Axial Magnification

Plane Geometry Approach

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2021

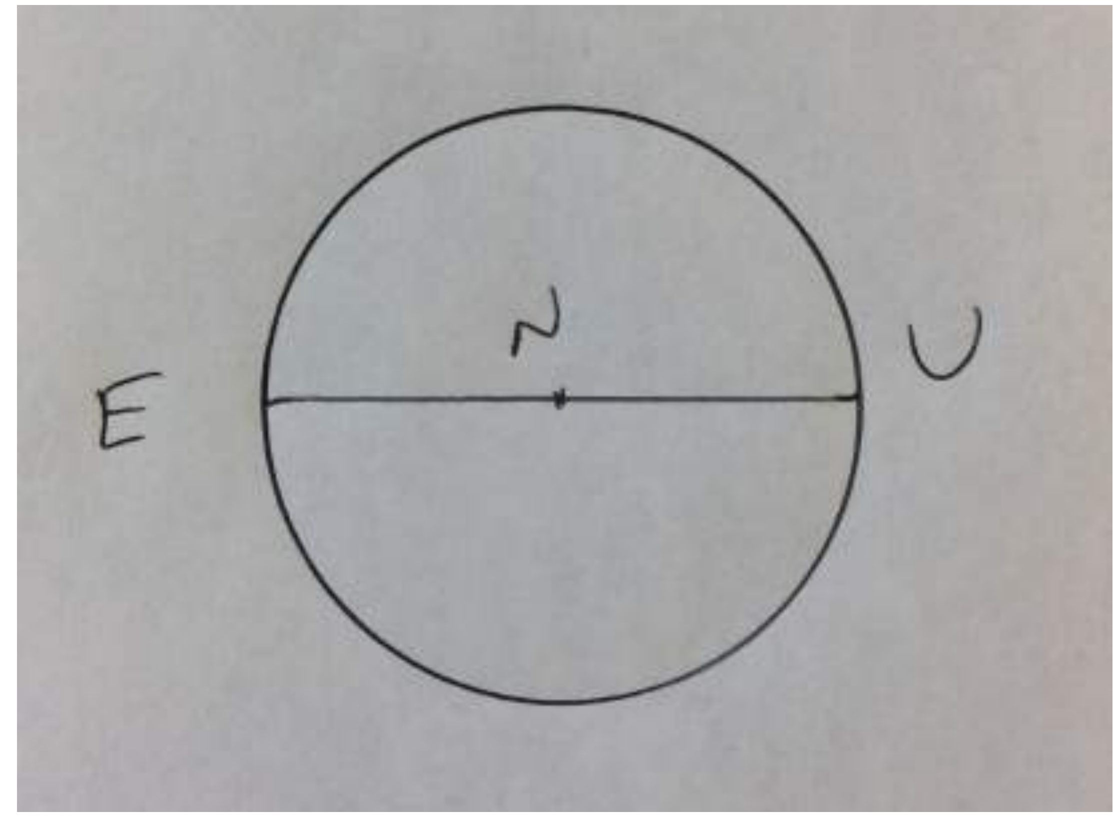
Dedicated to my Geometrical Optics professor, William Brown, OD, PhD, who always taught the geometry first.

Reference:
Isaac Barrows Optical Lectures, 1667
Translated by H.C. Fay
Edited by A.G. Bennett
Publisher:
The Worshipful Company of Spectacle Makers
London, England; 1987
ISBN # 0-951-2217-0-1

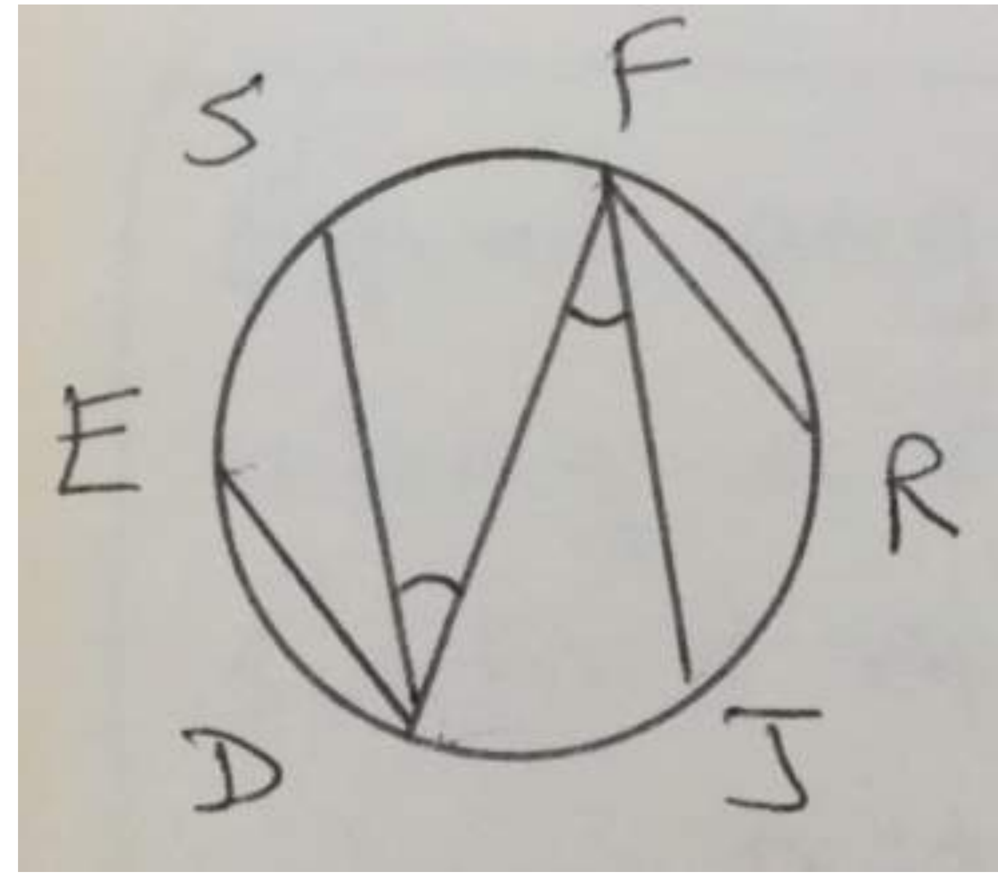
Contents

1). prerequisite geometry

On a circle with diameter EU and center N:



Two equal arcs ~SE and ~JR can be shown to subtend equal angles by drawing any two parallel lines SD and JF. Since parallel lines intercept equal arcs across a circle,



~SF = ~JD

~SE + ~SF = ~JR + ~JD

~EF = ~RD

ED || RF, and therefore:

∠SDE = ∠JFR

Since conversely, equal angles along a circle subtend equal arcs, any angle along any circle can be defined in terms of its subtended arc and the circle's diameter.

