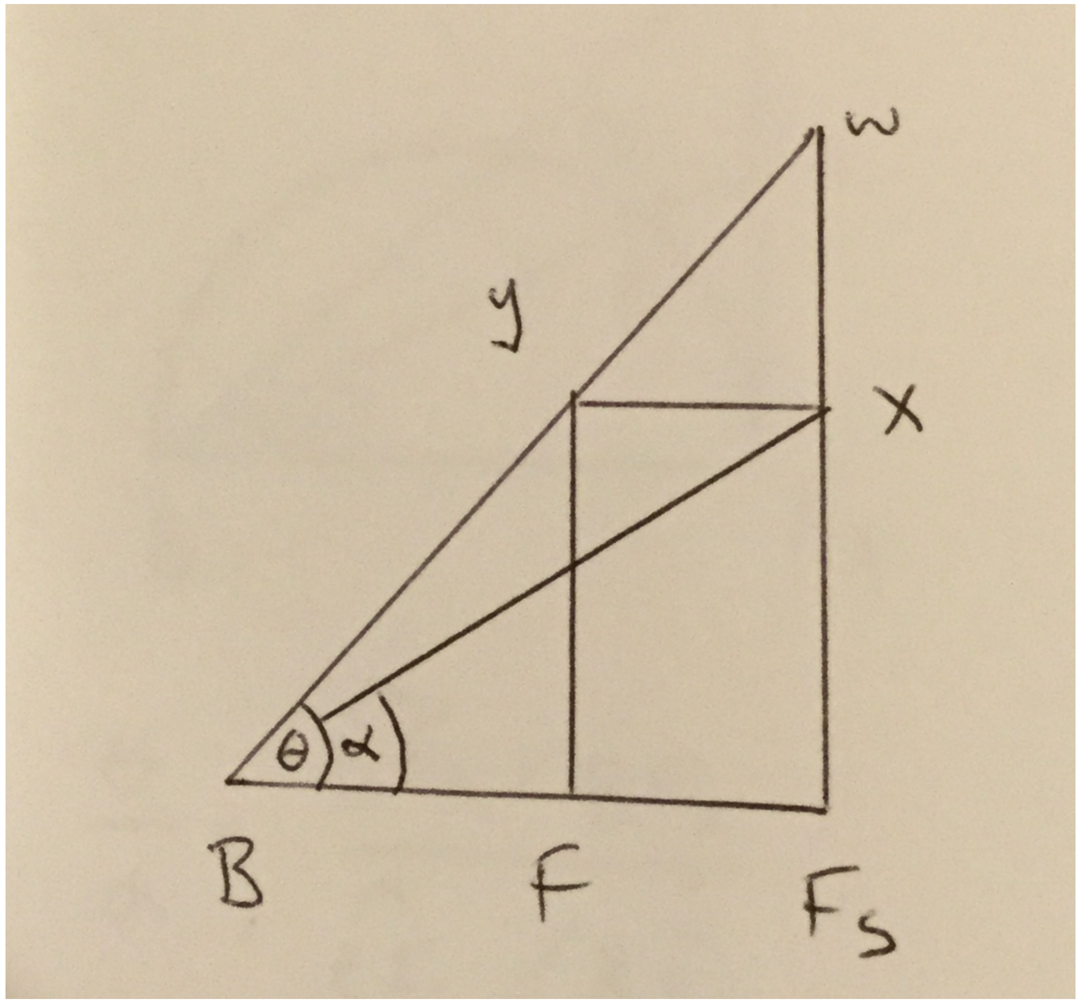
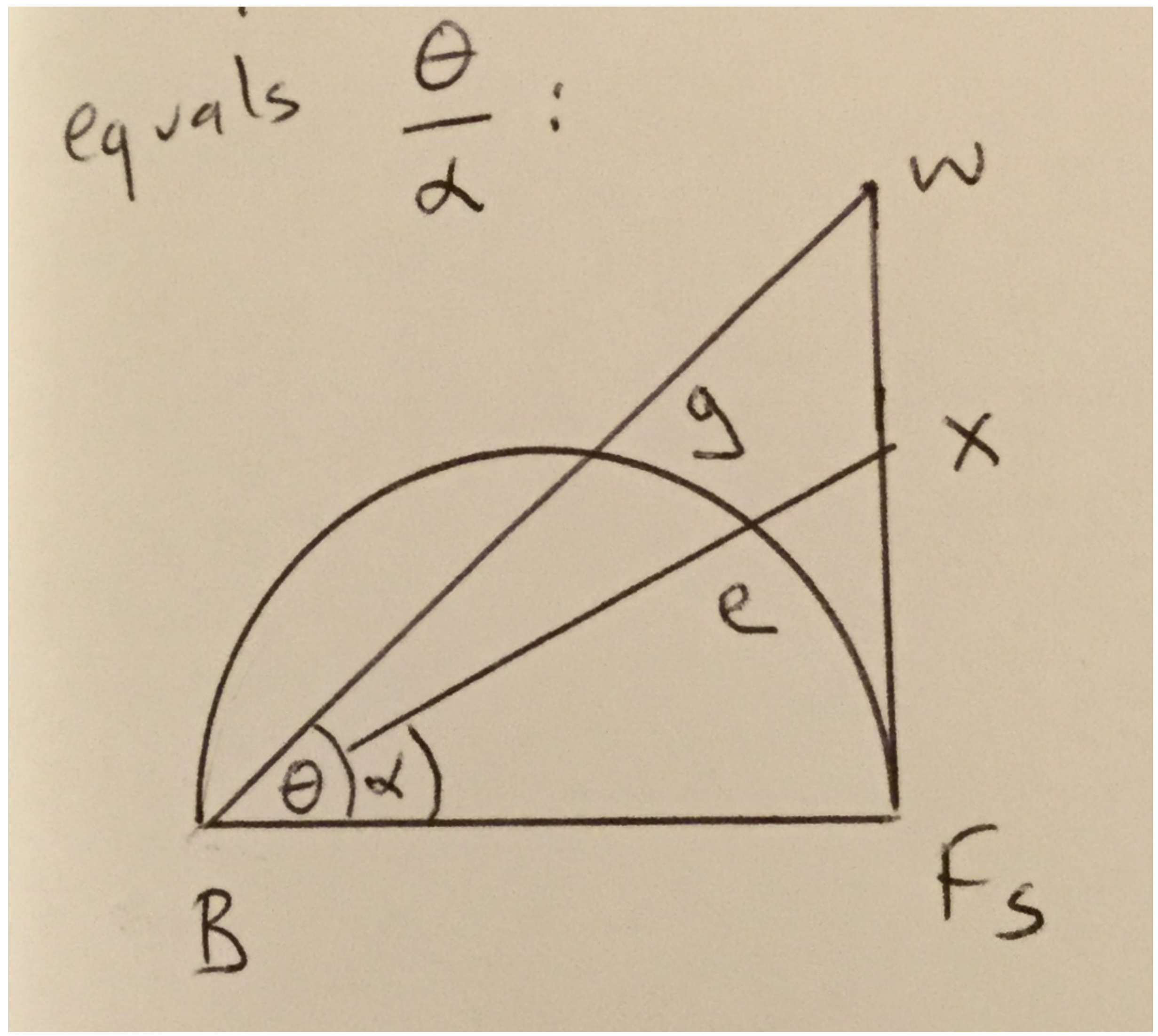


Figure 65:

The near object angular subtense magnification



$$\theta/\alpha = \frac{\sim gF_s/BF_s}{\sim eF_s/BF_s}$$

$$\text{as } yF = xF_s \Rightarrow 0$$

$$\theta/\alpha \Rightarrow \frac{wF_s}{xF_s} = \frac{wF_s}{yF} = \frac{BF_s}{BF}$$

which equals the axial near object angular subtense magnification.

Multiplying the axial near subtense magnification by the axial magnification of near correction produces:

$$\frac{BFs}{FDe} = \frac{D_2F_1}{D_2F} \; \frac{BFs}{D_1F_1}$$

Since the converging lens at D_2 creates a virtual image at F_1 of an object at F, so that the enlargement of an object at F created by D_2 equals D_2F_1/D_2F; when the diagram represents a stand magnifier with lens D_2 and stand height D_2F, and the reading spectacle add is D_1, (or the ocular accommodation is D_1 at B), the magnification produced by the stand magnifier is its (constant) enlargement factor, multiplied by that produced by D_1 alone.

The ratio describing near object axial angular subtense magnification:

$$\frac{BFs}{BF}$$

when combined with the ratio describing near magnification due to a single converging lens producing parallel light for an emmetropic eye:

$$\frac{FB}{FD}$$

produces a ratio product which factors out the object's actual distance to the eye, confirming that when a converging lens is used with its front focal point at the near object, (and therefore parallel light leaves the converging lens from the object), the image size is the same regardless of the object-to-eye distance.

Section 9

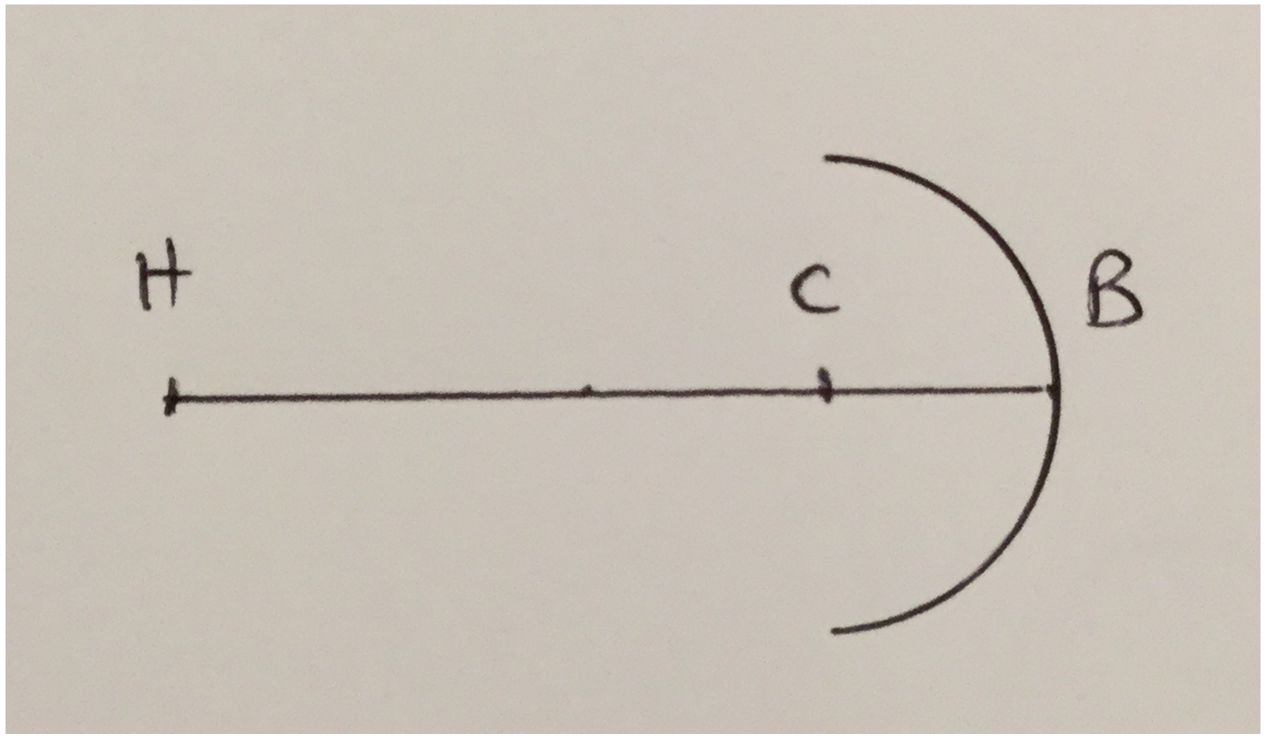
Crossed Cylinders

It is often useful to know the meridian of maximum axial refraction when combining the effects of two spherical cylinders at an oblique axis. To do this, we need to describe how their axial radii of curvature change with various meridional cross sections, and find expressions of those axial radii of curvature that are additive in terms of refraction. We then need to find the maximum sum of those expressions in terms of the meridional axis.

Meridional cross sections of a spherical cylinder are ellipses, (until they become parallel lines along the cylinder axis). Finding the axial radii of these ellipses would be difficult. Assuming a spherical cylinder is a parabolic cylinder, (and assuming cross sections of parabolic cylinders are parabolas, until they become parallel lines along the cylinder axis), allows for a much simpler determination of the axial radii of curvature of meridional cross sections.

This section works with these assumptions in order to provide approximations of axial radii of curvature for meridional cross sections of spherical cylinder. It also then uses expressions of these axial radii of curvature that are additive in terms of refraction, and demonstrates how to find the maximum sum of those expressions in terms of the meridional axis.

Figure 66:



ℝ = HB/HC

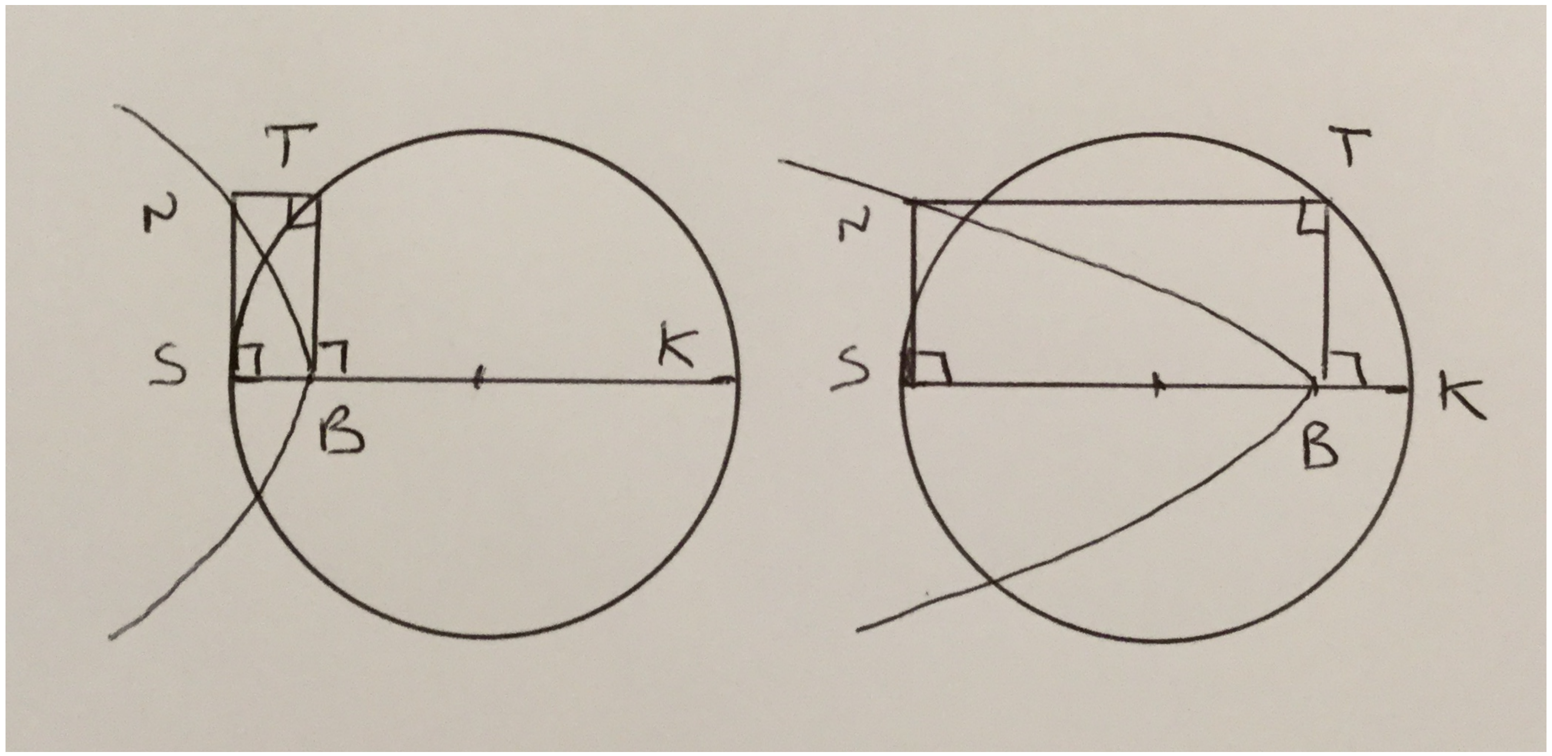
With any axial radius of curvature **CB**, and index of refraction ℝ, the axial image of a distant object lies at **H** when:

The axial refractive effects of compound refractive surfaces at **B** are additive only as their refractive "powers," which equal:

$$\frac{\mathbb{R}}{HB} = \frac{1}{HC} = \frac{(HB - HC)/HC}{CB} = \frac{(\mathbb{R} - 1)}{CB}$$

All parabolas have the same shape, in the same way that all circles have the same shape. However, while circles have a single (internal) determining constant, the radius of curvature, parabolas have both a determining constant internal and external to the curve, and can be defined by either.