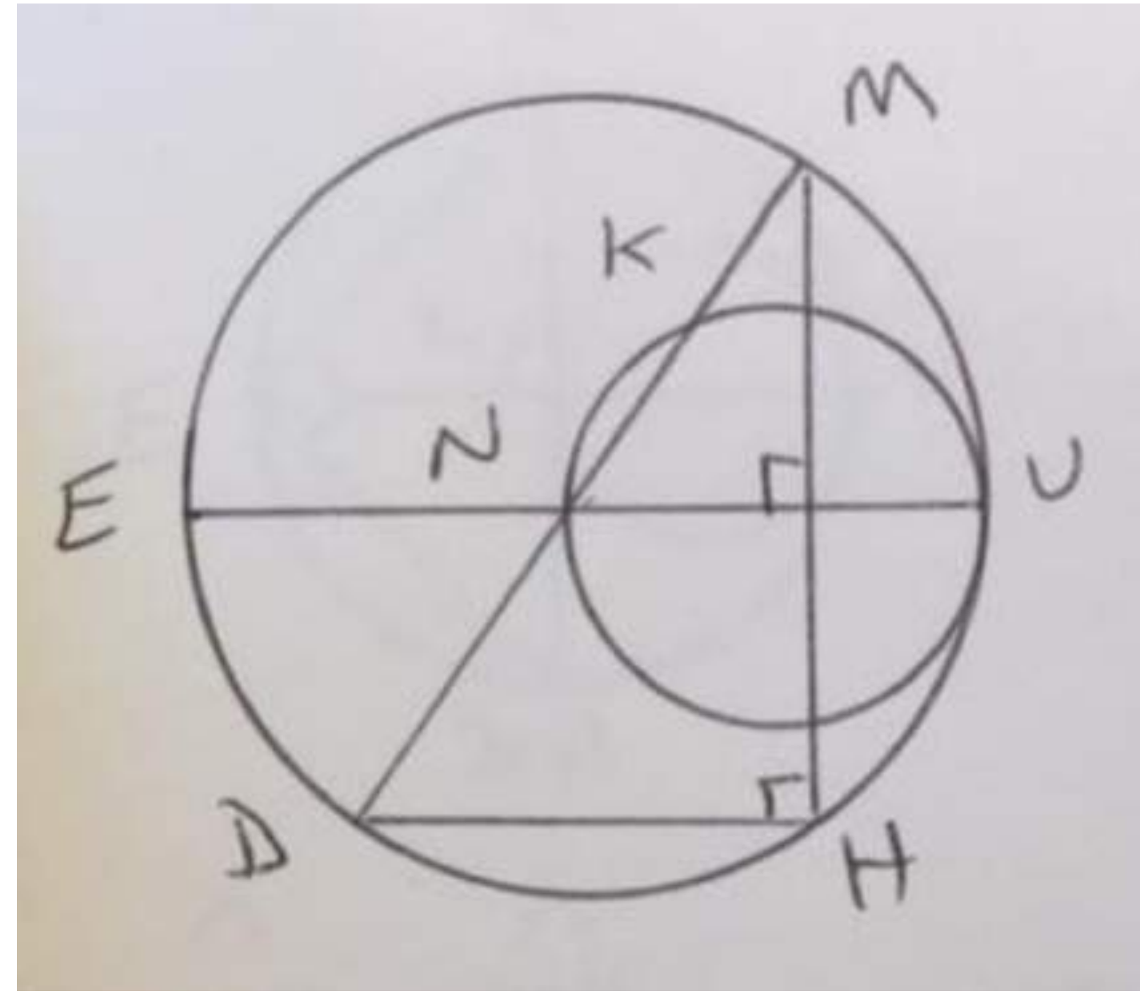
For example: ∠RFJ = ~RJ/EU

∠KNU = ∠MDH

∠MDH = ~MH/MD

= ~MH/UE = 2(~UM)/UE

= 2∠MEU



∠KNU = ~UK/UN

= 2(~UM)/2(UN)

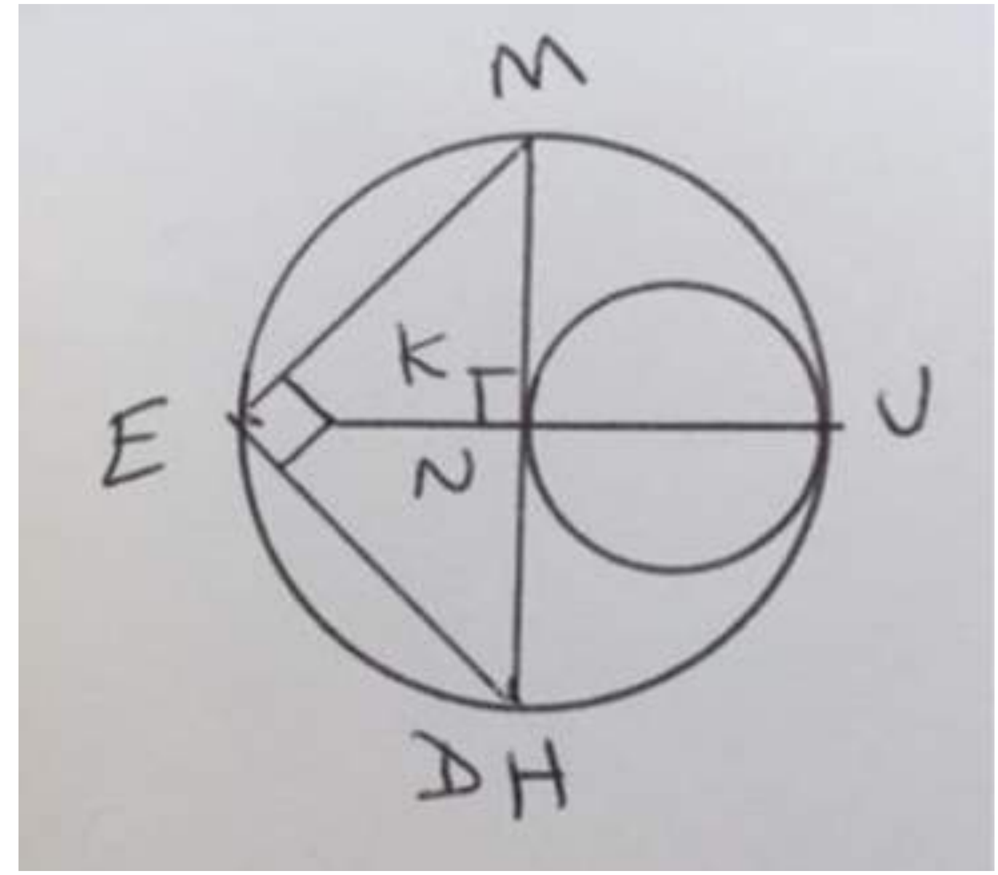
~UK = ~UM

Let K ⇒ N and D ⇒ H:

~UK/UN = ~MH/MD
= ~MH/UE = ∠MEH

~UK/UN = ∠MNU

2(~UK)/UN = ∠MNH = π



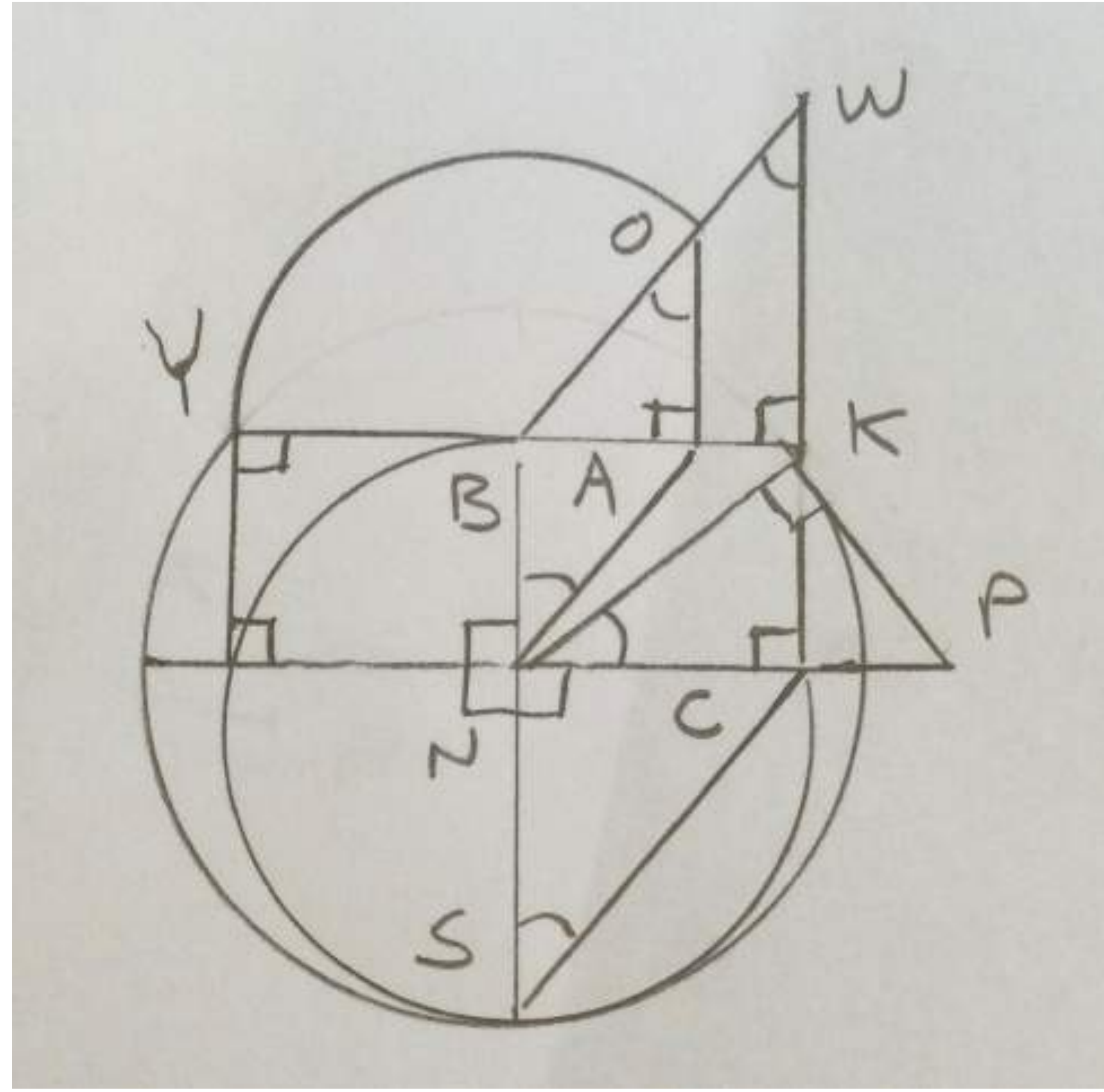
NS/NC = NC/NB
NK/NC = CN/CK

ΔNSC = ΔKWB = ΔKNP
NC = KP

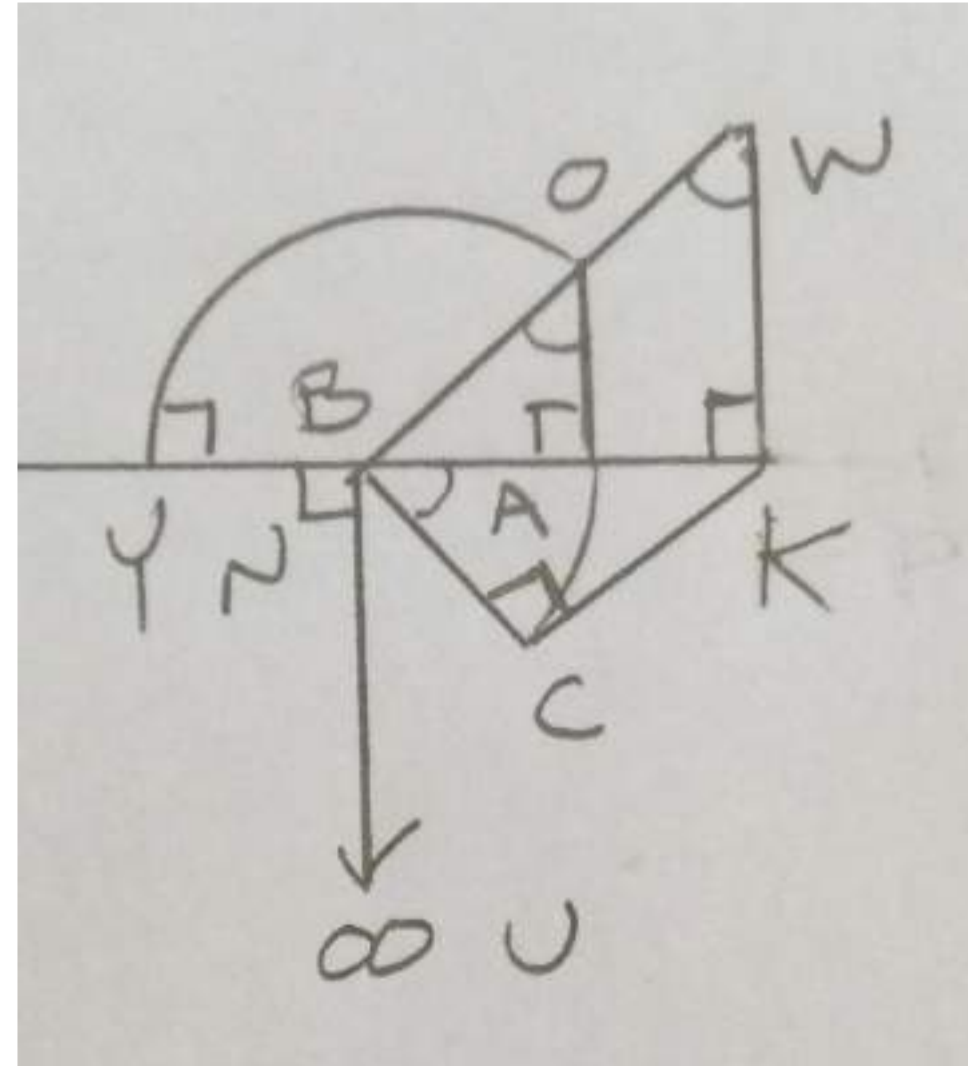
ΔCKP = ΔBNA = ΔAOB
NA = KP

NC = NA = OB
NC = KB = YB

WK = NS **= YN**



Keeping only:
NA = NC, and
ΔCNK ≅ ΔAOB ≅ ΔKWB:



As N ⇒ B, WK ⇒ YN

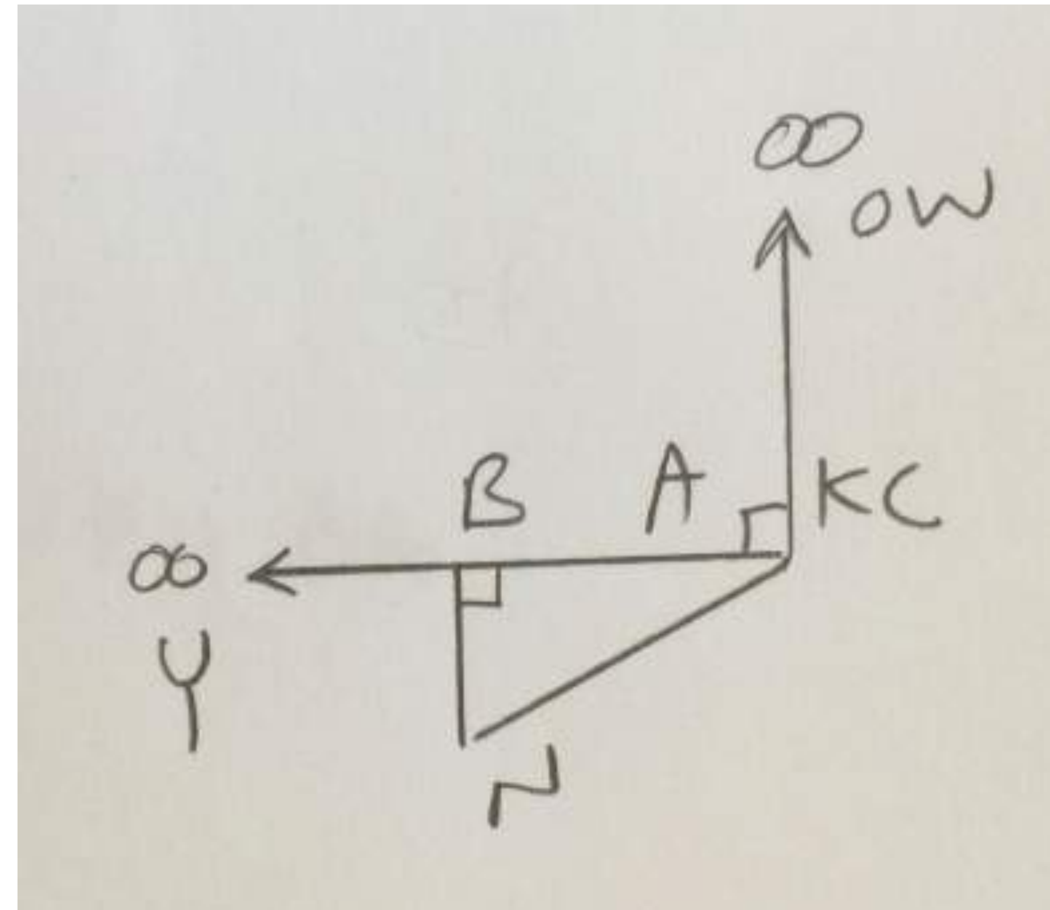
because:
WK/OA ⇒ NK/NA = NK/NC

= OB/OA = WB/WK

so that:
WK ⇒ OB ⇒ YN

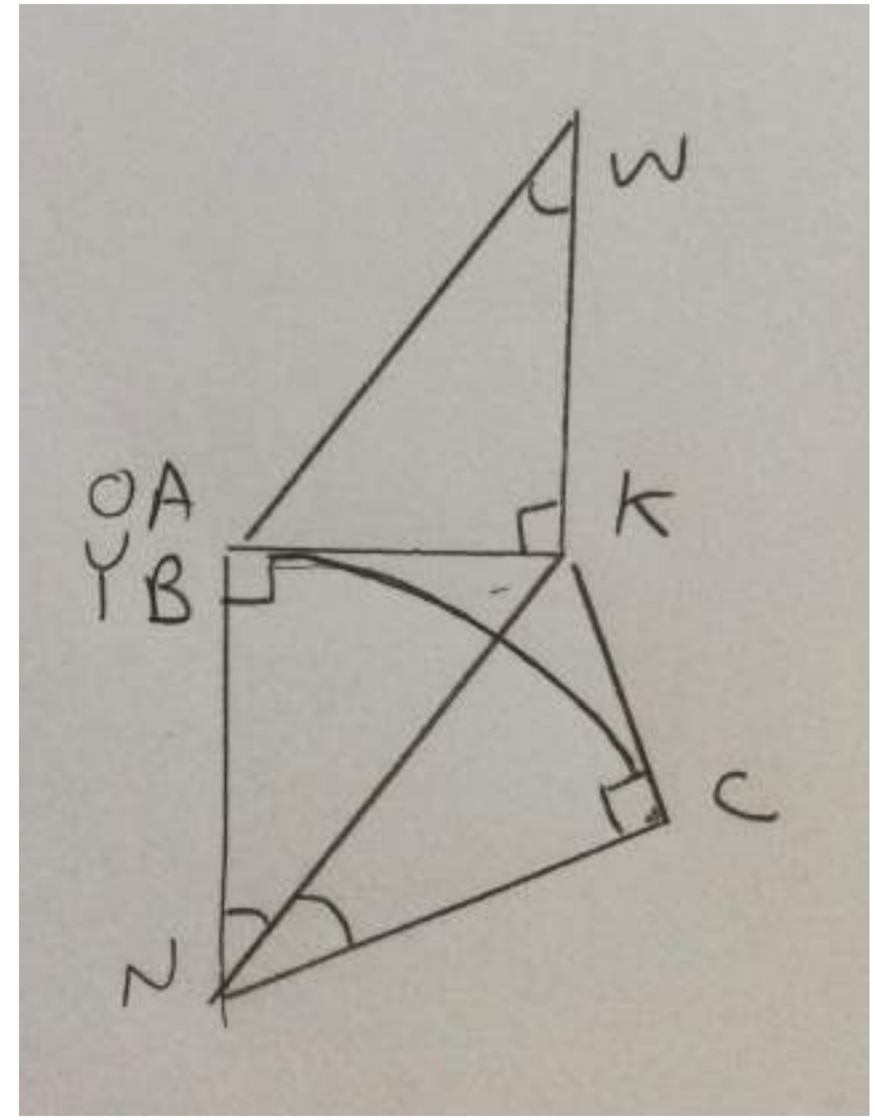
Keeping only:
NA = NC, and
ΔCNK ≅ ΔAOB ≅ ΔKWB:

As A ⇒ K, WK ⇒ YN

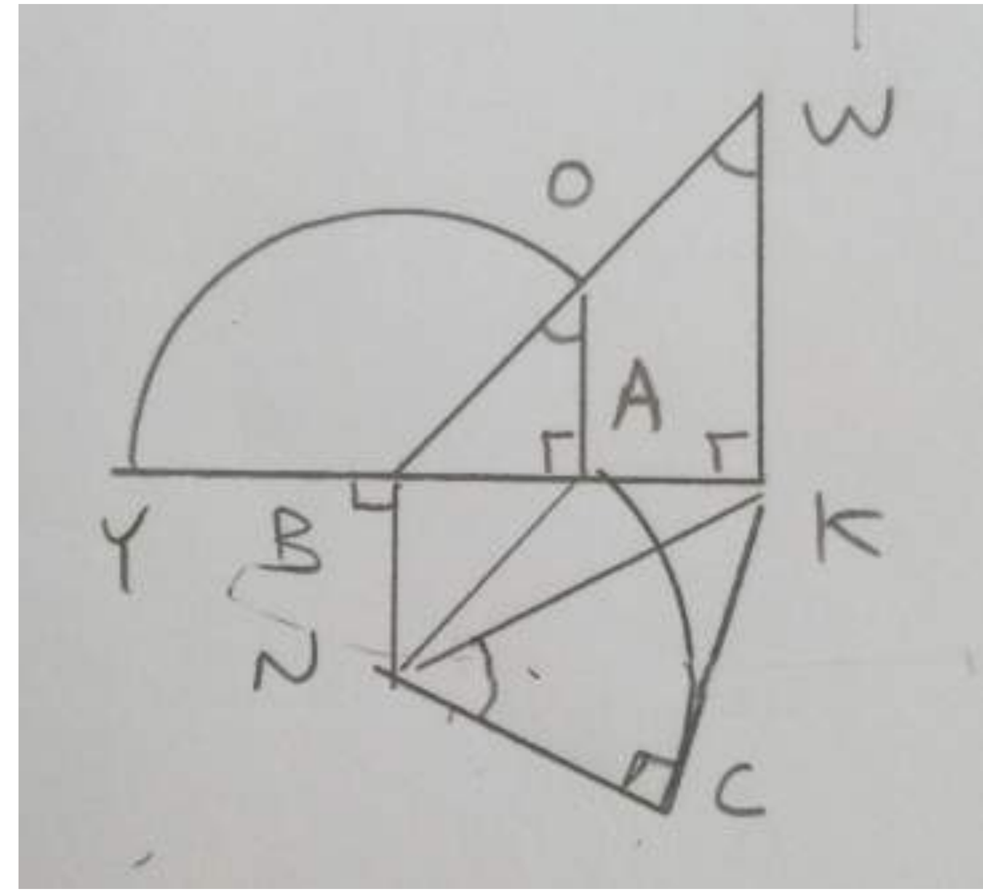


Keeping only:
NA = NC, and
ΔCNK ≅ ΔAOB ≅ ΔKWB:

As A ⇒ B, WK ⇒ YN



We can therefore assume that whenever A lies on BK, given right triangle KBN, if NA = NC, and ΔCNK ≅ ΔAOB ≅ ΔKWB as shown, then:

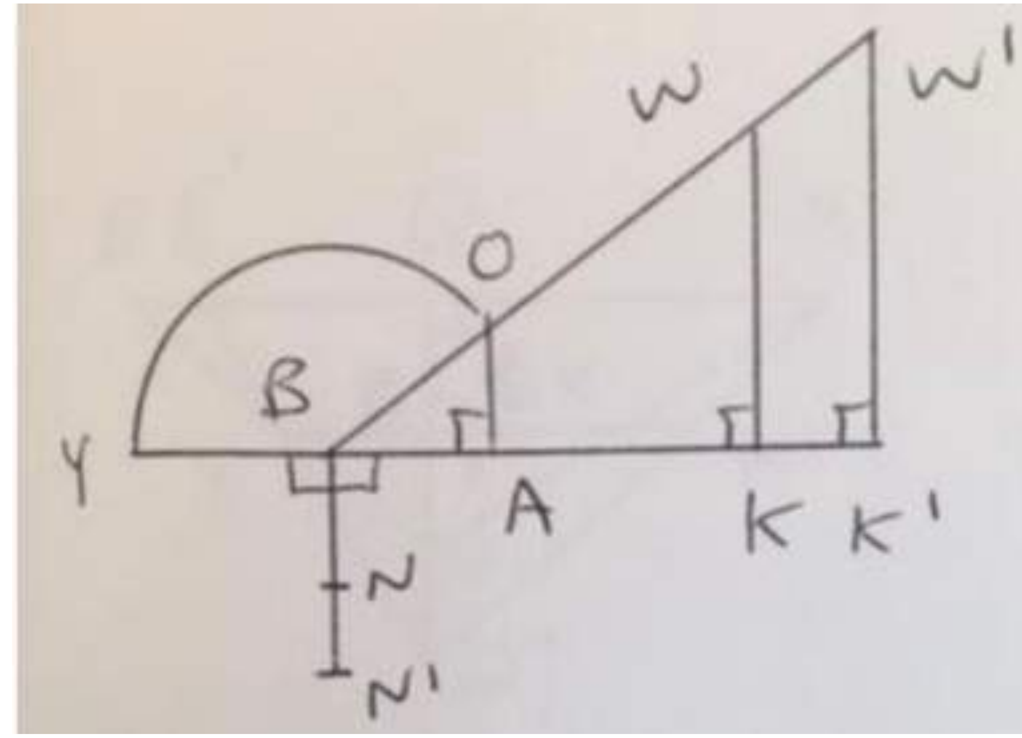


WK = YN

OB/OA = NK/NA
= N'K'/N'A

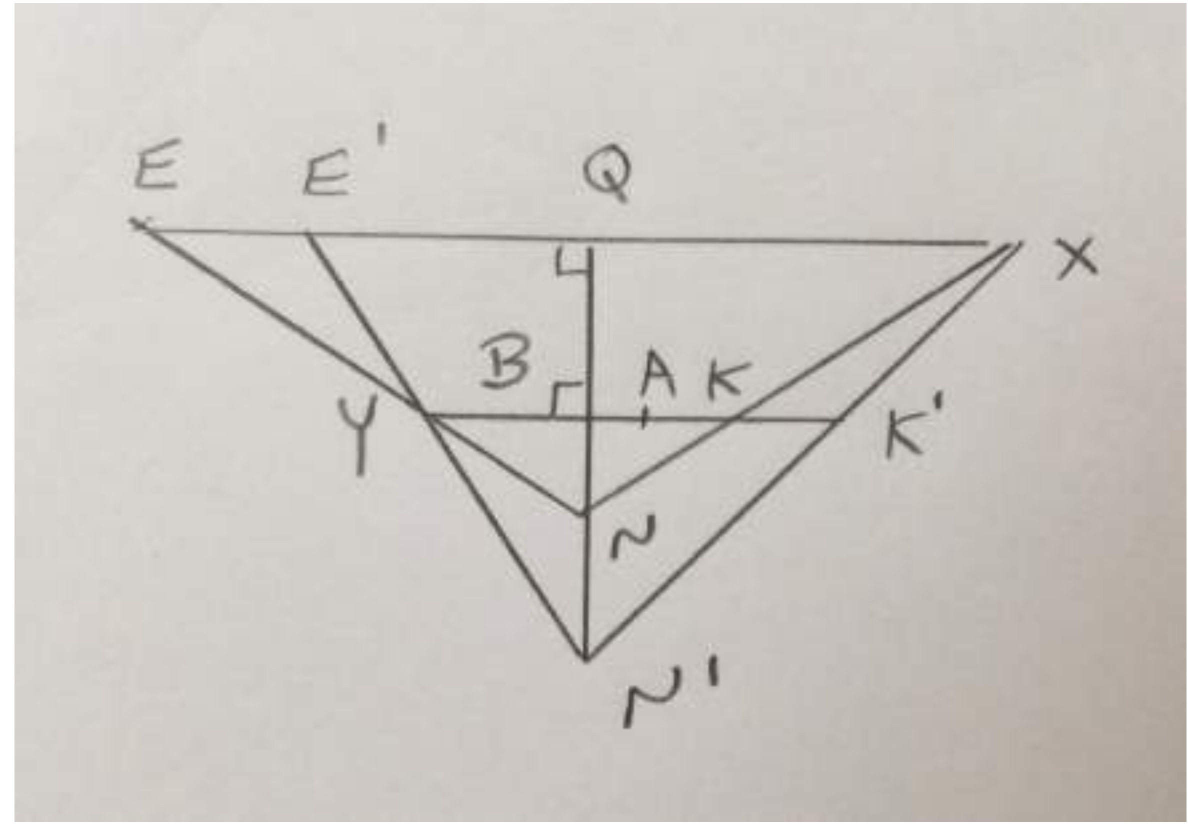
KW = YN
K'W' = YN'