Figure 67:

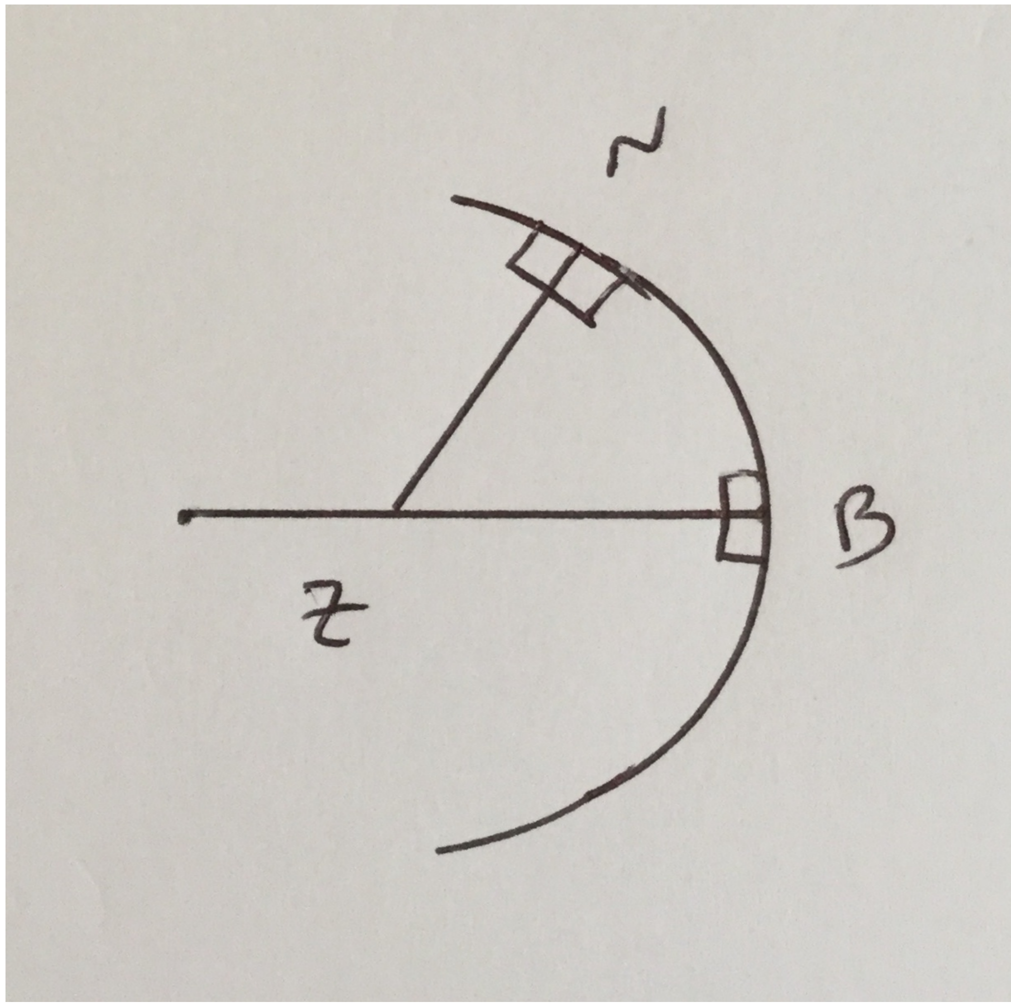


For example, a parabola's external determining constant equals **BK** when:

$$\frac{SB}{BT} = \frac{BT}{BK}$$

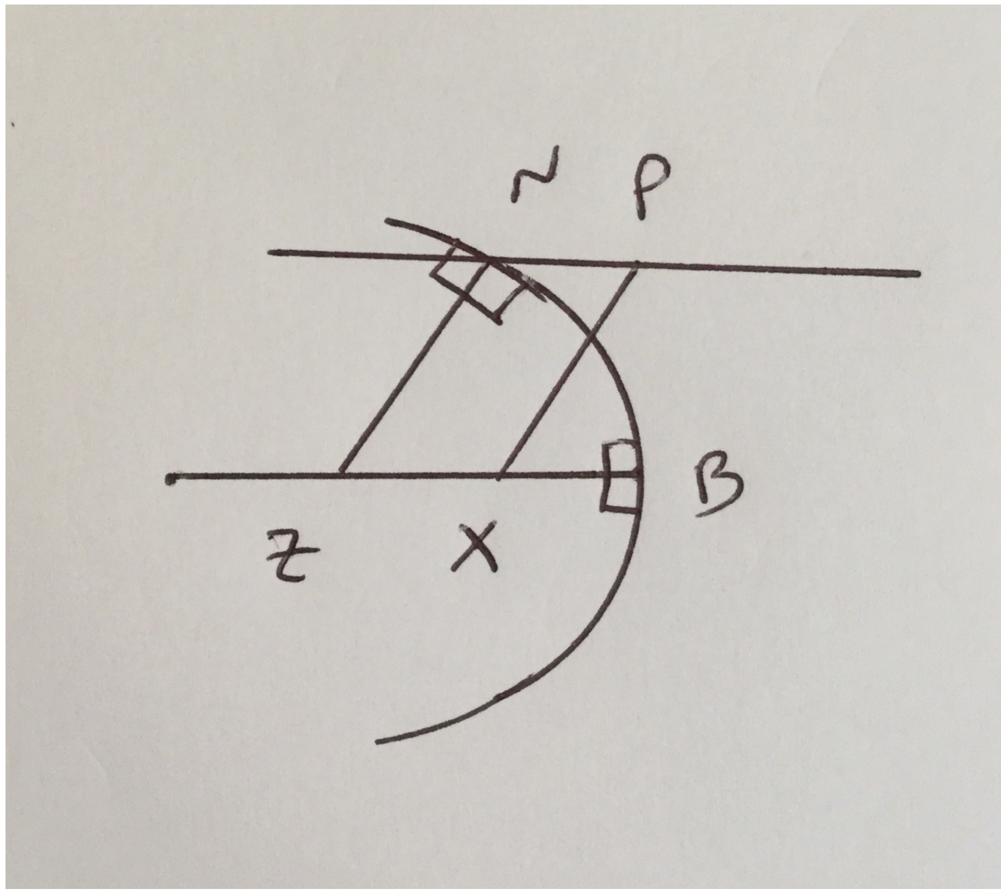
[2(**SN**) equals the sagitta corresponding to the sagittal depth **SB**].

Figure 68:



We can set up the necessary off-axis conditions to determine a parabola's axial center of curvature in terms of its internal determining constant **XB**, by involving **ZN** in the geometric solution for **XB**.

Figure 69:



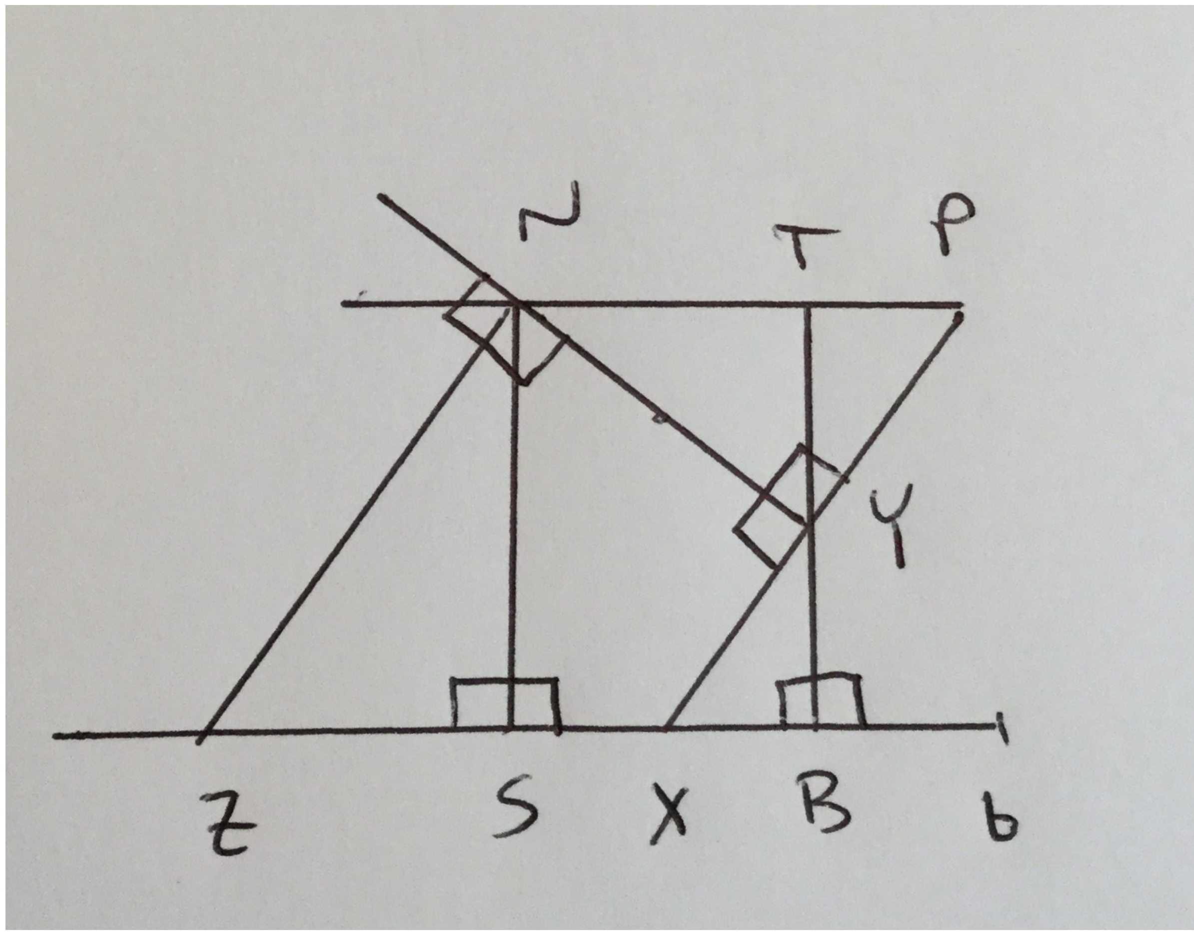
In order to keep the determining geometrical relationships axial as **N** ⇒ **B**,

they should also depend on line **NP** being parallel to the axis, and **XP** being parallel to **ZN**.

We know **X** lies between **Z** and **B**, since parabolas flatten in their periphery.

Since as **N** ⇒ **B**, **Z** ⇒ **C** by definition, and since **XP** = **ZN**, **P** will remain external to the curve, and **X** can therefore not be its axial center of curvature, but must instead lie somewhere along **CB**.

Figure 70:



In order to maintain **ZN** perpendicular to the parabola at **N** as **N** ⇒ **B**,

the same geometrical relationships must exist that allow for that when **N** lies at **B**.

In other words:

YP = YX and
Bb = BX so
CB = 2(XB)

Since:

$$\frac{TN}{TB} = \frac{TN}{2(TY)} = \frac{YB}{2(XB)} = \frac{YB}{CB} = \frac{TB}{2(CB)}$$

We know the external determining constant **BK** equals 2(**CB**), and the internal determining constant **XB** equals (**CB**)/2.

Axial refracting power equals $\frac{(\mathbb{R} - 1)}{CB}$

Since for a parabola:

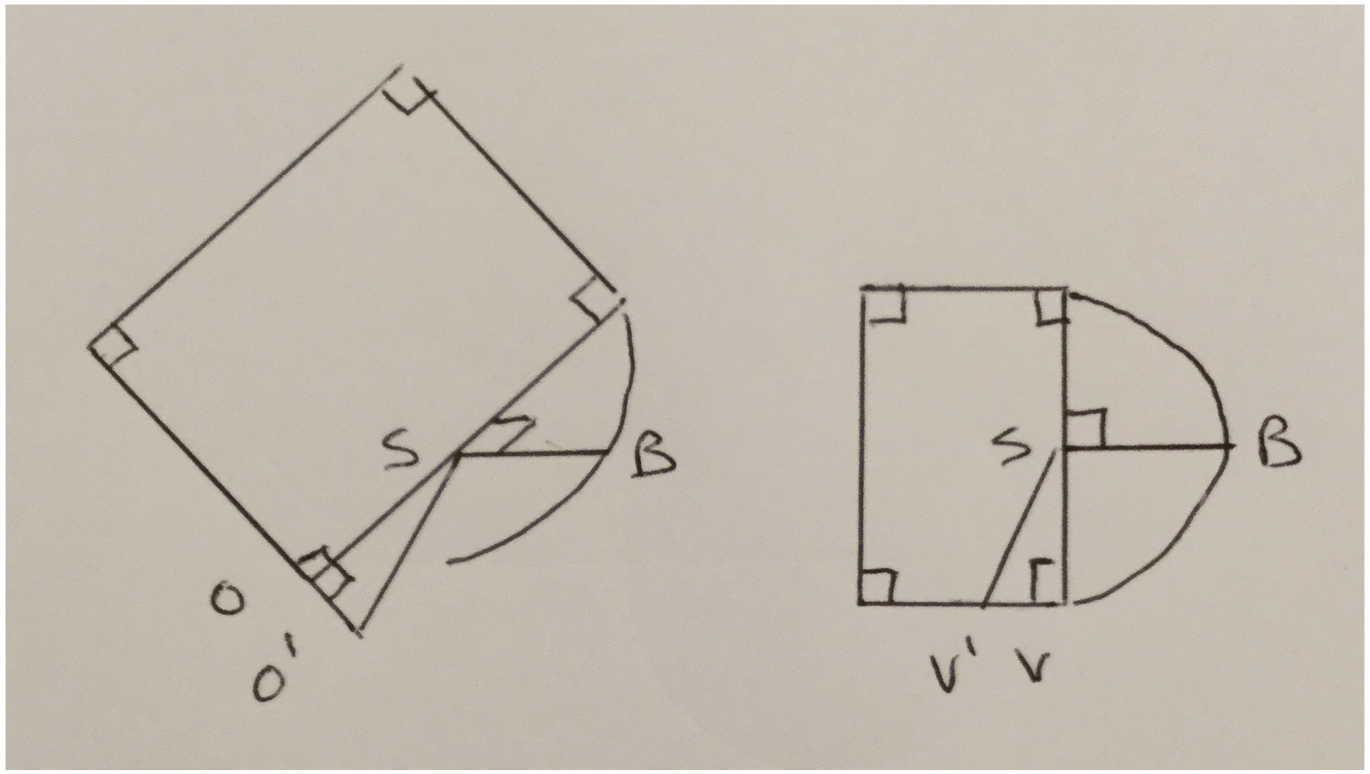
$$\frac{SB}{SN} = \frac{SB}{TB} = \frac{TB}{2(CB)}$$

If $\mathbb{R} = 1.5$

The axial refracting power of a parabola equals:

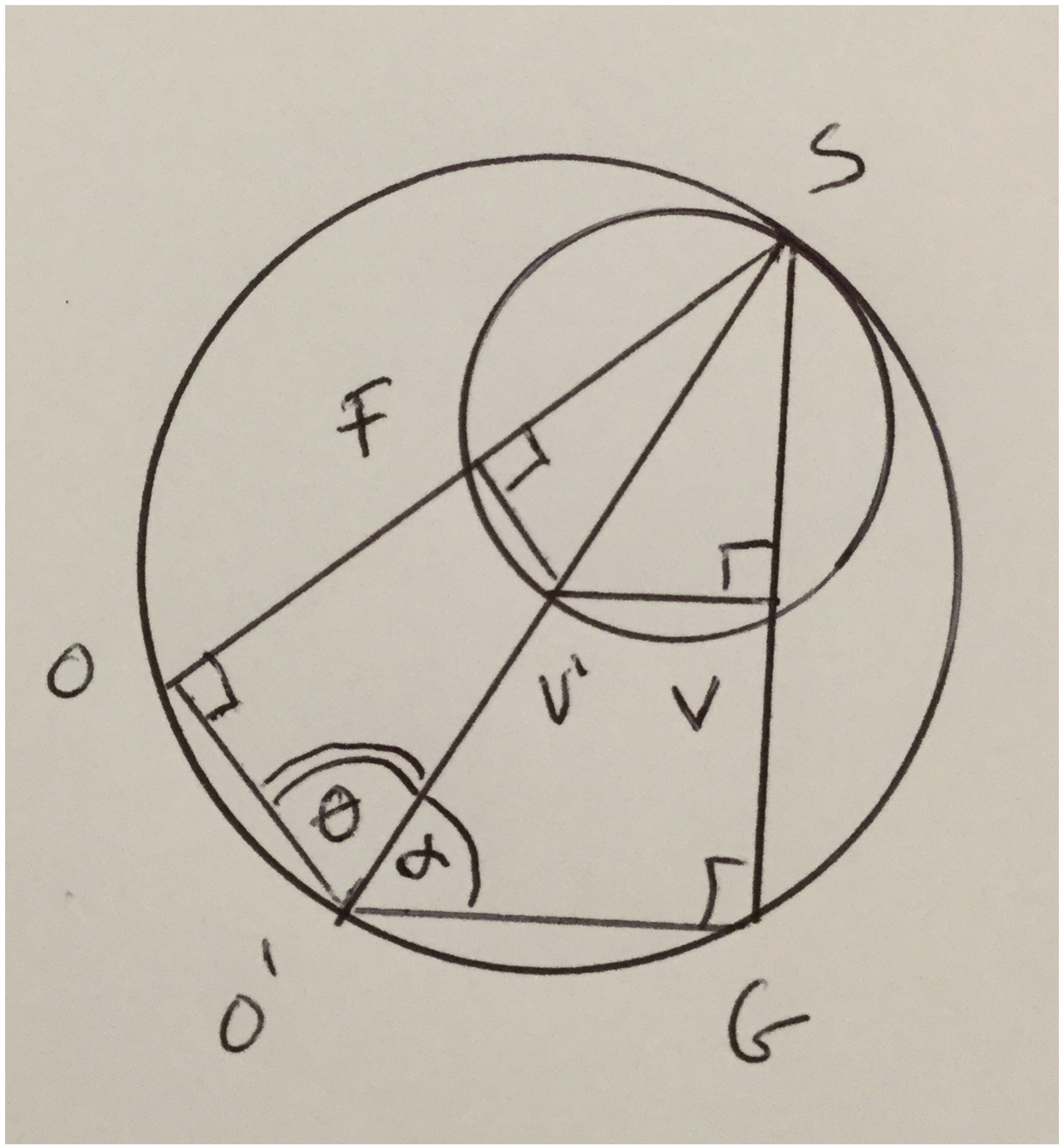
$$\frac{1}{2(CB)} = \frac{SB}{SN^2} = \frac{1}{BK}$$

Figure 71:



When 2(**SO**) equals the minimum sagitta of an oblique parabolic cylinder, and when with equal sagittal depth **SB**, 2(**SV**) equals the minimum sagitta of a more highly curved parabolic cylinder with a horizontal axis:

Figure 72:

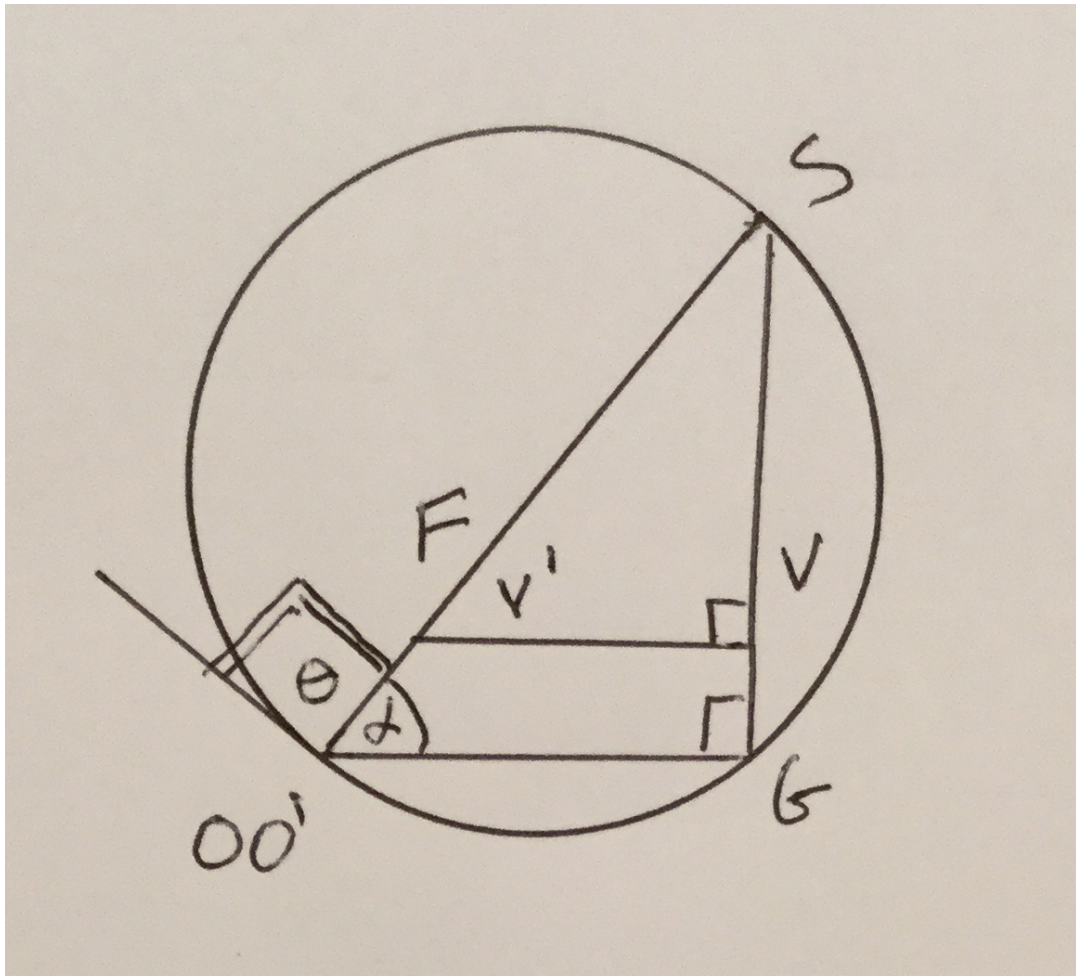


Keeping Δ**OSV** constant, as we rotate circle **SOG** with variable diameter **SV'O'** around point **S**:

∠OO'G is constant because ∠OSG is constant,

so Δθ = -Δα

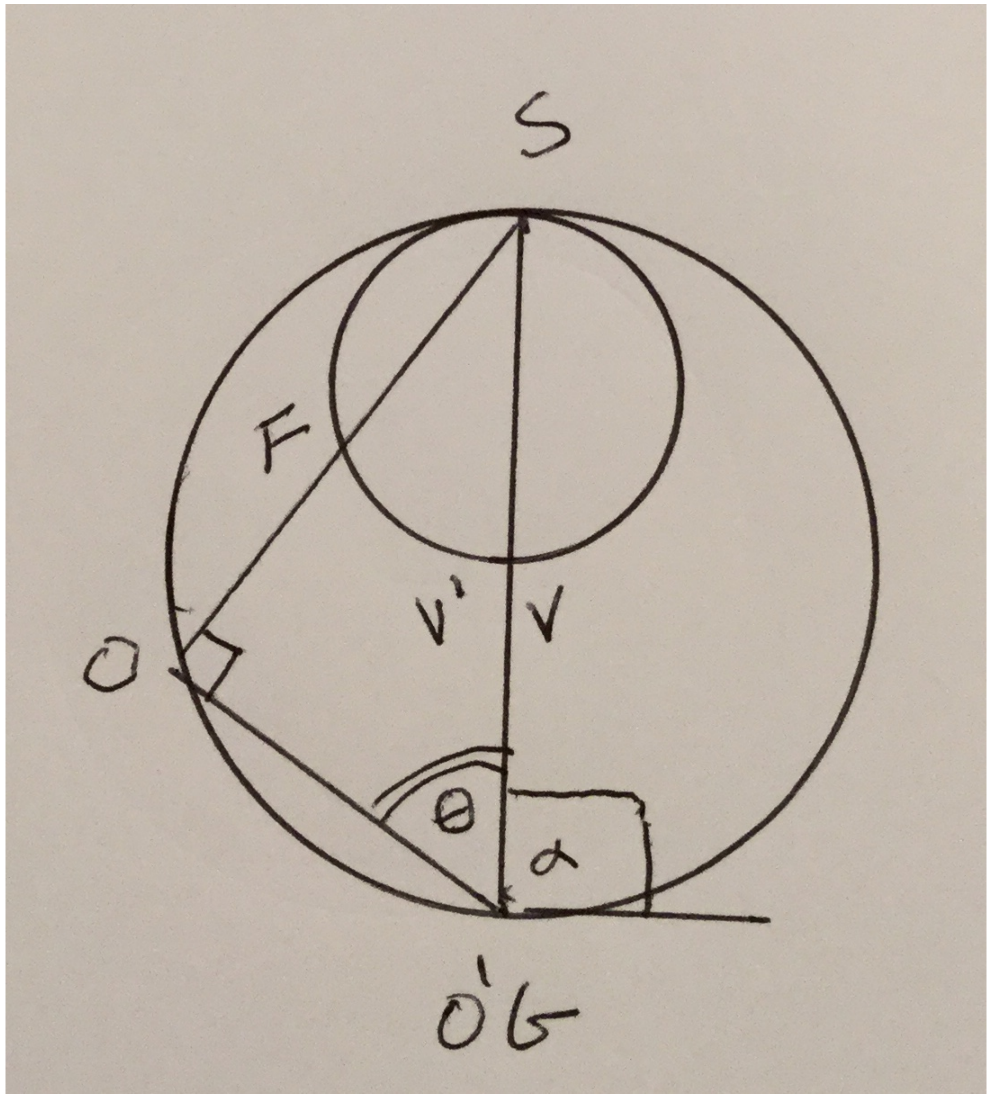
Figure 73:



As O' ⇒ O

SV' increases more than SO' decreases

Figure 74:



As V' ⇒ V

SO' increases more than SV' decreases

Since the sum (**SO' + SV'**) increases when either:

O' ⇒ **O**,   or **V'** ⇒ **V**

there must be a specific **SV'O'** within Δ**OSV** producing a minimum sum (**SO' + SV'**),
which must be near where small rotations produce only minimal changes in (**SO' + SV'**).

Since as when one term of the sum (**SO' + SV'**) increases, the other always decreases, this process can be taken to its limits to determine the meridian with minimum (**SO' + SV'**) using:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta(SO') = \lim_{\Delta\alpha \Rightarrow 0} \Delta(SV')$$

However, the combined effects of refraction are additive only as refractive powers, which, when ℝ = 1.5, equal:

$$\frac{SB}{(SO')^2} \quad \text{and} \quad \frac{SB}{(SV')^2}$$

Therefore, the meridian with the maximum combined effects of this refraction can be found using:

$$\lim_{\Delta\theta \Rightarrow 0} \Delta \frac{SB}{(SO')^2} = \lim_{\Delta\alpha \Rightarrow 0} \Delta \frac{SB}{(SV')^2}$$

To solve this equation, all variables must be expressed in terms of the variables approaching zero, so:

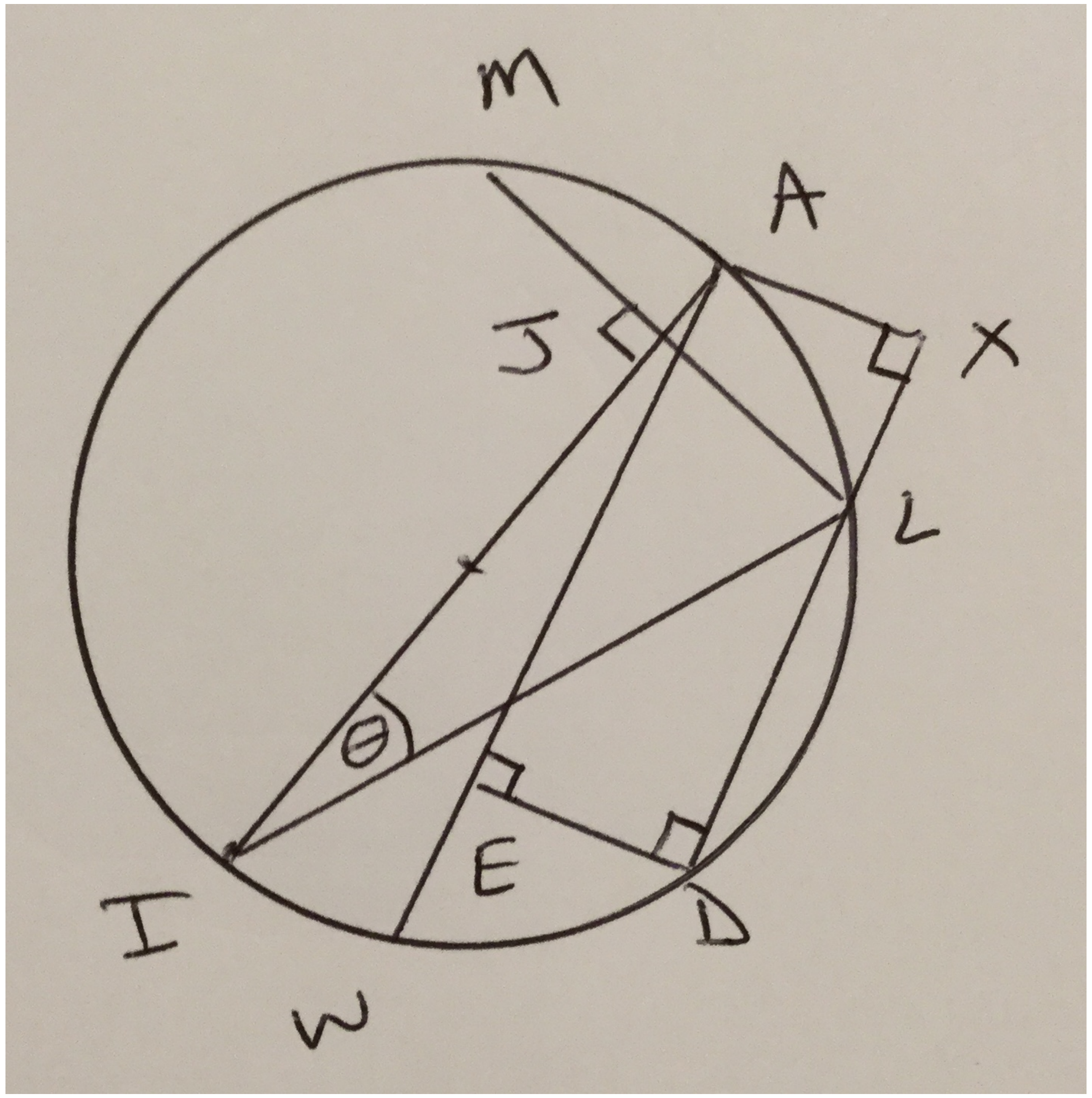
$$\lim_{\Delta\theta \Rightarrow 0} \Delta \frac{SB(SO/SO')^2}{(SO)^2} = \lim_{\Delta\alpha \Rightarrow 0} \Delta \frac{SB(SV/SV')^2}{(SV)^2}$$

$$\lim_{\Delta\theta \Rightarrow 0} \Delta \frac{(SB)\sin^2}{(SO)^2} \theta = \lim_{\Delta\alpha \Rightarrow 0} \Delta \frac{(SB)\sin^2}{(SV)^2} \alpha$$

$$\frac{SB}{SO^2} \lim_{\Delta\theta \Rightarrow 0} \Delta \sin^2 \theta = \frac{SB}{SV^2} \lim_{\Delta\alpha \Rightarrow 0} \Delta \sin^2 \alpha$$

$$\frac{\lim_{\Delta\theta \Rightarrow 0} \Delta \sin^2 \theta}{\lim_{\Delta\alpha \Rightarrow 0} \Delta \sin^2 \alpha} = \frac{SO^2}{SV^2}$$

Figure 75:



Solve for

Limit $\Delta \sin^2 \theta$
$\Delta\theta \Rightarrow 0$

on the reference circle:

AW ≥ LD ‖ AW

$\angle ALD = \sim \frac{AID}{AI} \geq \sim \frac{AI}{AI} = \pi$

Establish the necessary functions of θ in terms of line segments and chords.

$$\theta = \sim\frac{AL}{AI} \quad ; \quad \sin^2\theta = \frac{AL^2}{AI}$$

$$\Delta\,\theta = \sim\frac{LD}{AI} \quad ; \quad \sin^2\Delta\,\theta = \frac{LD^2}{AI}$$

$$(\theta + \Delta\,\theta) = \sim\frac{ALD}{AI} \quad ; \quad \sin^2(\theta + \Delta\,\theta) = \frac{AD^2}{AI}$$

$$\cos\theta = \frac{IL}{AI} \quad ; \quad \cos(\theta + \Delta\,\theta) = \frac{DI}{AI}$$

$$\sin\theta = \frac{AL}{AI} = \frac{JL}{IL} \quad ; \quad \sin\theta\cos\theta = \frac{JL}{IL}\,\frac{IL}{AI}$$

$$2\,(\sin\theta\cos\theta) = \frac{ML}{AI} = \sin 2\theta$$

Then consider the following property of the cyclic quadrilateral circle **ALDW**: $AD(LW) = AL(DW) + LD(AW)$

$$\Delta DIA \cong \Delta EWD = \Delta XLA \;;\; AD^2 = AL^2 + LD(AW)$$

$$AW = LD + 2(AL)\frac{LX}{LA} \;;\; AW = LD + 2(AL)\frac{ID}{IA}$$

$$AD^2 - AL^2 = LD^2 + 2(LD)(AL)\frac{ID}{IA}$$

$AI\ [\sin^2(\theta + \Delta\theta) - \sin^2\theta] =$

$AI\ [\sin^2\Delta\theta] + 2(LD)(AL)\cos(\theta + \Delta\theta) =$

$AI\ [\sin^2\Delta\theta] + 2(LD)\ [(AI)\sin\theta]\ \cos(\theta + \Delta\theta)$

Divide both sides by **AI**:

$\sin^2(\theta + \Delta\theta) - \sin^2\theta = \sin^2\Delta\theta + 2(LD)\ \sin\theta\ \cos(\theta + \Delta\theta)$

$$\lim_{\Delta\theta \Rightarrow 0} \frac{\Delta(\sin^2\theta)}{\sim LD} = 2\sin\theta\,(\cos\theta) = \sin 2\theta$$

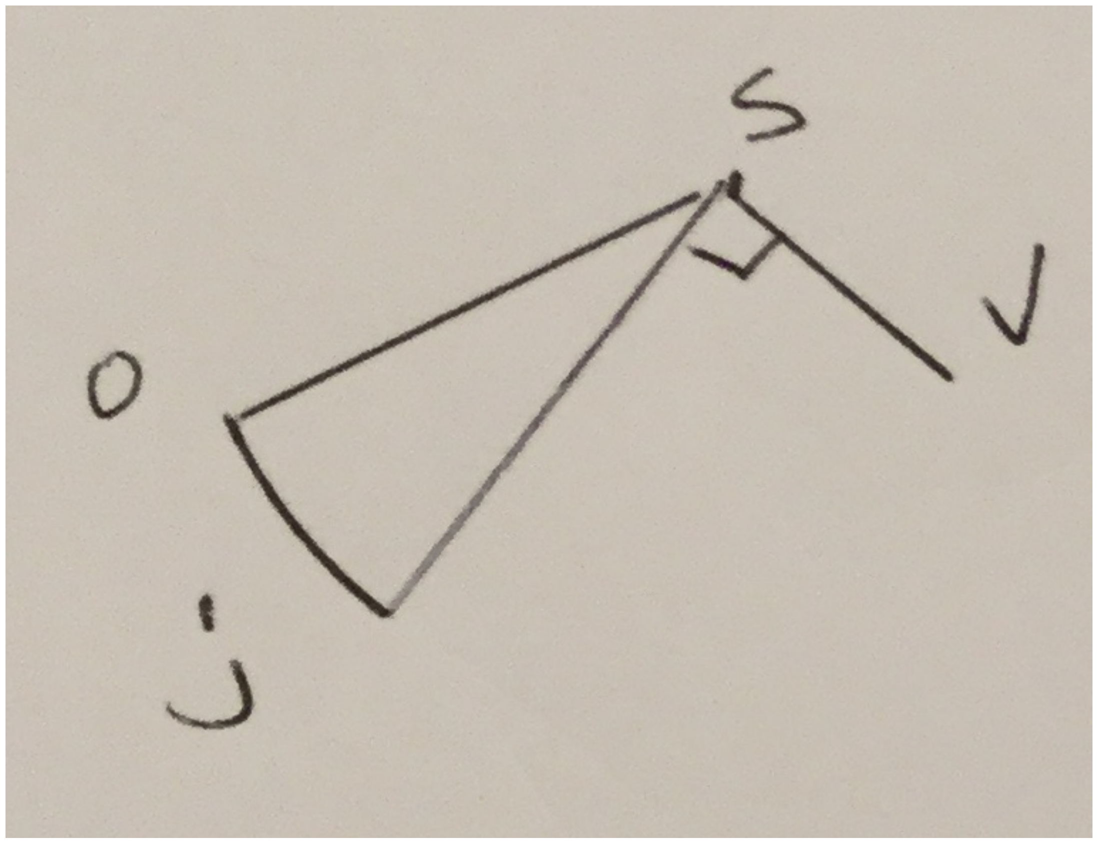
Therefore, the meridian with the maximum combined effects of refraction can be found using:

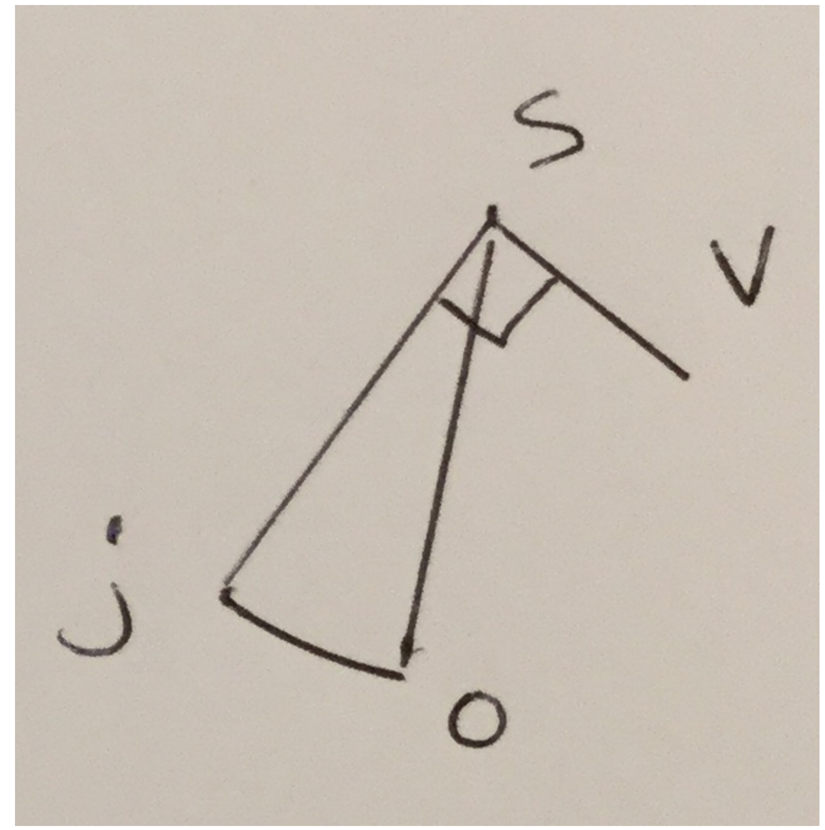
$$\frac{\sin 2\theta}{\sin 2\alpha} = \frac{SO^2}{SV^2}$$

The first step to solve this problem is to divide **SV** into **SaV** so that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

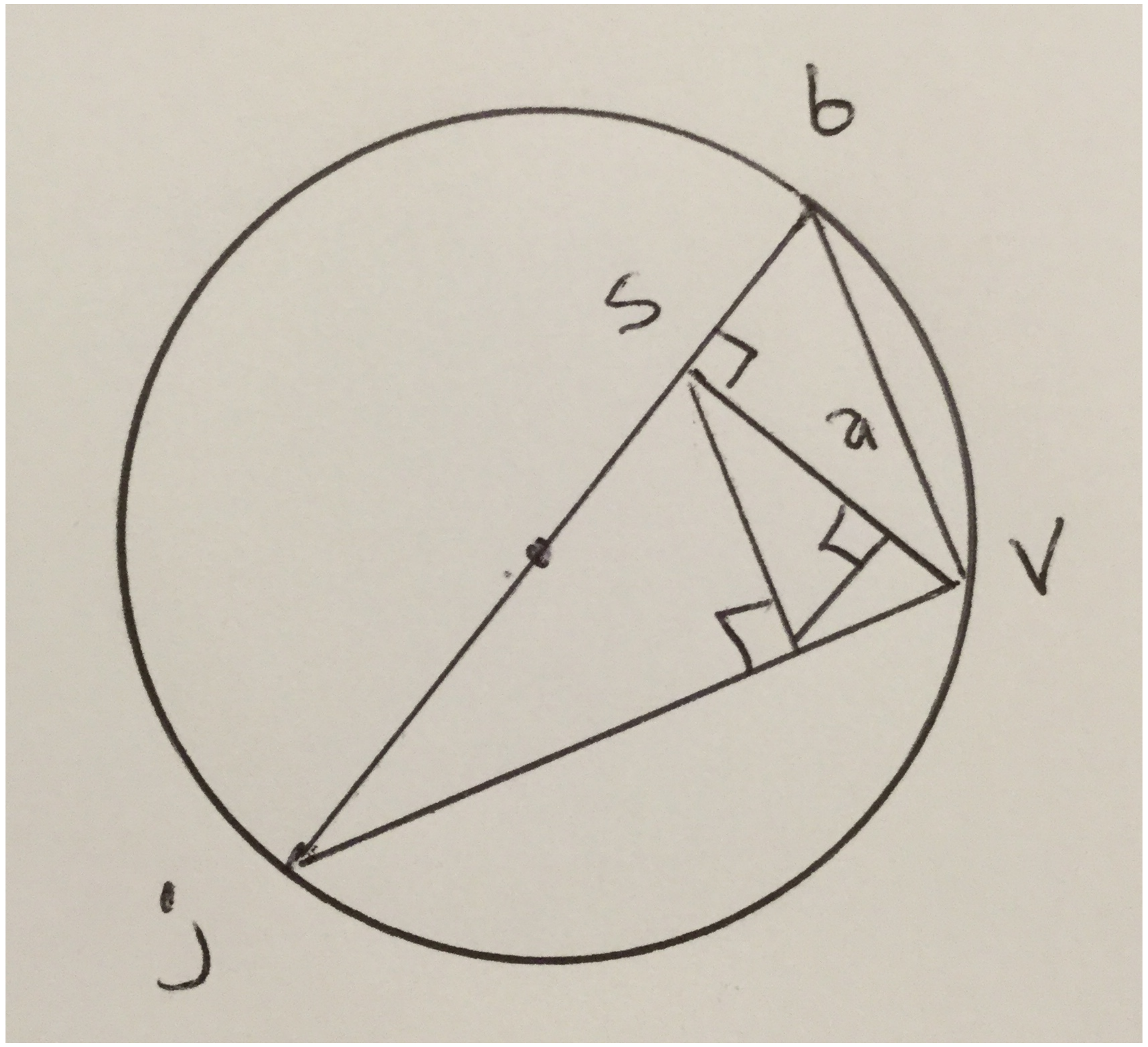
Figure 76:





Make **SO** = **Sj** ⊥ **SV** to construct:

Figure 77:

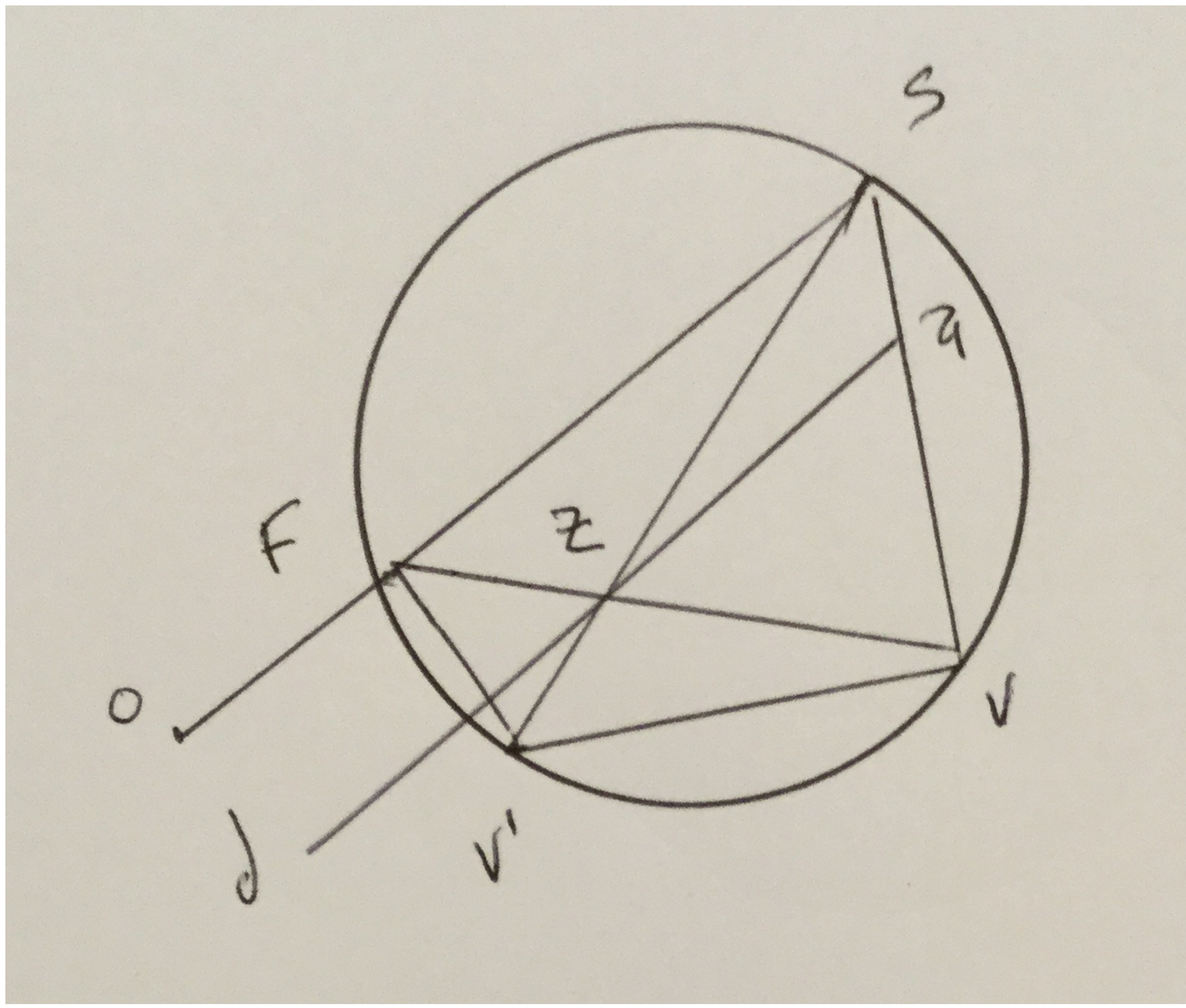


Similar triangles show that:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV}$$

$$\frac{Sj}{SV} = \frac{SV}{Sb} \quad ; \quad \frac{Sj^2}{SV^2} = \frac{Sj}{Sb} = \frac{SO^2}{SV^2}$$

Figure 78:



Draw ad ∥ SO
Choose a circle through **S** and **V** with a variable diameter **SV'** so that **FZV** lies on a common chord.

The easiest way to do this involves a template of various circles, each with the location of their diameters already marked.

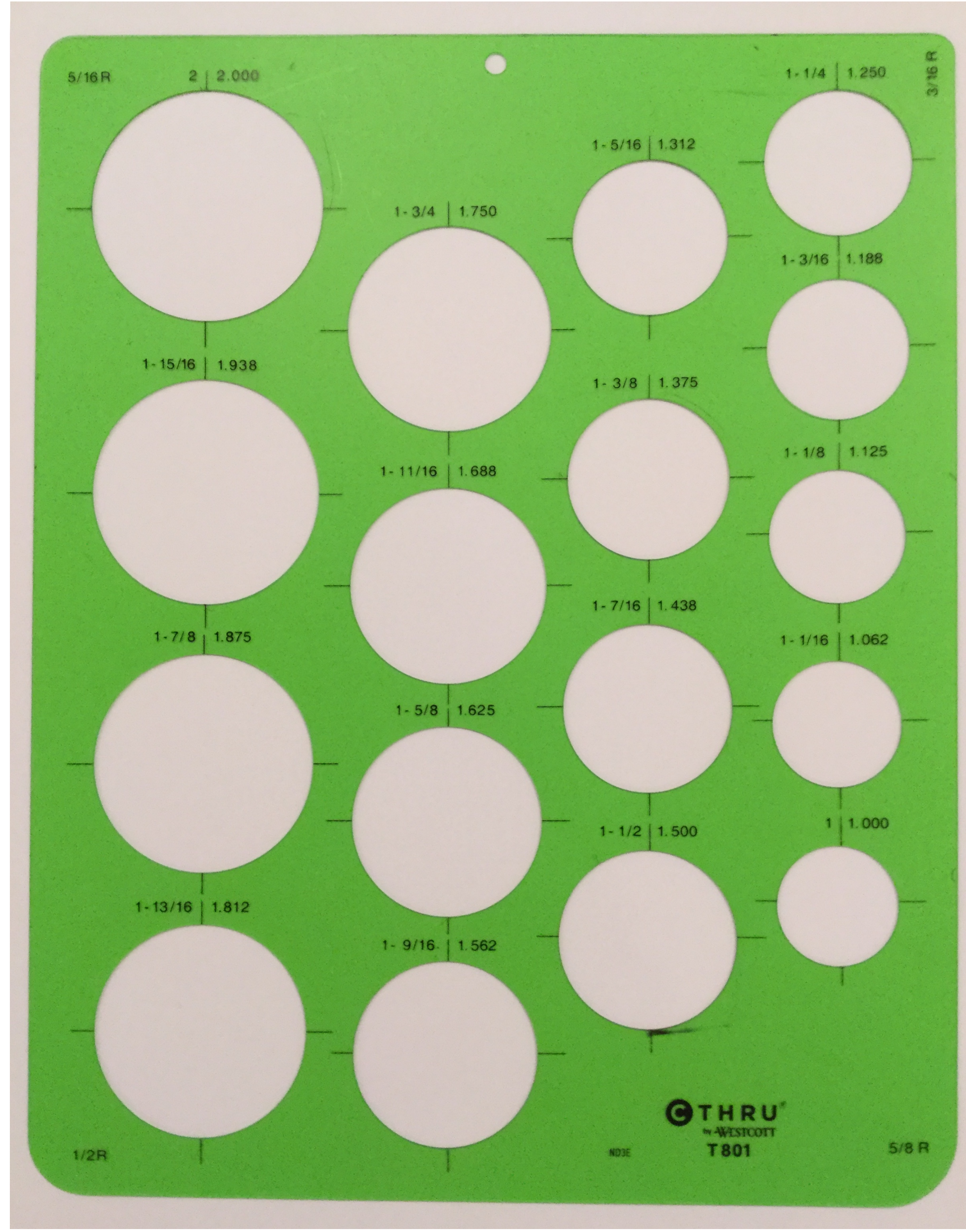
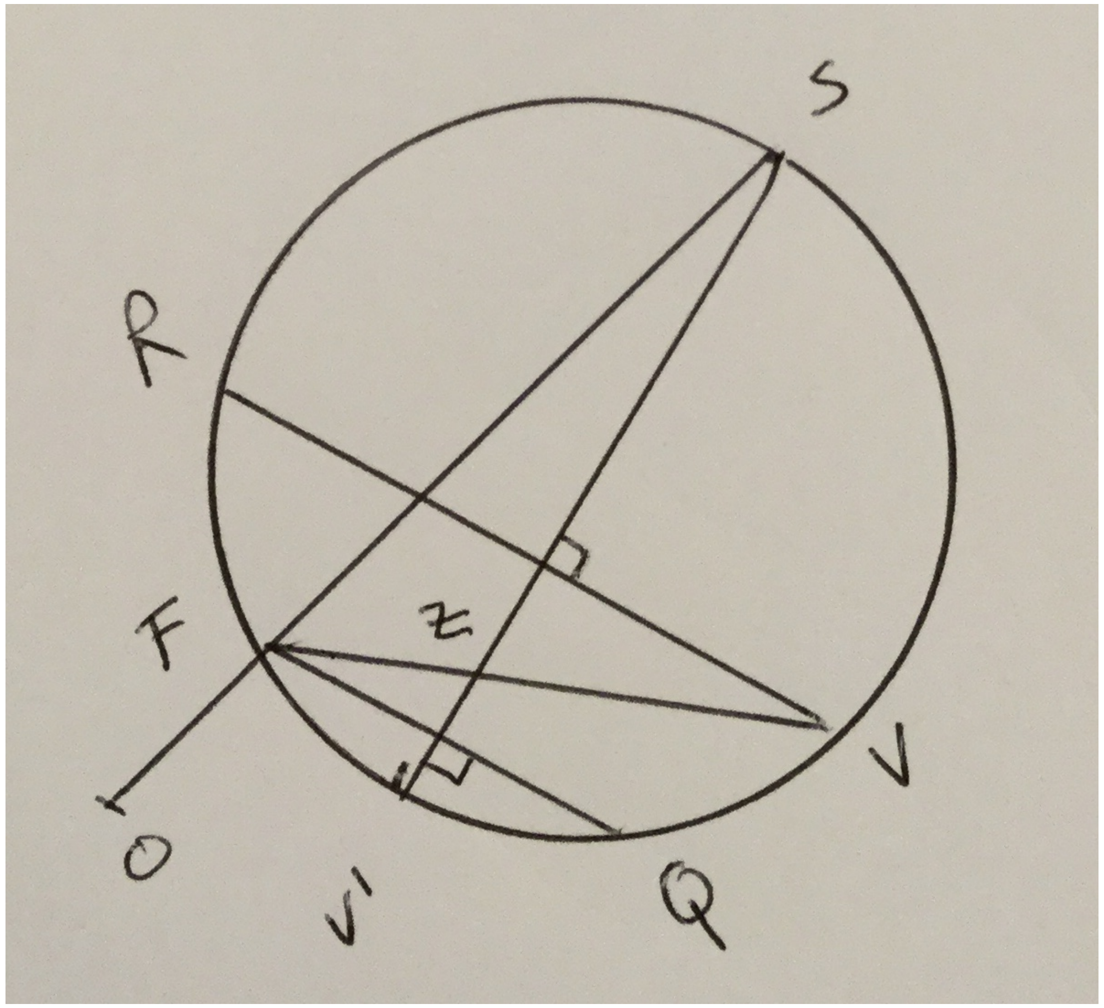