KB/YN = K'B/YN'



QX/EN = KB/YN
= K’B/YN’ = QX/E’N’

EN = E’N’

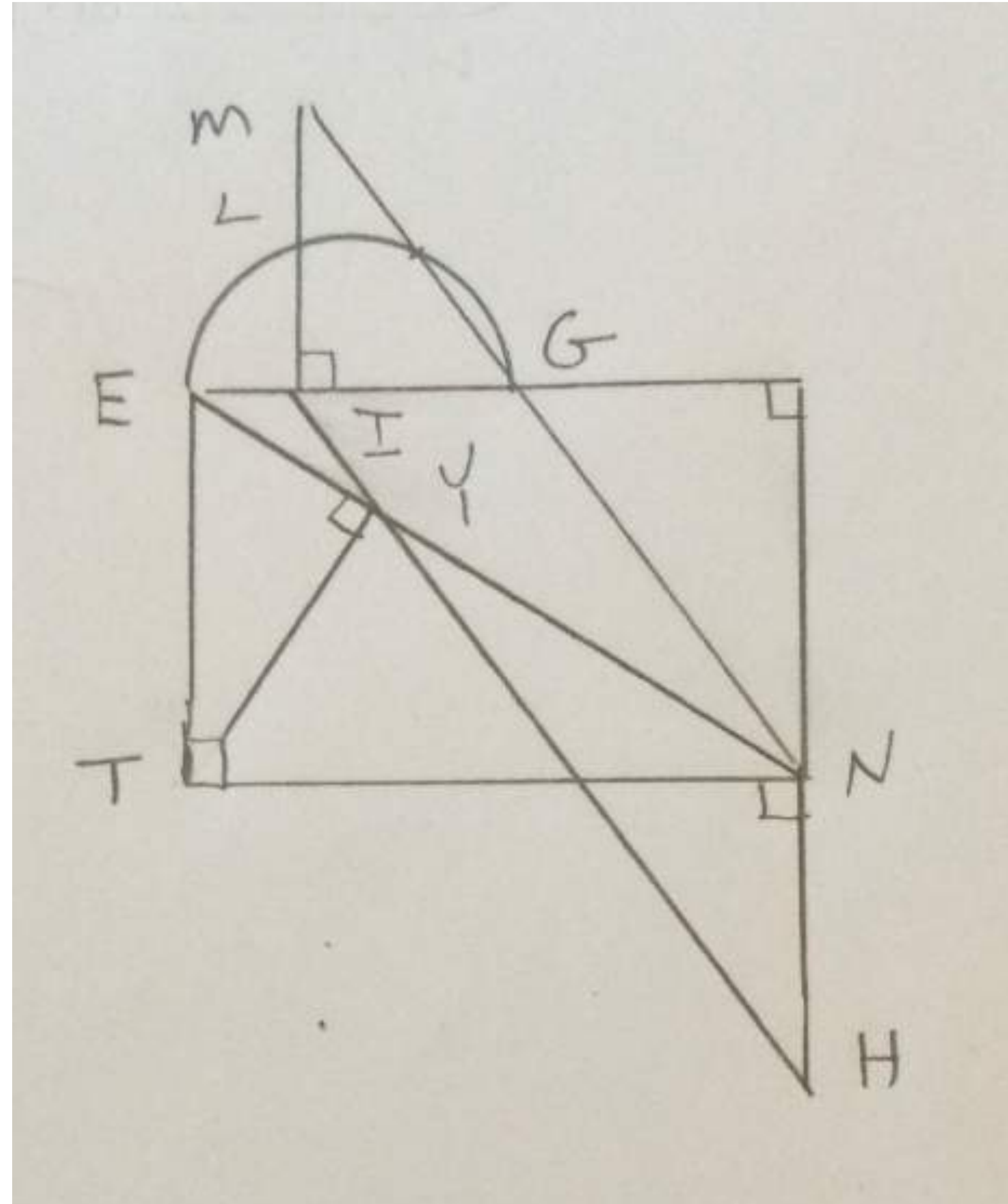


Only one N’K’X exists for NKX since only one E’N’ exists equal to EN. When EN is the smallest segment through Y included in the right angle EQN, E’ lies at E, and N’ lies at N.

NE || GL
TY || EL
HI || NM
HI = NM
NM > NL

NL is the hypotenuse of right triangle NEL

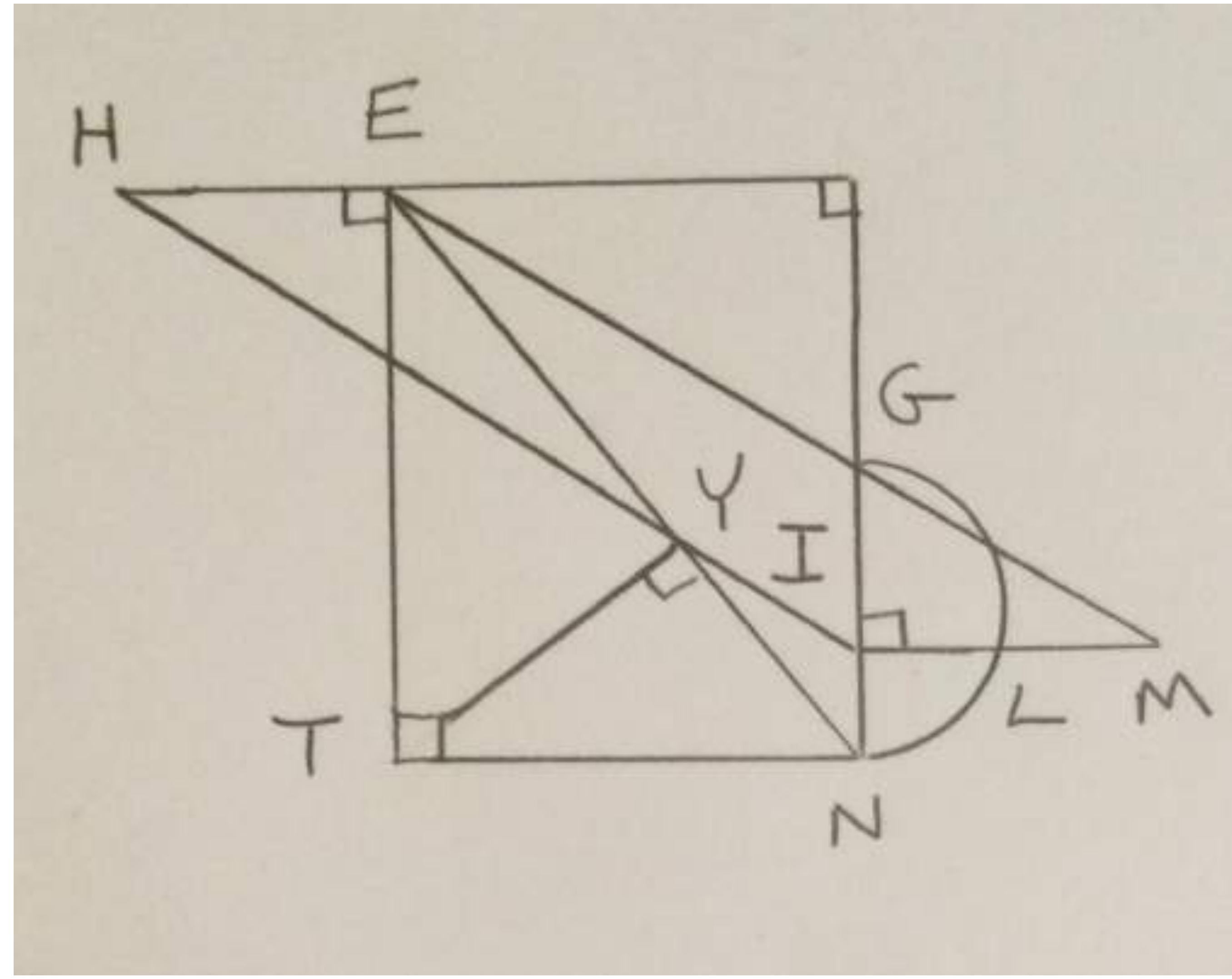
NL > NE
HI > NE



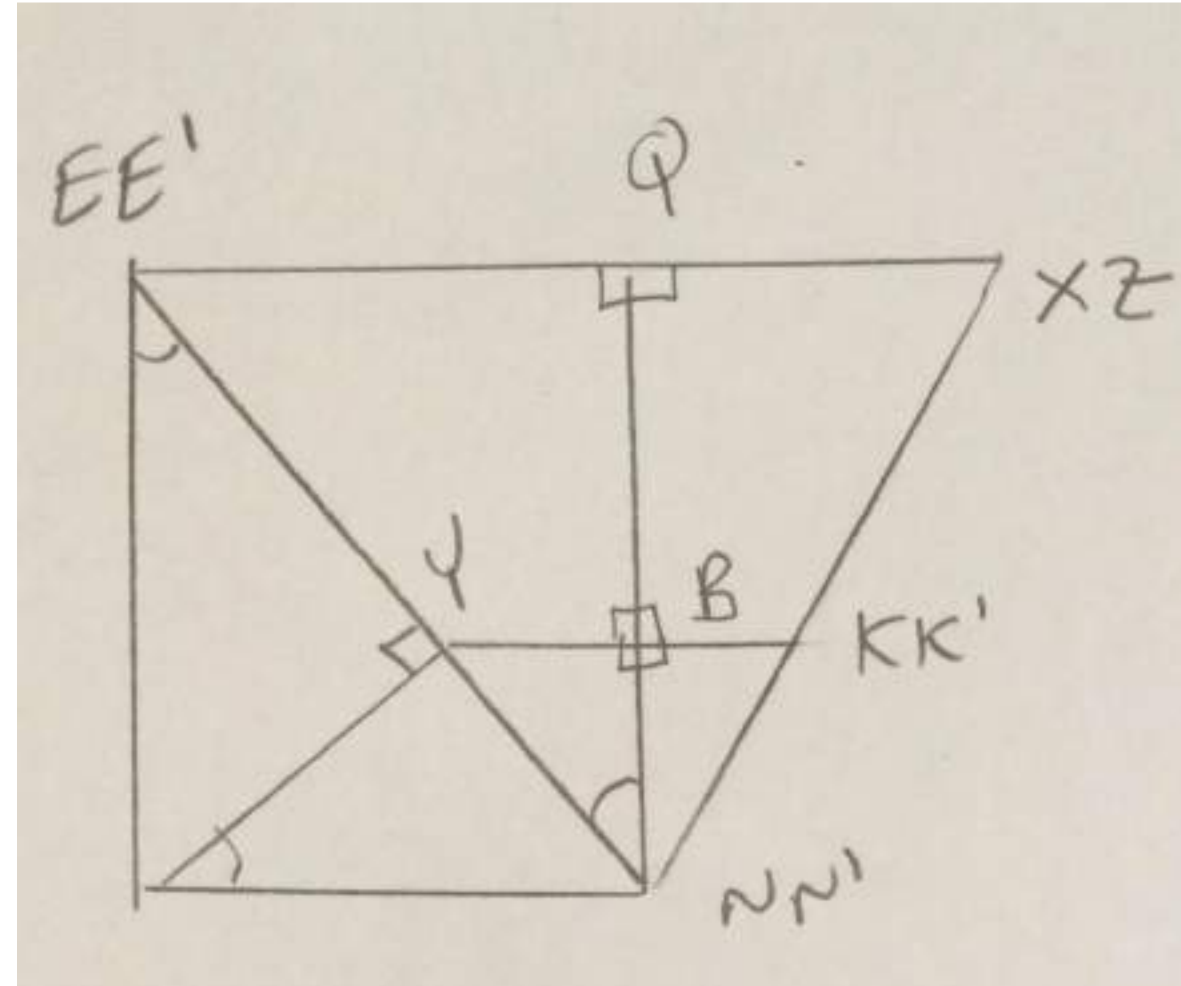
NE || GL
TY || NL
HI || EM
HI = EM
EM > EL