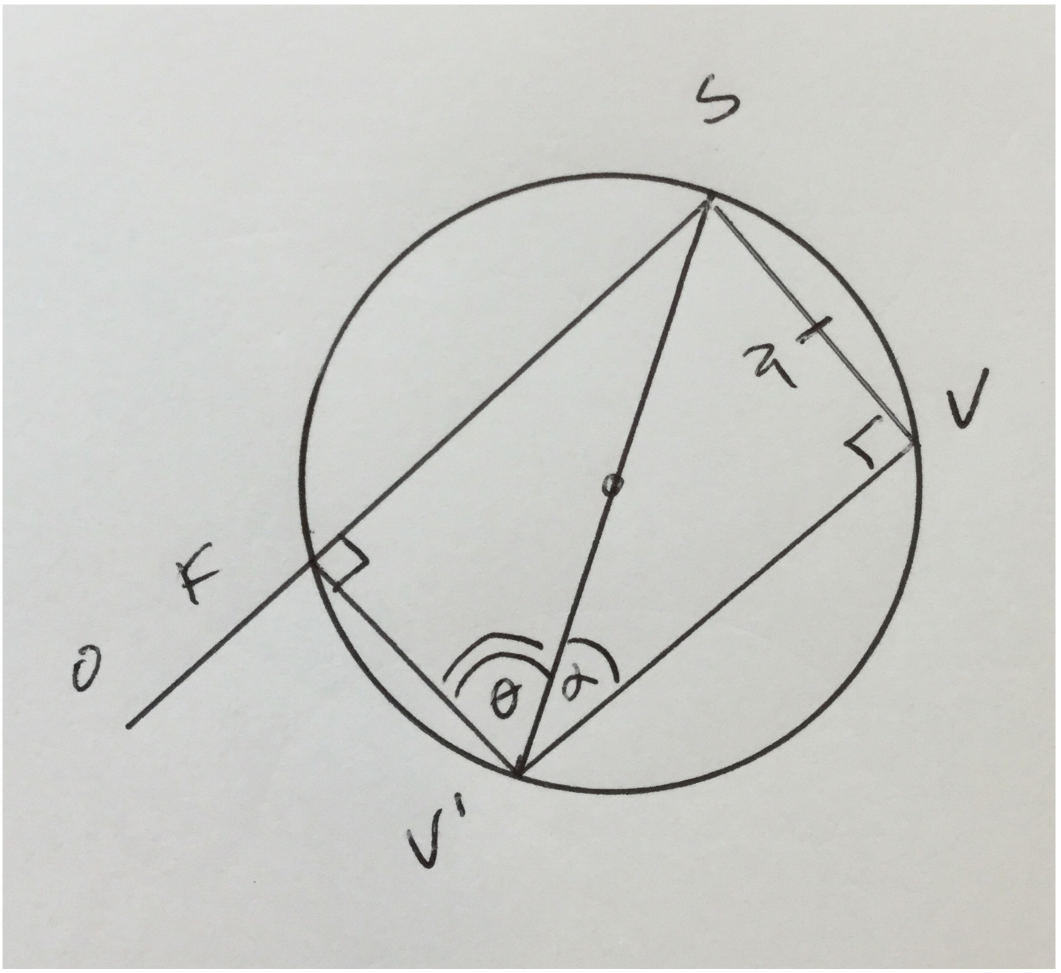
Figure 79:



SV' is the meridian with the maximum combined effects of refraction because:

$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{FZ}{ZV} = \frac{FQ/2}{RV/2} = \frac{FQ}{RV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

Figure 80: **Double-angle Method**



Given constant Δ**OSV**:

∠**FSV** is constant

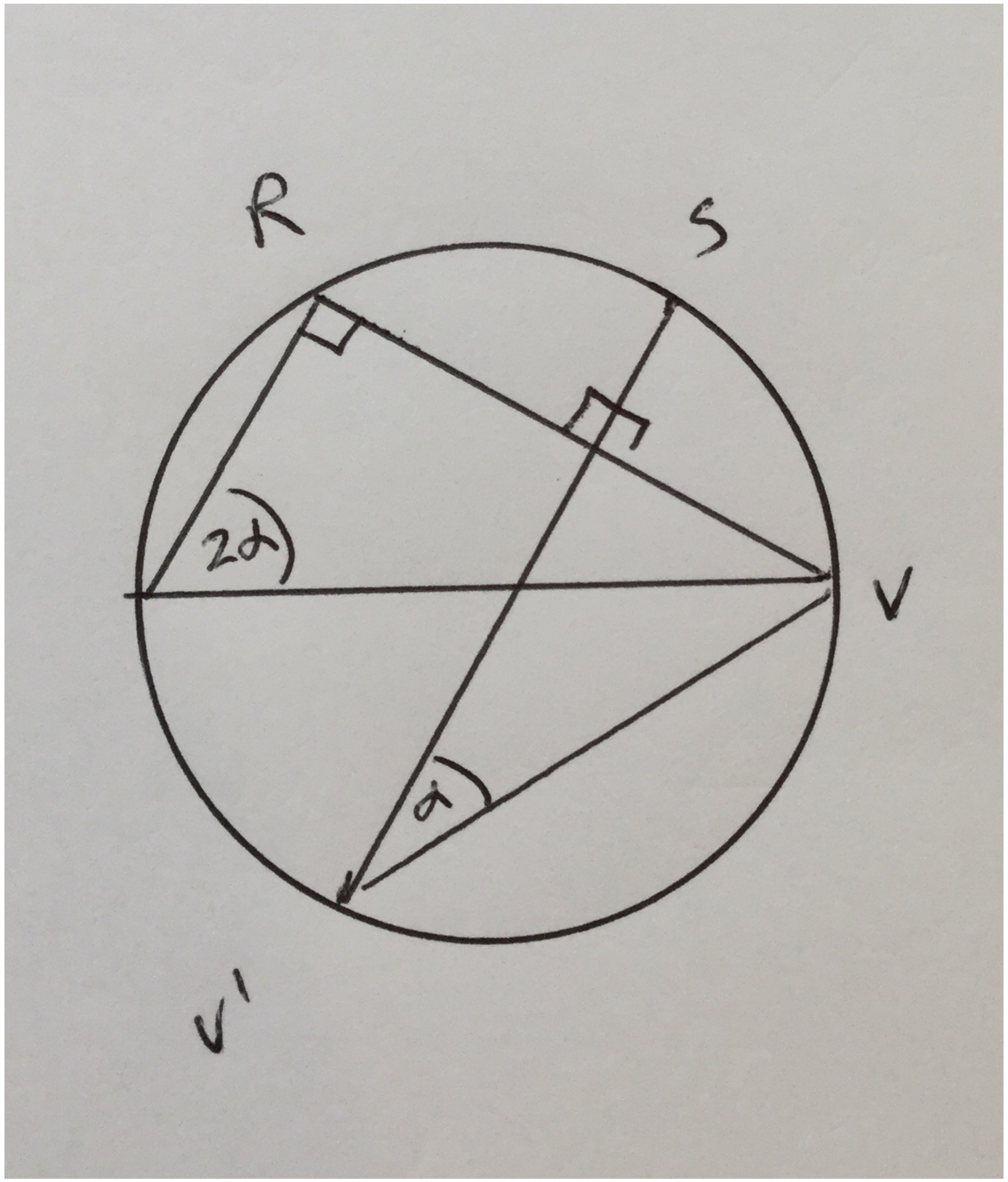
∠**FSV** + (θ + α) = π

(θ + α) Is constant

We have already shown how to find single angles θ + α so that:

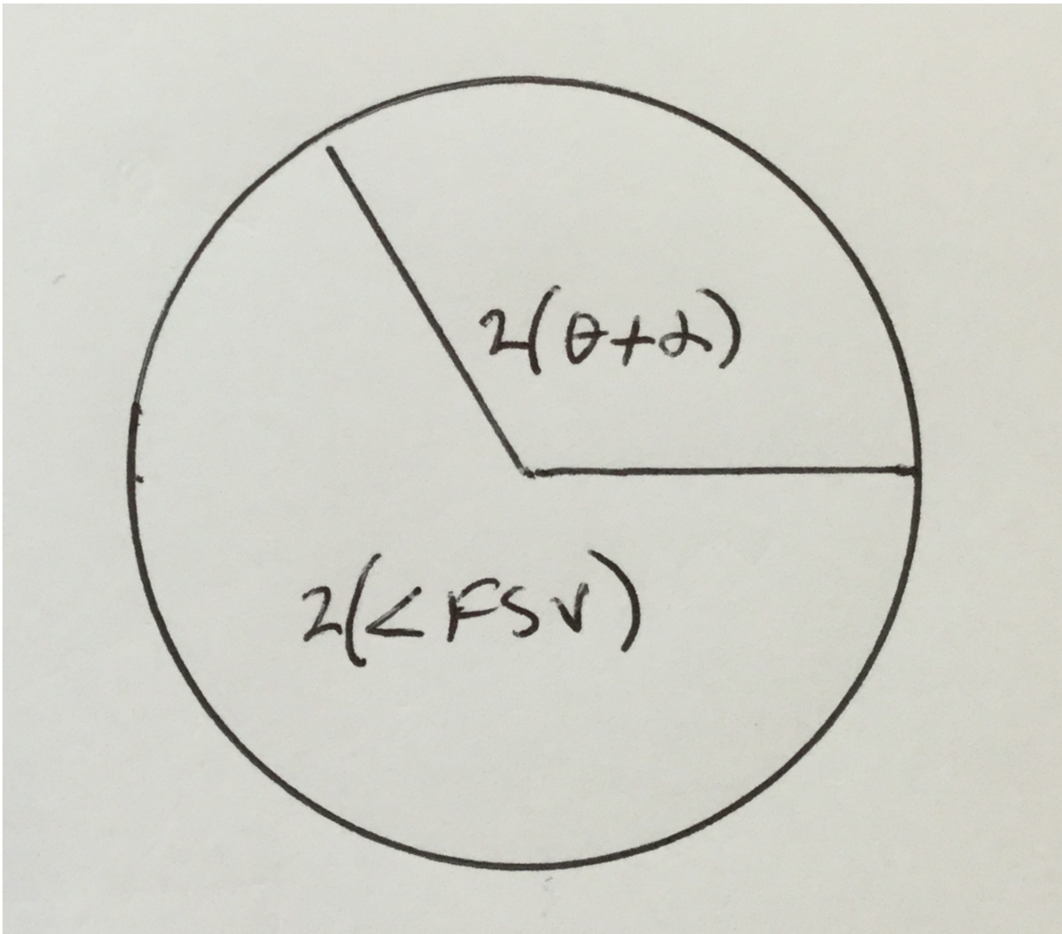
$$\frac{SO^2}{SV^2} = \frac{aS}{aV} = \frac{\sin 2\theta}{\sin 2\alpha}$$

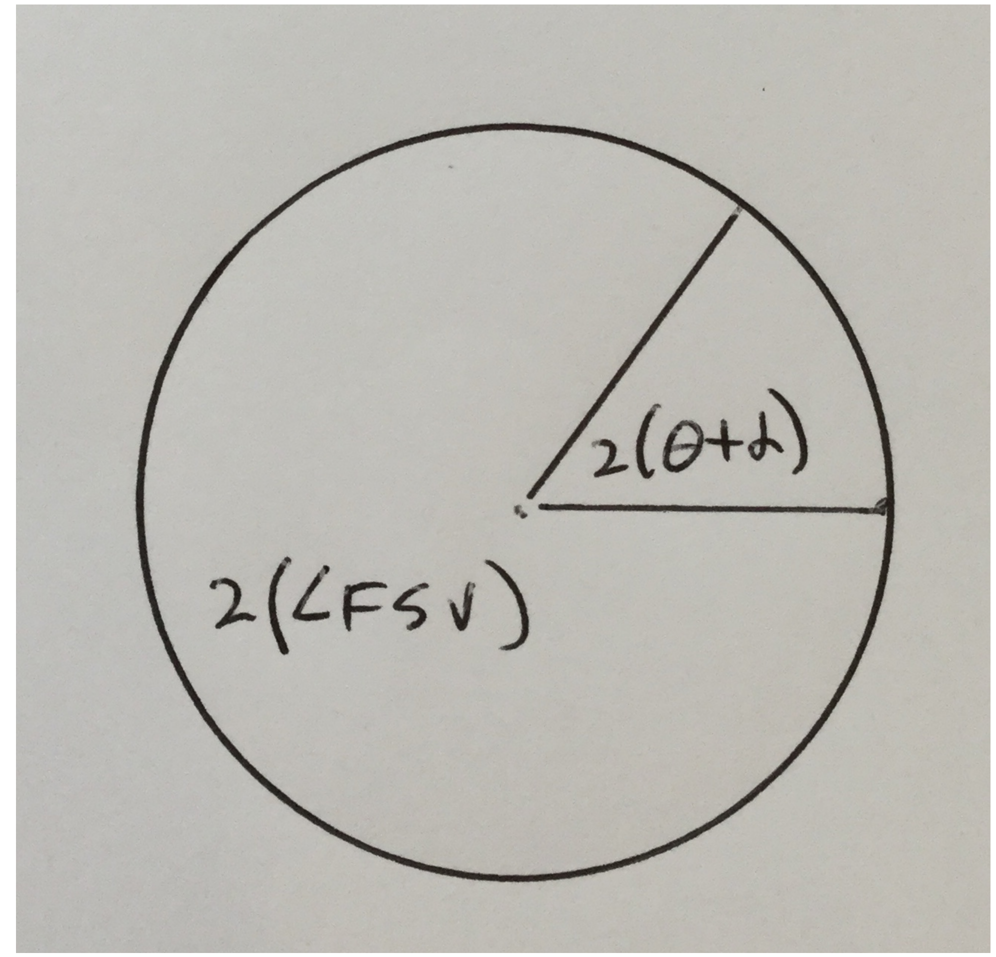
Figure 81:



An angle on a circle equals its inscribed arc, divided by the arc's diameter. Since the sum of all angles measured on a circle's circumference add to π, when measured from a circle's center they add to 2π.

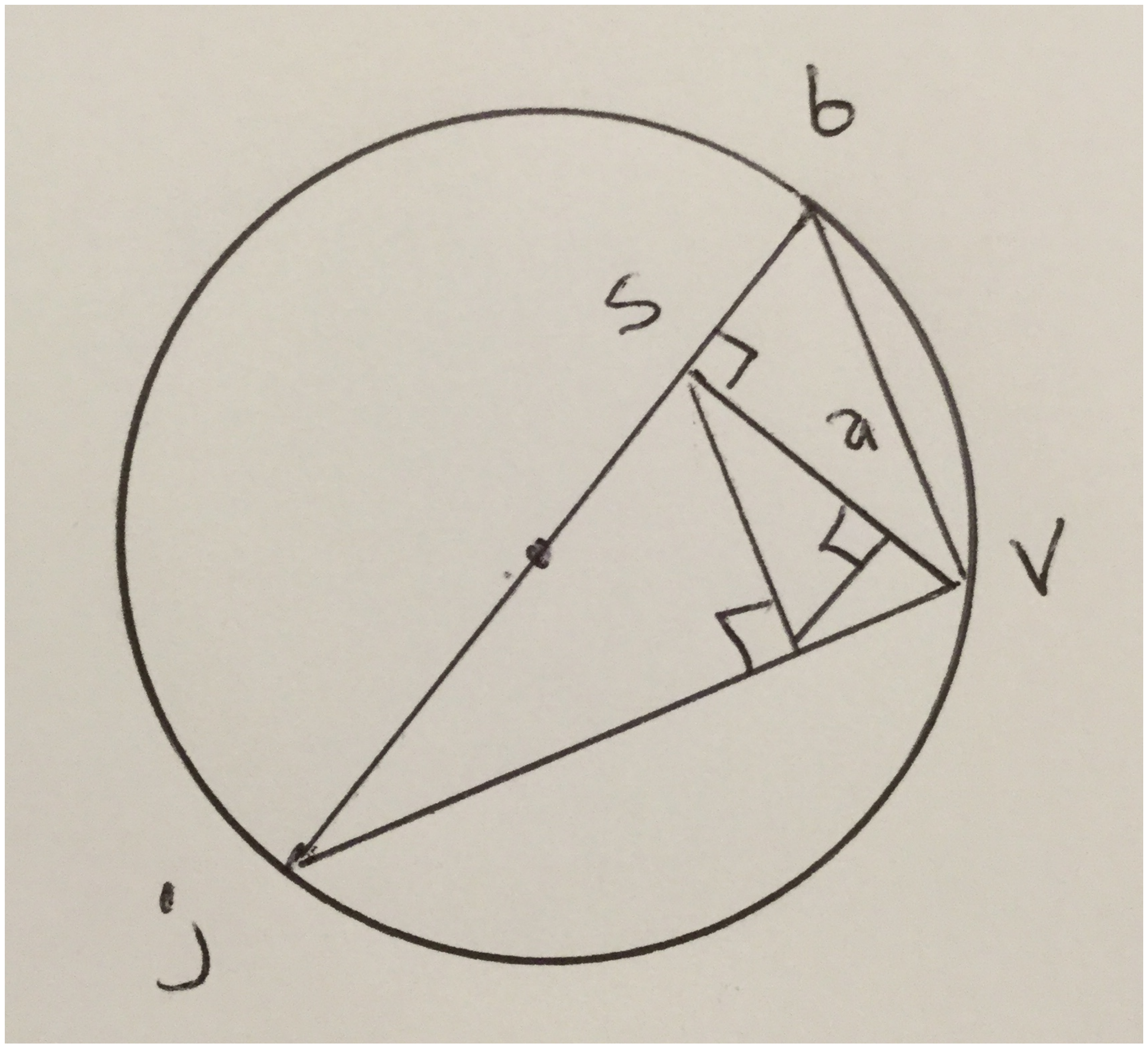
Figure 82:





Therefore: $2(\angle FSV) + 2\,(\theta + \alpha) = 2\pi$

Figure 83:

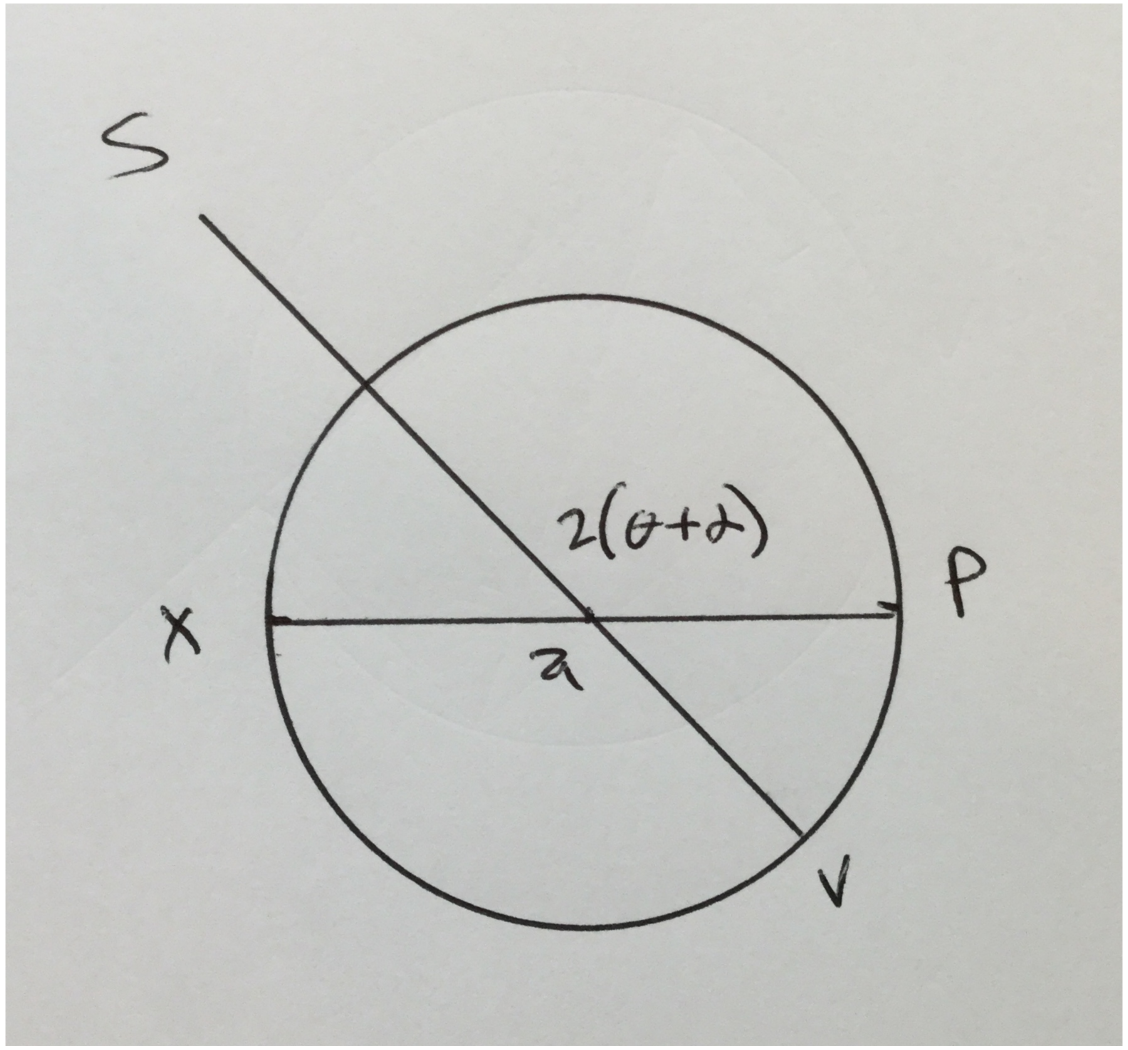


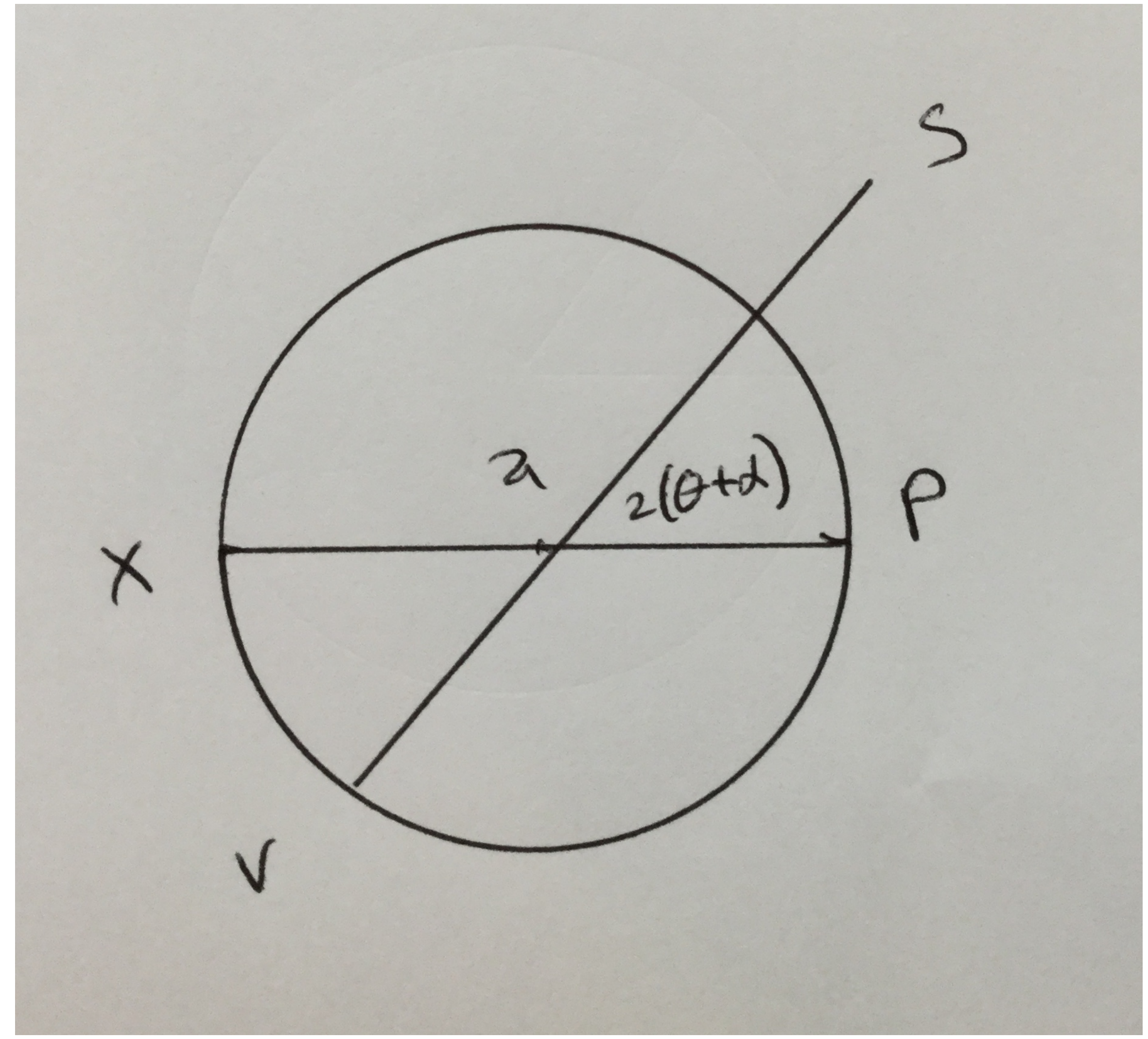
When:

$$\frac{SO^2}{SV^2} = \frac{Sj^2}{SV^2} = \frac{aS}{aV}$$

as drawn:

Figure 84:

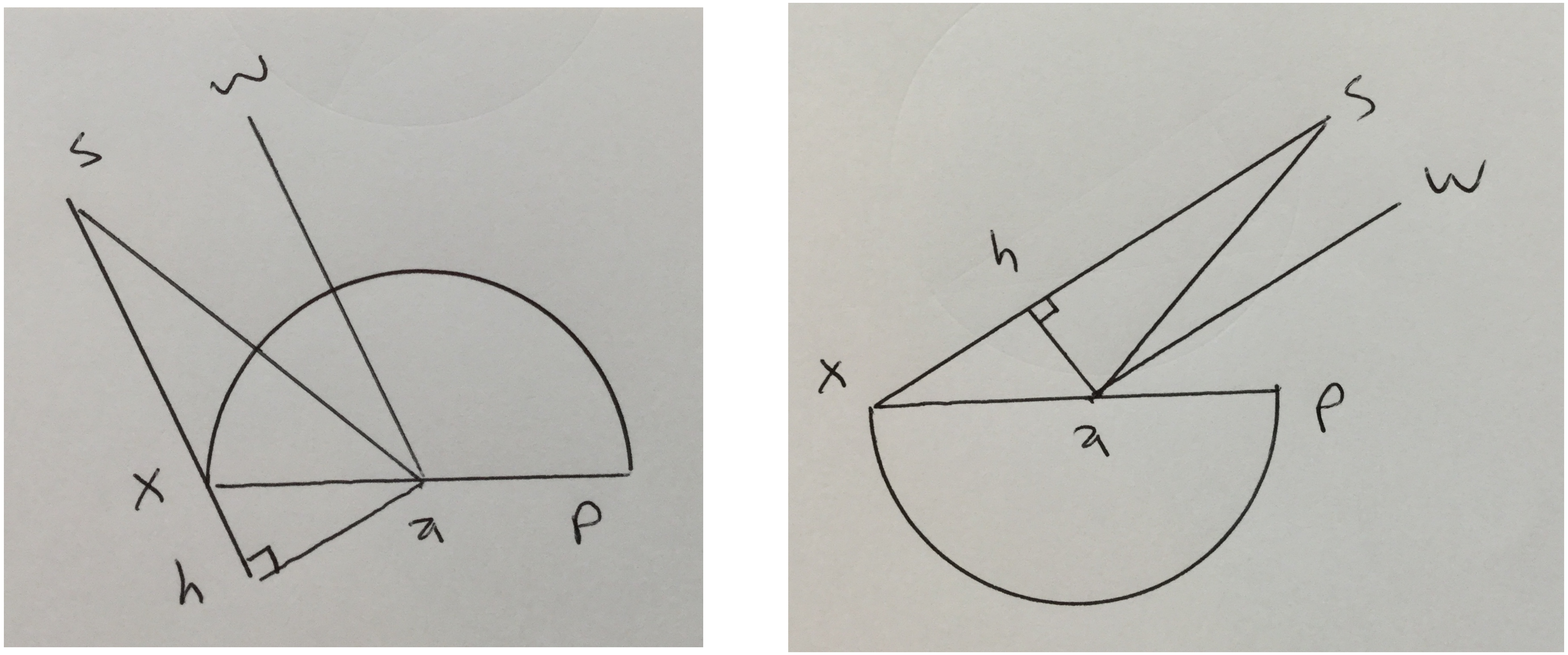




If we draw diameter **XaP** so:

aX = aV,   and   ∠SaP = 2 (θ + α)

Figure 85:



$$\frac{SO^2}{SV^2} = \frac{aS}{aX} = \frac{ah/aX}{ah/aS} = \frac{\sin 2\theta}{\sin 2\alpha}$$

When aw ∥ sX, we have divided the doubled angle

$2(\theta + \alpha) = \angle SaP$

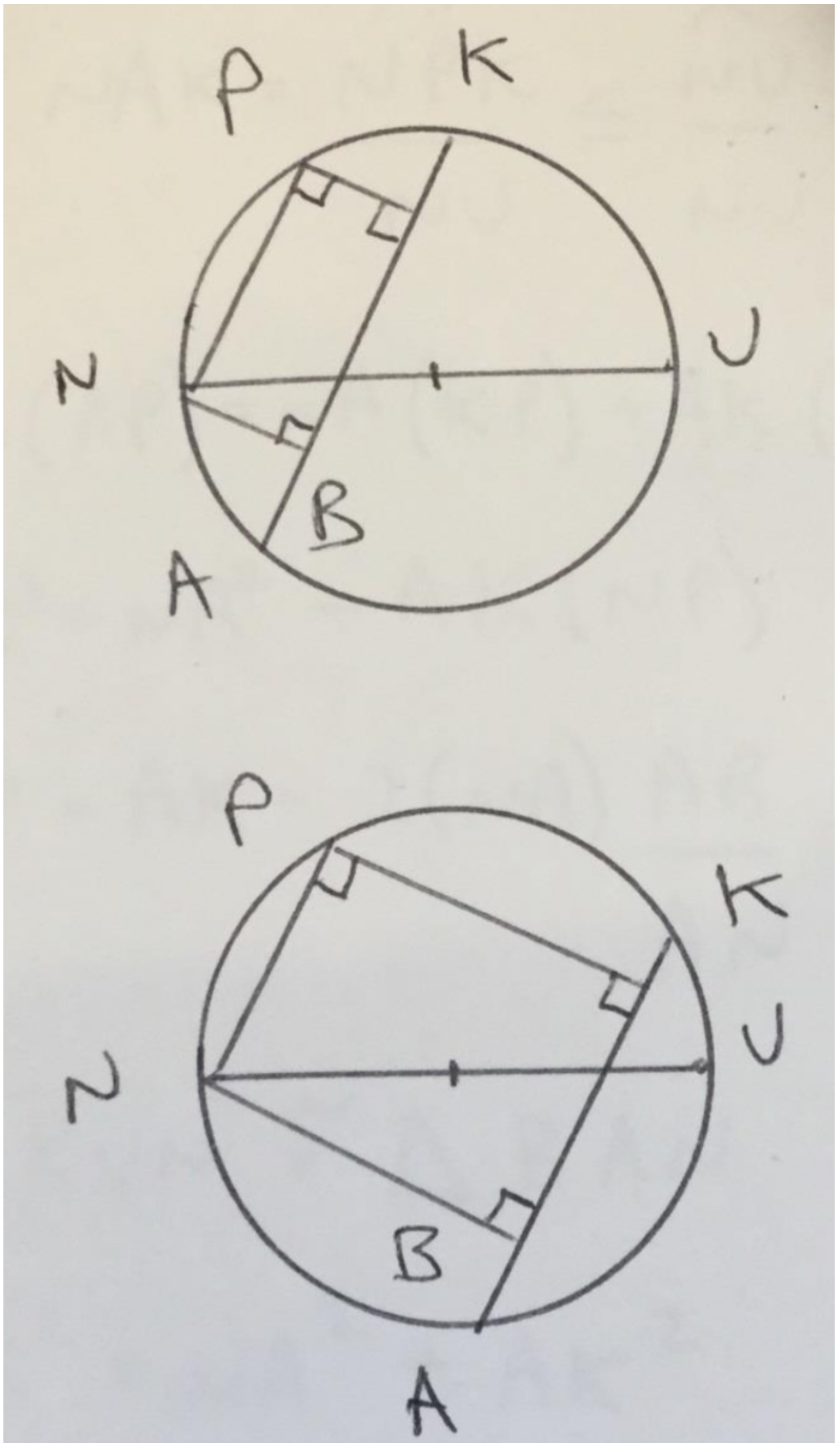
into $2\theta = \angle WaP$, and $2\alpha = \angle WaS$.

The approximate meridian of maximum refraction of two crossed spherical cylinders can be visualized by first examining the parabolic sagitta of each component cylinder in various cross meridians using the same sagittal depth **SB**. Although the meridian with the minimum sagittal sum does not represent the meridian of maximum refractive effect, a geometrical determination of that meridian can be determined once axial refractive power is expressed in terms of parabolic sagitta.

Appendix

The Law of Cosines approach to further illustrate that YN = KW in figures 9 - 13:

Figure 86:



$$AK \geq NP \parallel AK$$

$$\angle NAK = \sim\frac{NPK}{NU} \leq \sim\frac{NU}{NU} = \frac{\pi}{2}$$

$$NK.AP = NA.KP + AK.NP$$

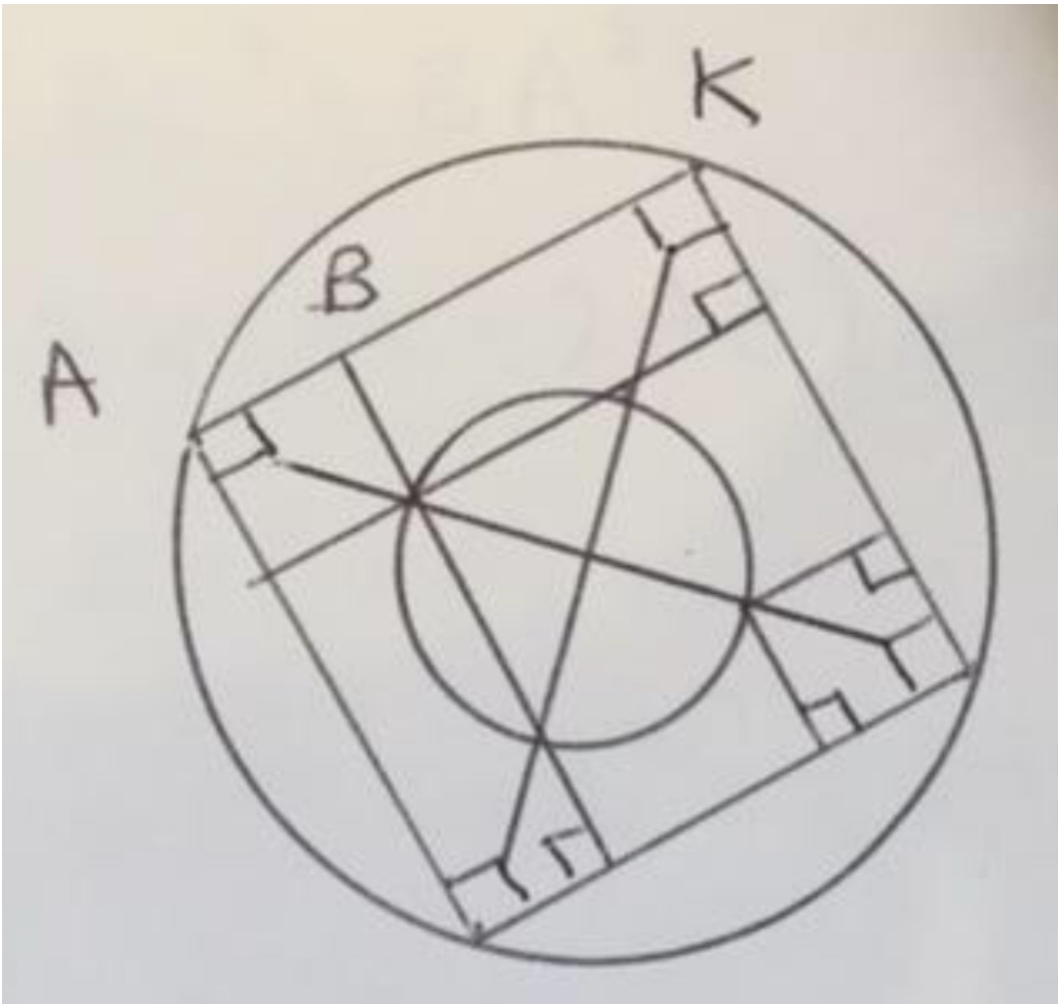
$$NK^2 = NA^2 + AK.NP$$

$$NP = AK - 2(NA)\frac{AB}{AN}$$

$$\Delta KUN \cong \Delta BAN$$

$$NK^2 = NA^2 + AK^2 - 2(AK)NA.\frac{UK}{UN}$$

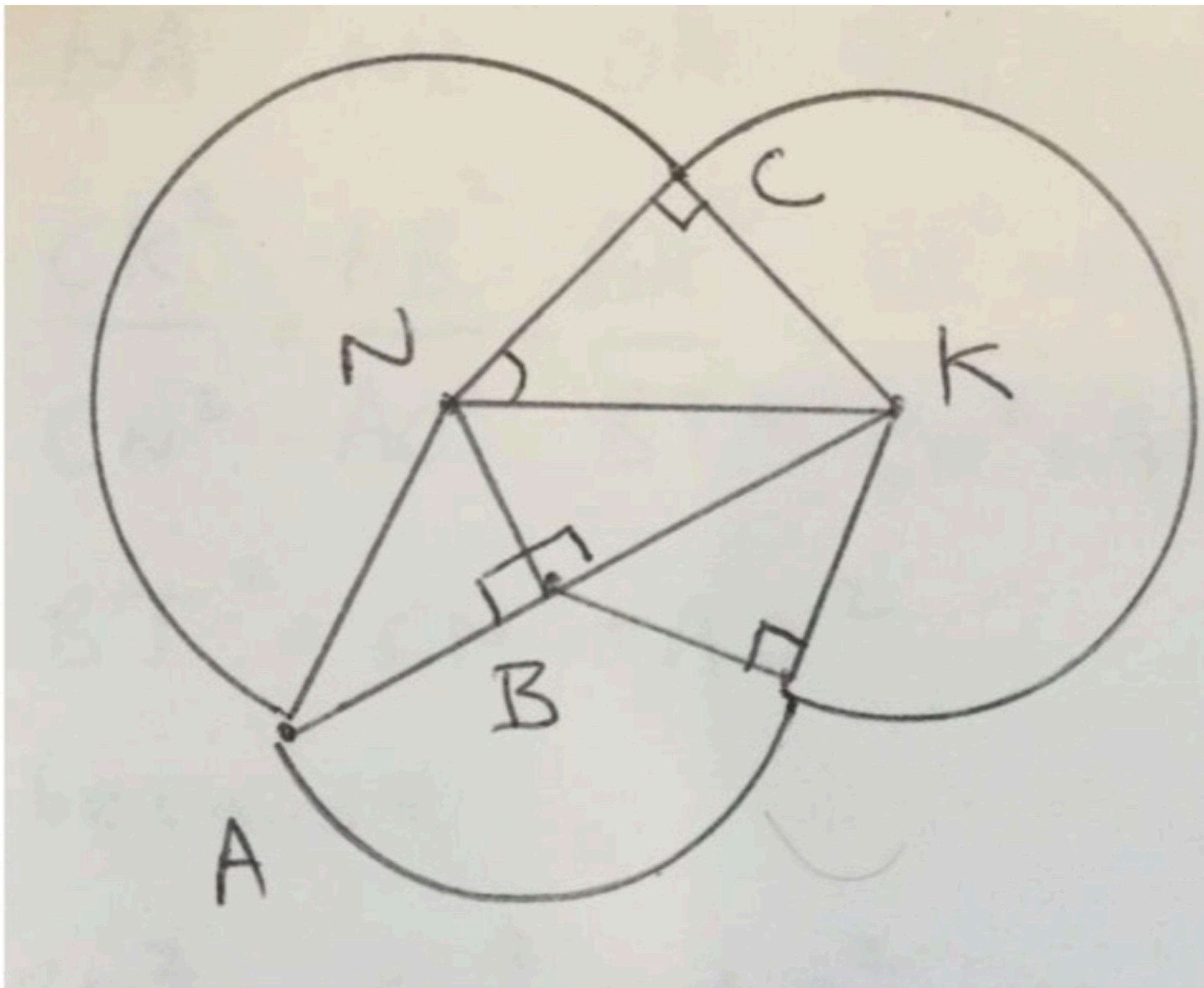
Figure 87:



$$BK^2 - BA^2 = AK^2 - 2(AK)AB =$$

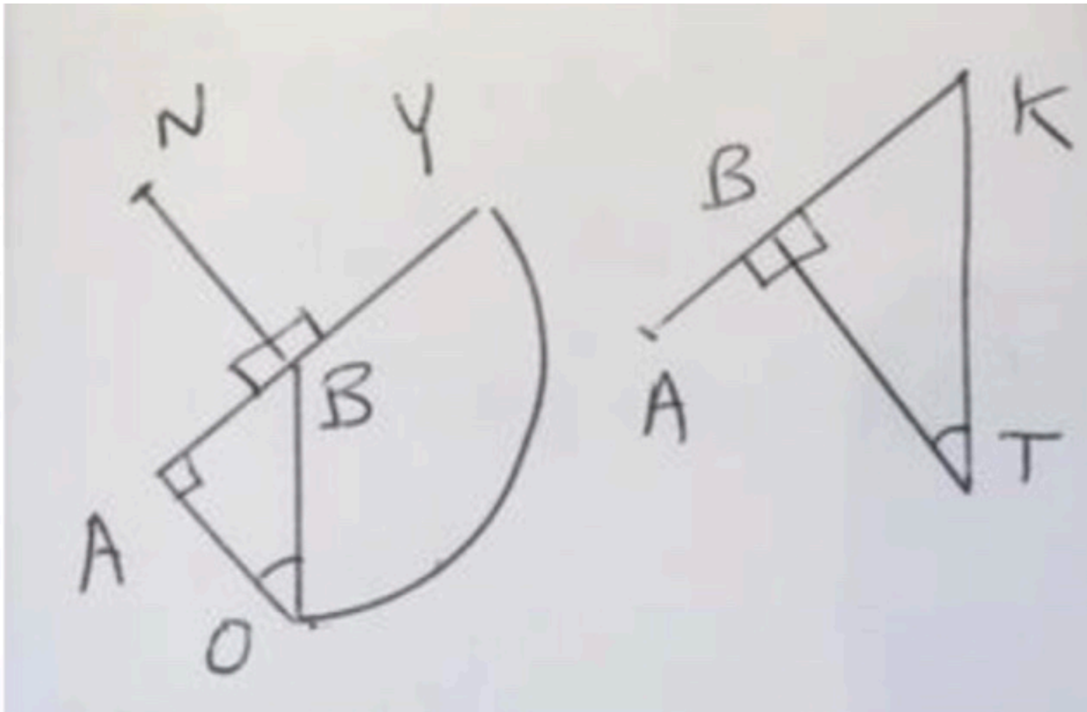
$$AK.NP = NK^2 - NA^2$$

Figure 88:



$BK^2 - BA^2 = NK^2 - NA^2 = CK^2$

Figure 89:



$$\frac{NK}{NA} = \frac{NK}{NC} = \frac{OB}{OA} = \frac{TK}{TB}$$

$$\frac{CK^2}{CN^2} = \frac{AB^2}{AO^2} = \frac{BK^2}{BT^2} = \frac{CK^2 + AB^2}{CN^2 + AO^2}$$

because:

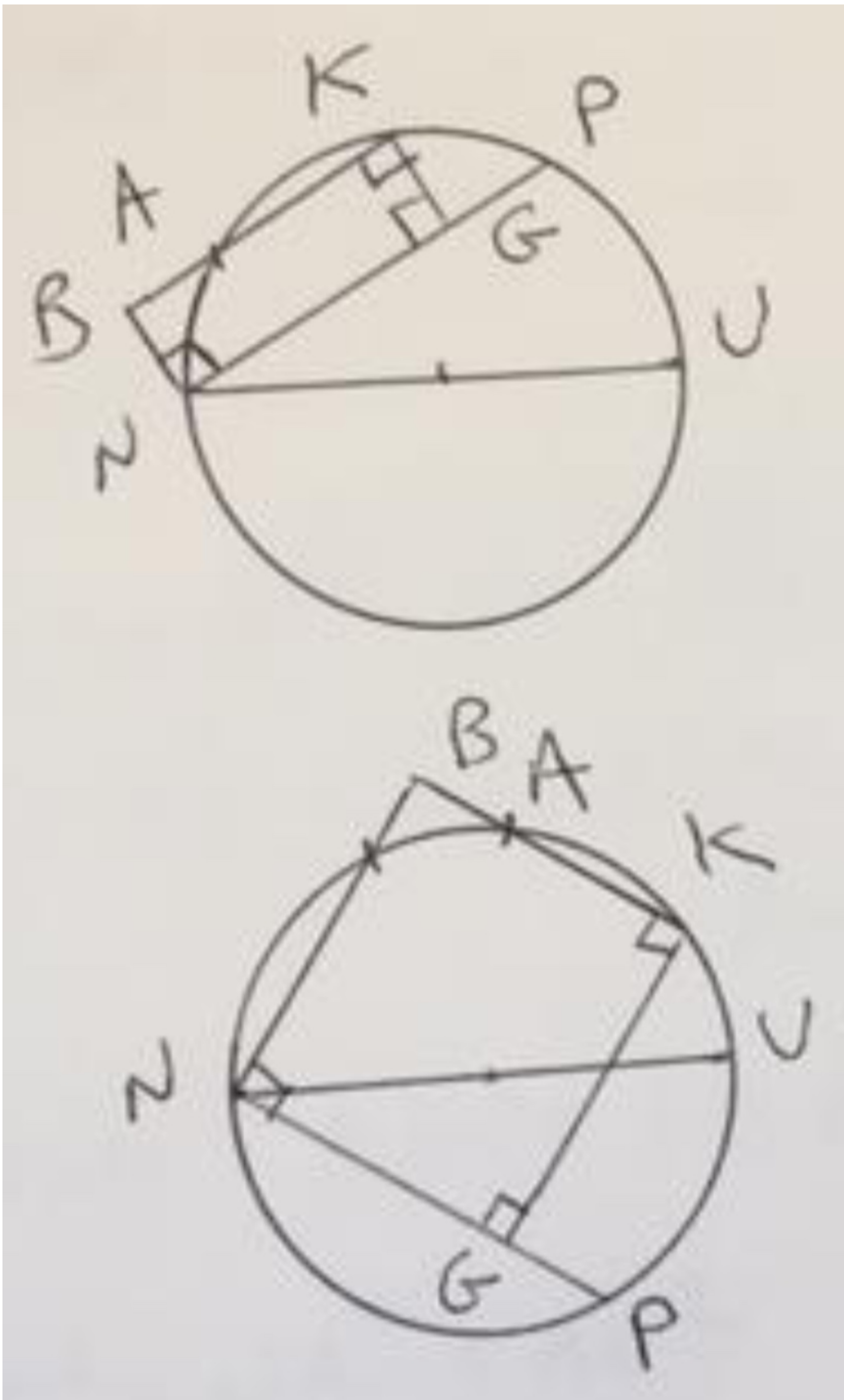
$$BK^2 = CK^2 + AB^2$$

$$\begin{aligned} BT^2 &= CN^2 + AO^2 \\ &= AN^2 + AO^2 \\ &= BN^2 + AB^2 + BO^2 - AB^2 \\ &= NY^2 \end{aligned}$$

$$BT = NY$$

given ΔBAO
use ΔKBT to find ΔYBN
and use ΔYBN to find ΔKBT

Figure 90:



$$NP \geq AK \parallel NP$$

$$\angle NAK = \frac{\sim NUK}{NU} \geq \frac{\sim NU}{NU} = \frac{\pi}{2}$$

$$NK.AP = NA.KP + AK.NP$$

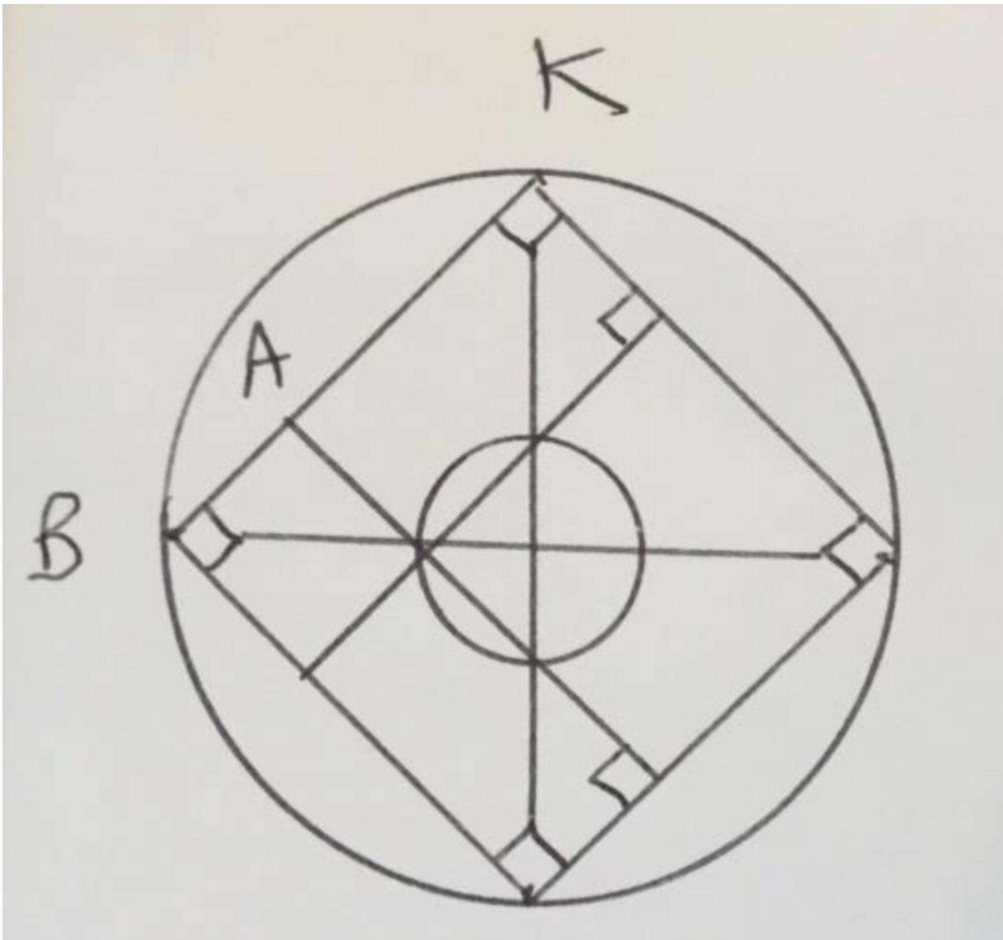
$$NK^2 = NA^2 + AK.NP$$

$$NP = AK + 2(NA)\frac{AB}{AN}$$

$$\Delta KUN \cong \Delta GPK \cong \Delta BAN$$

$$NK^2 = NA^2 + AK^2 + 2(AK)NA.\frac{UK}{UN}$$

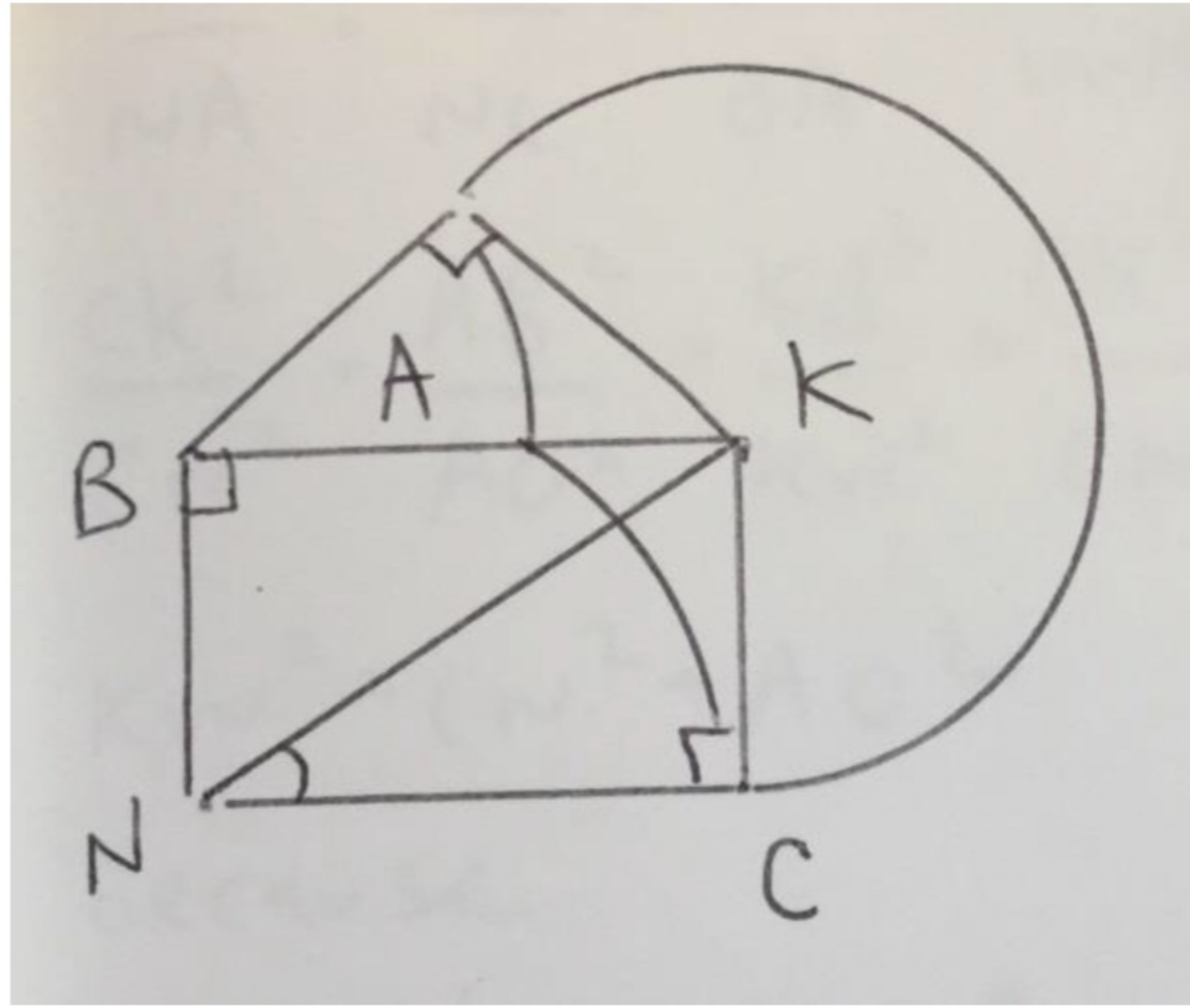
Figure 91:



$$BK^2 - BA^2 = AK^2 + 2(AK)AB =$$

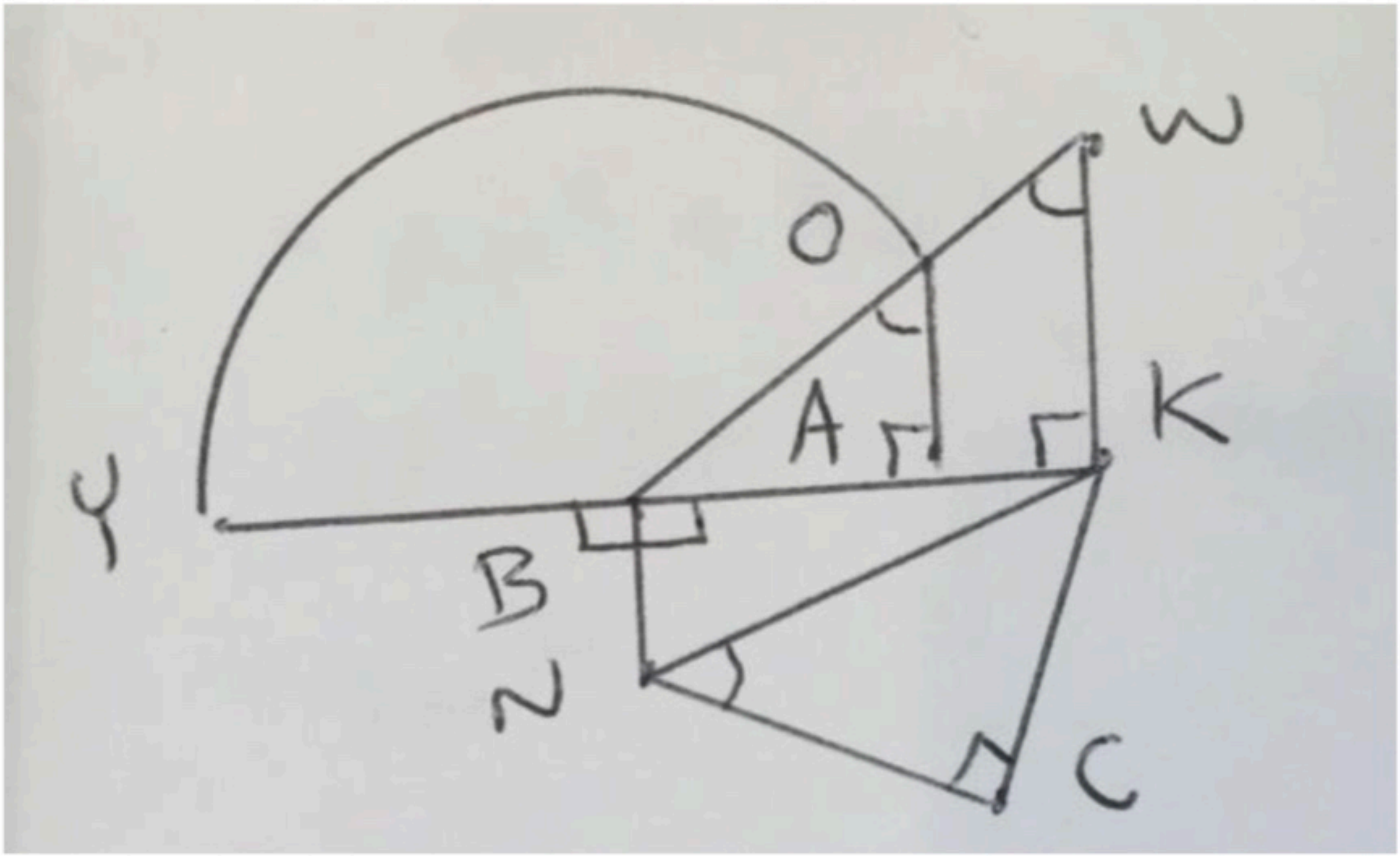
$$AK.NP = NK^2 - NA^2$$

Figure 92:



$$BK^2 - BA^2 = NK^2 - NA^2 = CK^2$$

Figure 93:



$$\frac{NK}{NA} = \frac{NK}{NC} = \frac{OB}{OA} = \frac{WB}{WK}$$

$$\frac{CK^2}{CN^2} = \frac{AB^2}{AO^2} = \frac{KB^2}{KW^2} = \frac{CK^2 + AB^2}{CN^2 + AO^2}$$

because:

$$KB^2 = CK^2 + AB^2$$

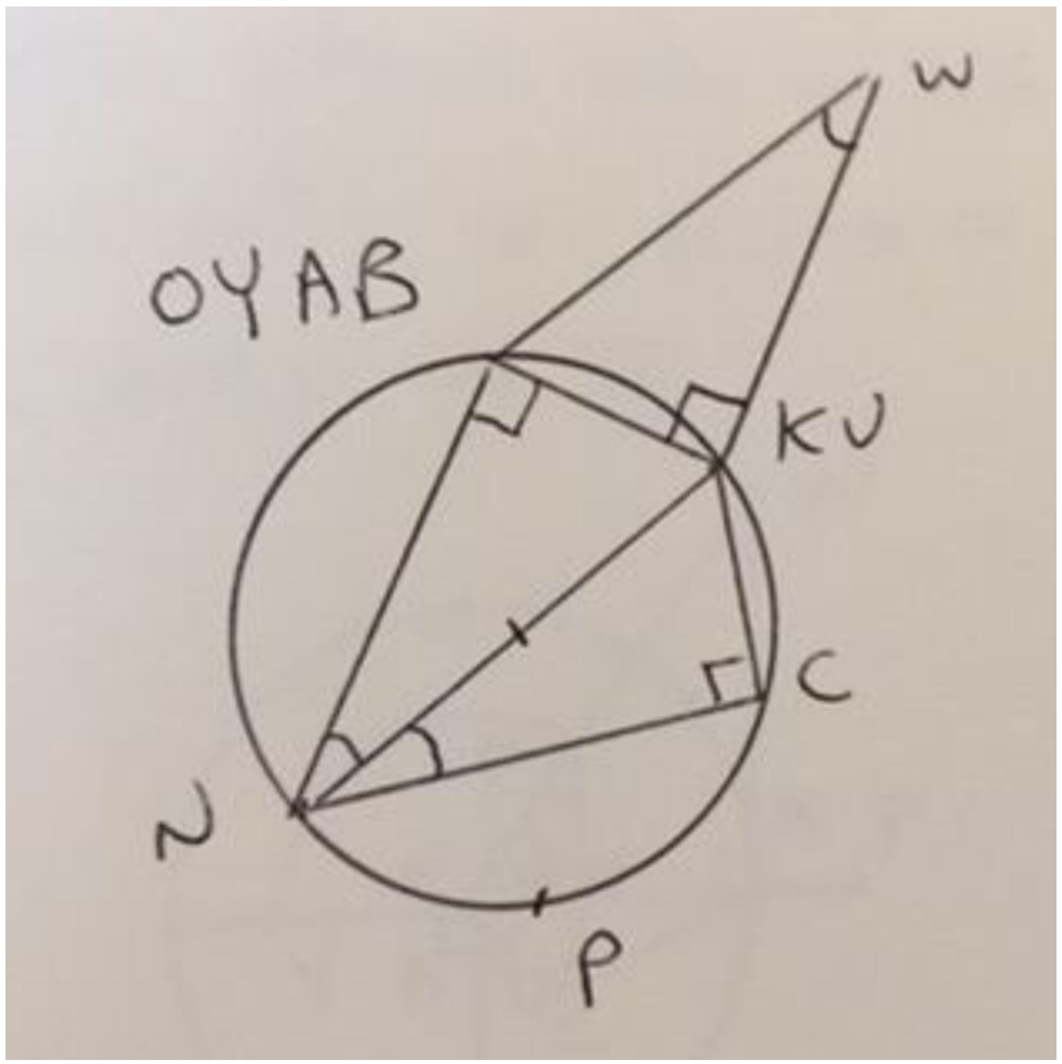
$$\begin{aligned} KW^2 &= CN^2 + AO^2 \\ &= AN^2 + AO^2 \\ &= BA^2 + BN^2 + BO^2 - BA^2 \\ &= YN^2 \end{aligned}$$

$$KW = YN$$

given ΔBAO
use ΔBKW to find ΔYBN
and use ΔYBN to find ΔBKW

Figure 94:

With NK constant:



let circle NPKA shrink
and rotate counter-clockwise around N
so that:

$$U \Rightarrow K, \text{ and } \angle NAK \Rightarrow \angle NBK = \frac{\pi}{2}$$

or, with NA constant
let circle NPKA expand
and rotate clockwise
around N
so that:

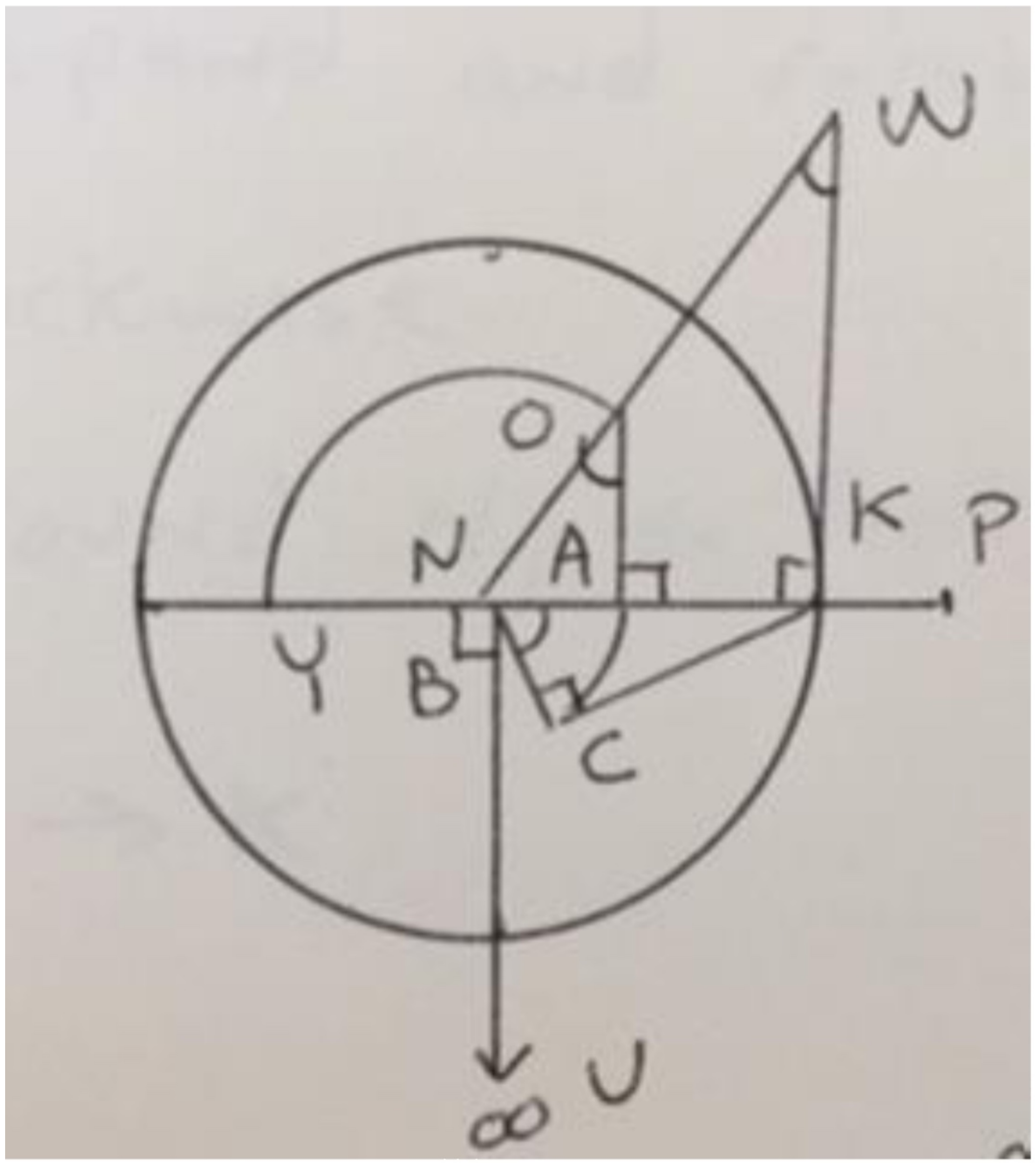
$$K \Rightarrow U, \text{ and } \angle NAK \Rightarrow \angle NBK = \frac{\pi}{2}$$

$$\frac{NK}{NA} = \frac{NK}{NC} = \frac{WB}{WK}$$

$$KW = YN$$

Figure 95:

With either NK or NA constant:



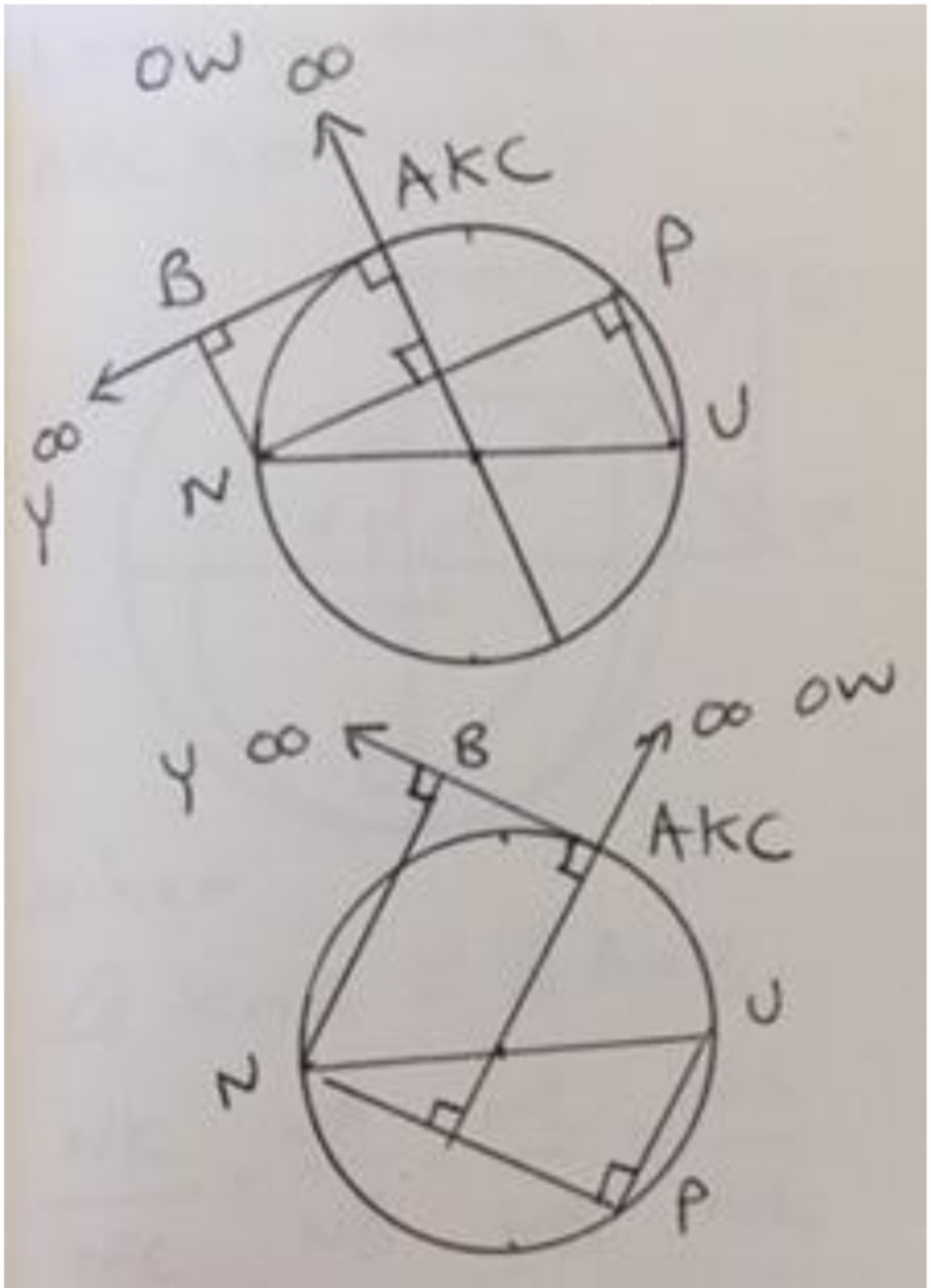
as NU $\Rightarrow \infty$

$\angle$NAK $\Rightarrow \pi$

$$\frac{(KW)}{(OA)} = \frac{NK}{NA} = \frac{NK}{NC} = \frac{OB}{OA} = \frac{WB}{WK}$$

KW (=OB) = YN

Figure 96:



with NK constant
let circle NPKA expand and
rotate clockwise around N
so that:
A⇒K

or, with NA constant
let circle NPKA shrink
and rotate counter-
clockwise around N so that:
K⇒A

$$\frac{NK}{NA} = \frac{NK}{NC} = \frac{OB}{OA} = \frac{WB}{WK}$$

$$KW = YN$$