EL is the hypotenuse of right triangle ENL

EL > EN
HI > EN

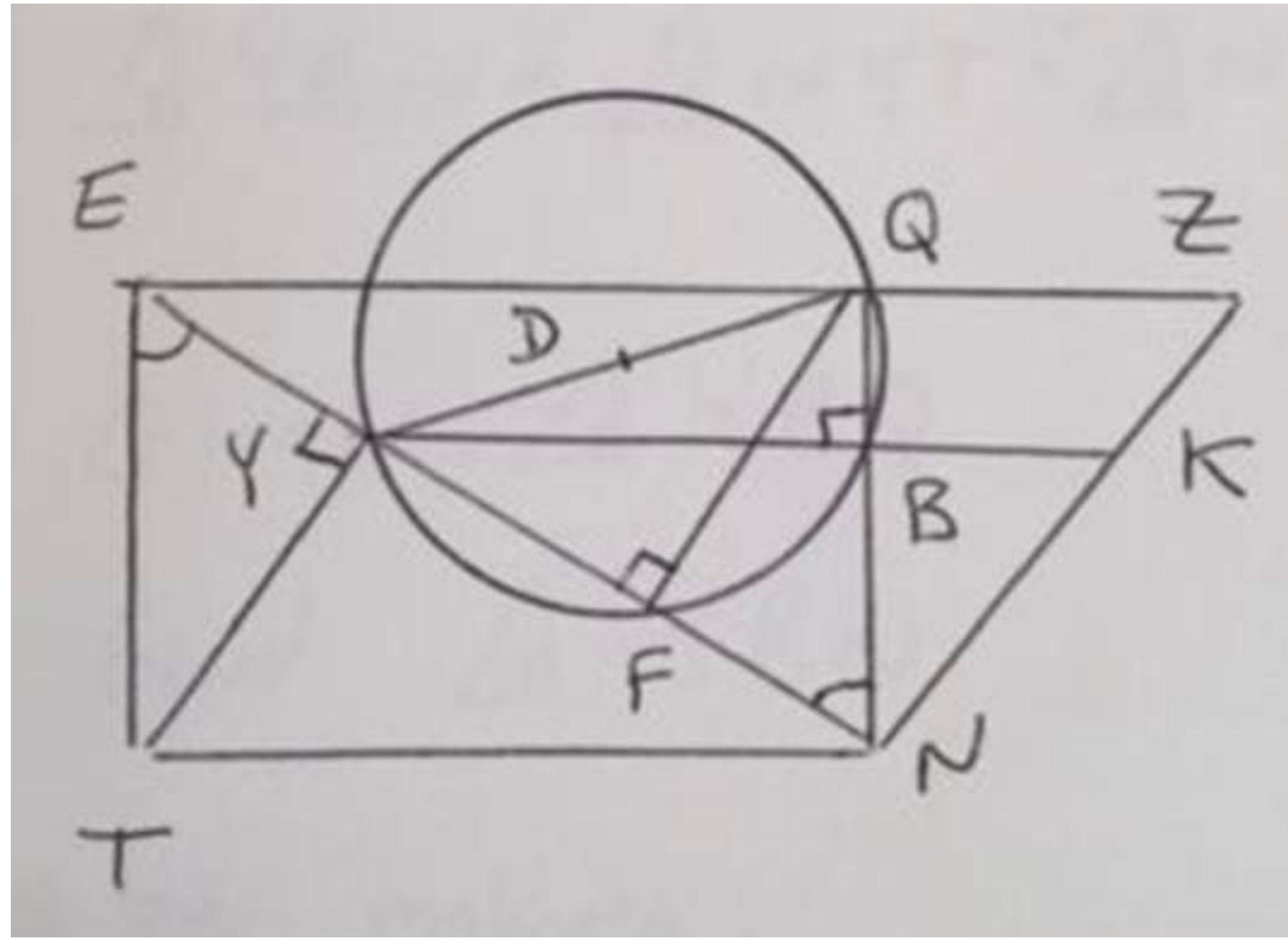


X = Z when EN is the shortest segment through Y included in right angle EQN

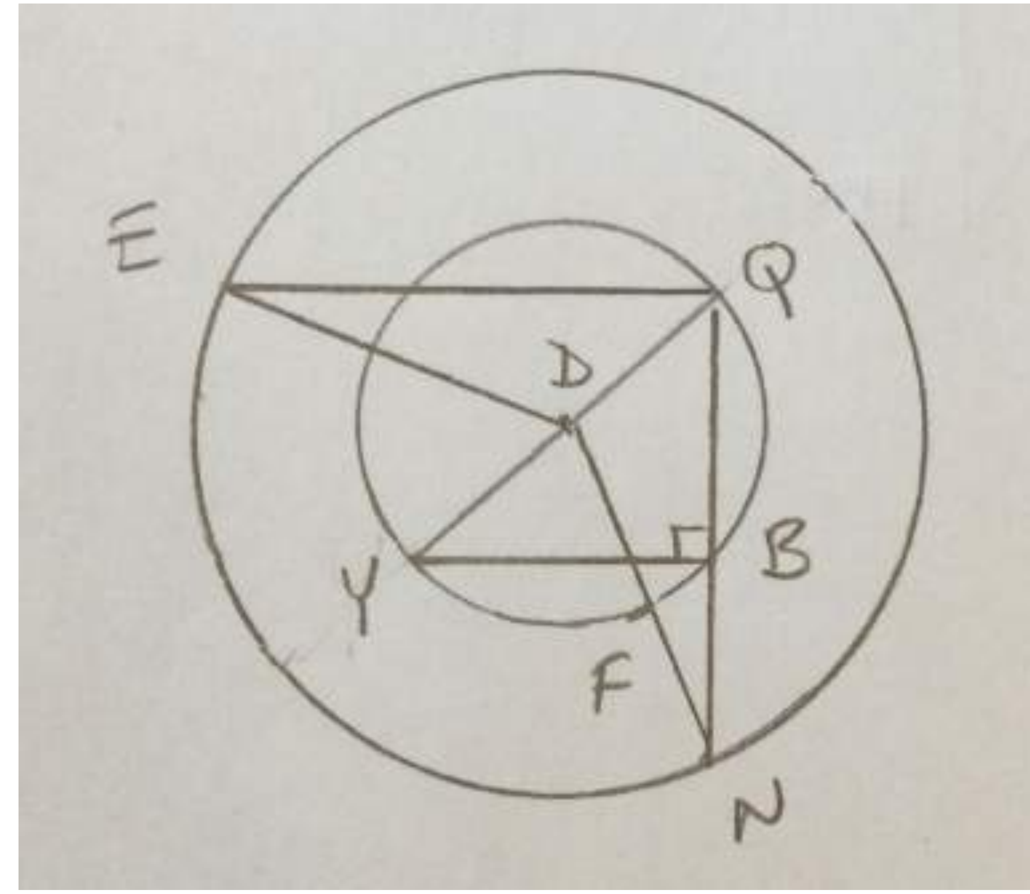


In order to find Z given ΔYBN and NK, we must find E using:

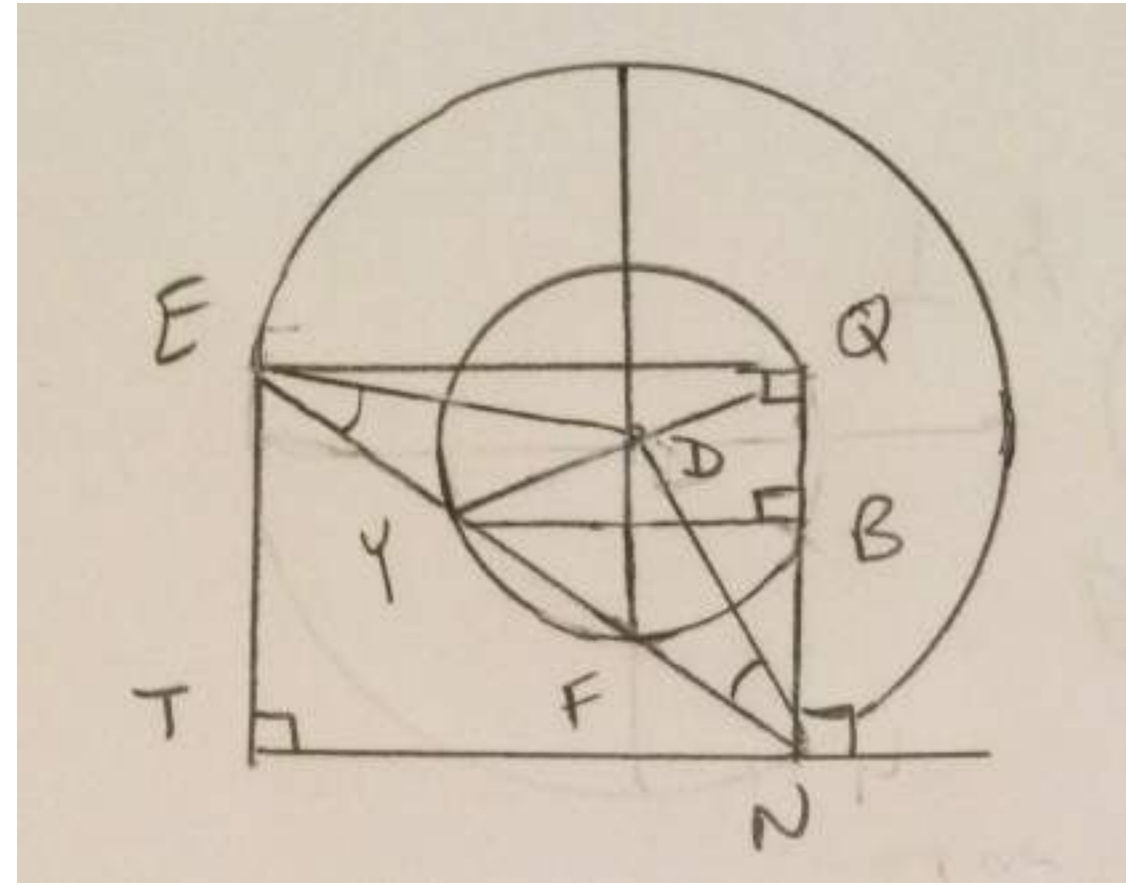
ΔYBN
≅ ΔNYT
≅ ΔNTE



In order to find Z given ΔYBQ,
we must find EN so that:
right triangle ΔTYE = ΔQFN
by drawing a circle concentric
with ⊙Y(F)BQ
around its center D
containing arc ~EN
so that YF lies on chord EN.



Not only does:
DY = DF, but also:
ED = ND and therefore
ΔEDY = ΔNDF
so EY = NF



Since ΔQFN is a right triangle, so is ΔTYE. Once we have found EN, we must also find NK in order to find Z.

2). refraction along a line

$\Delta N_oNK \cong \Delta KNA$

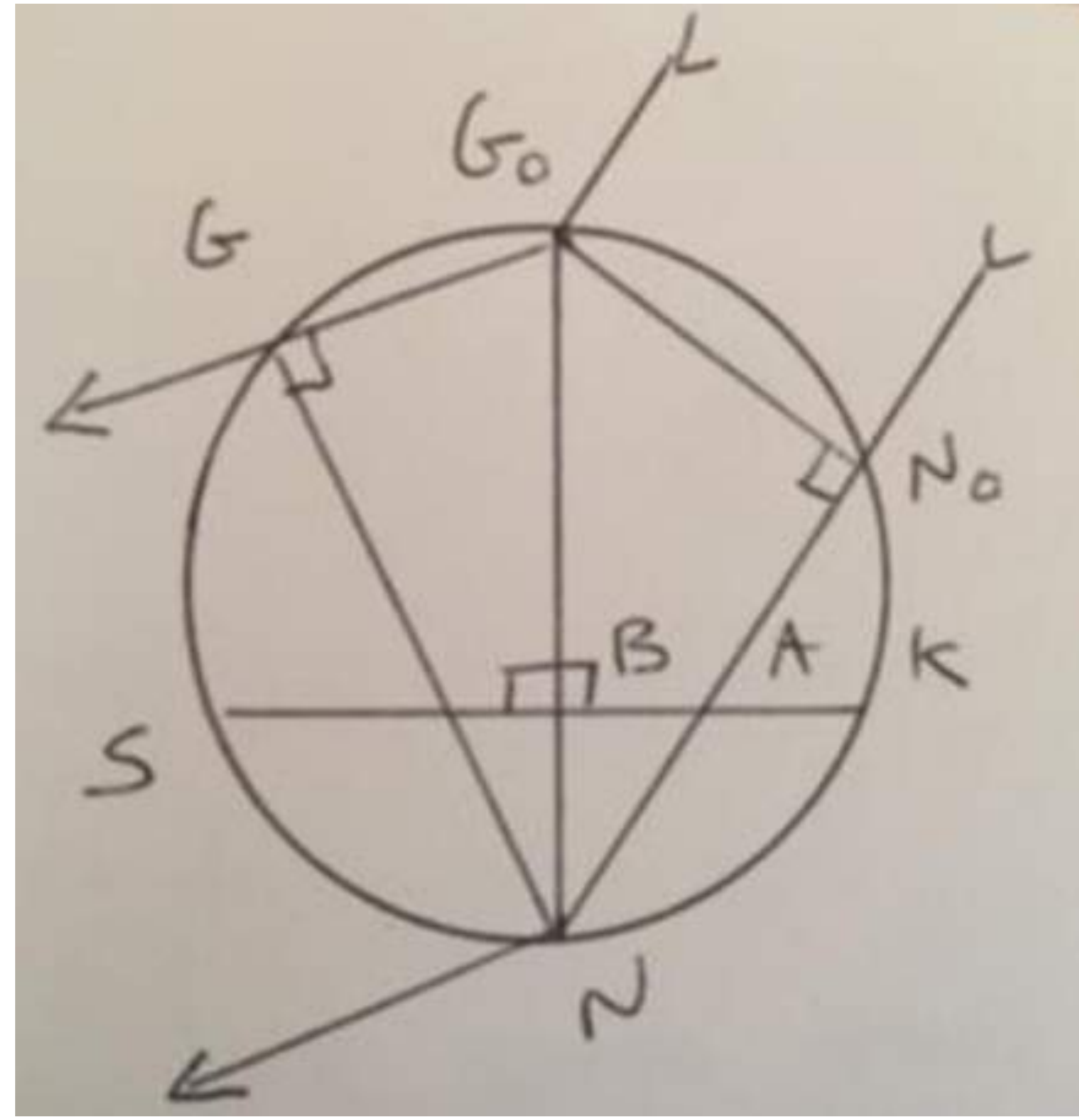
because:

~NS = ~NK

Wavefront G_oN_o refracts into wavefront GN along G_oN, because it travels G_oG in the same time it travels N_oN.

$\boldsymbol{R} = NN_o/GG_o$

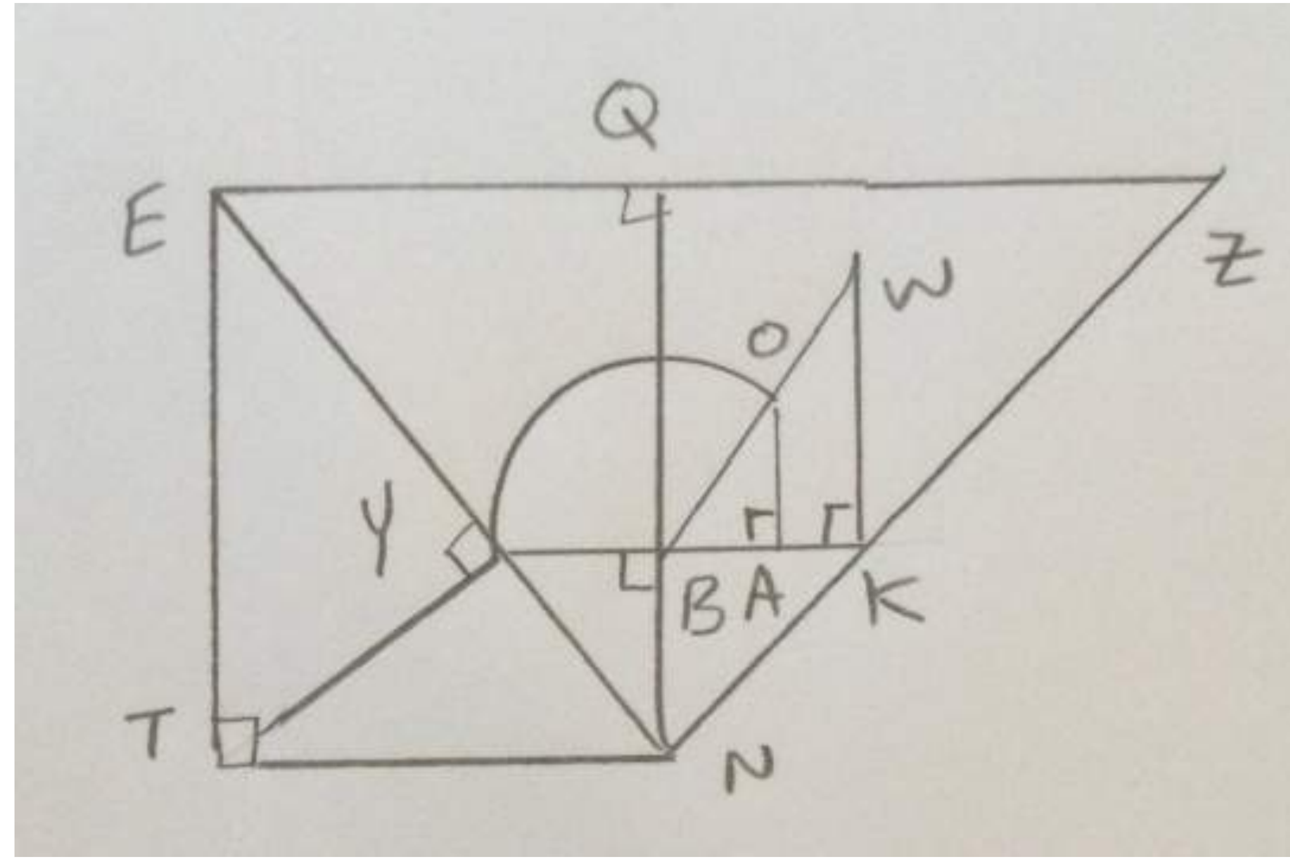
$= NN_o/NK = NK/NA$



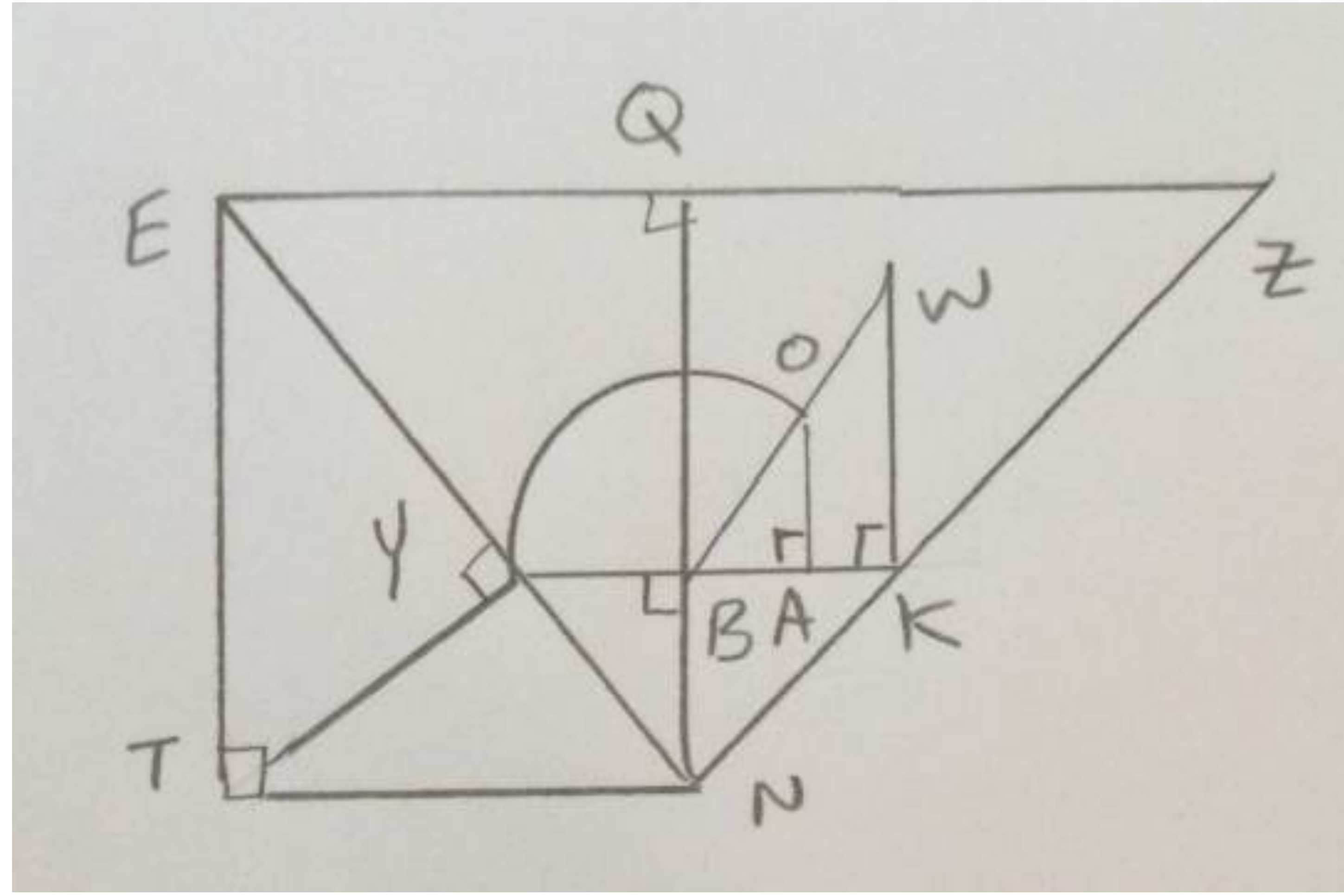
If ***R*** = OB/OA,

and KW = YN:

R = NK/NA

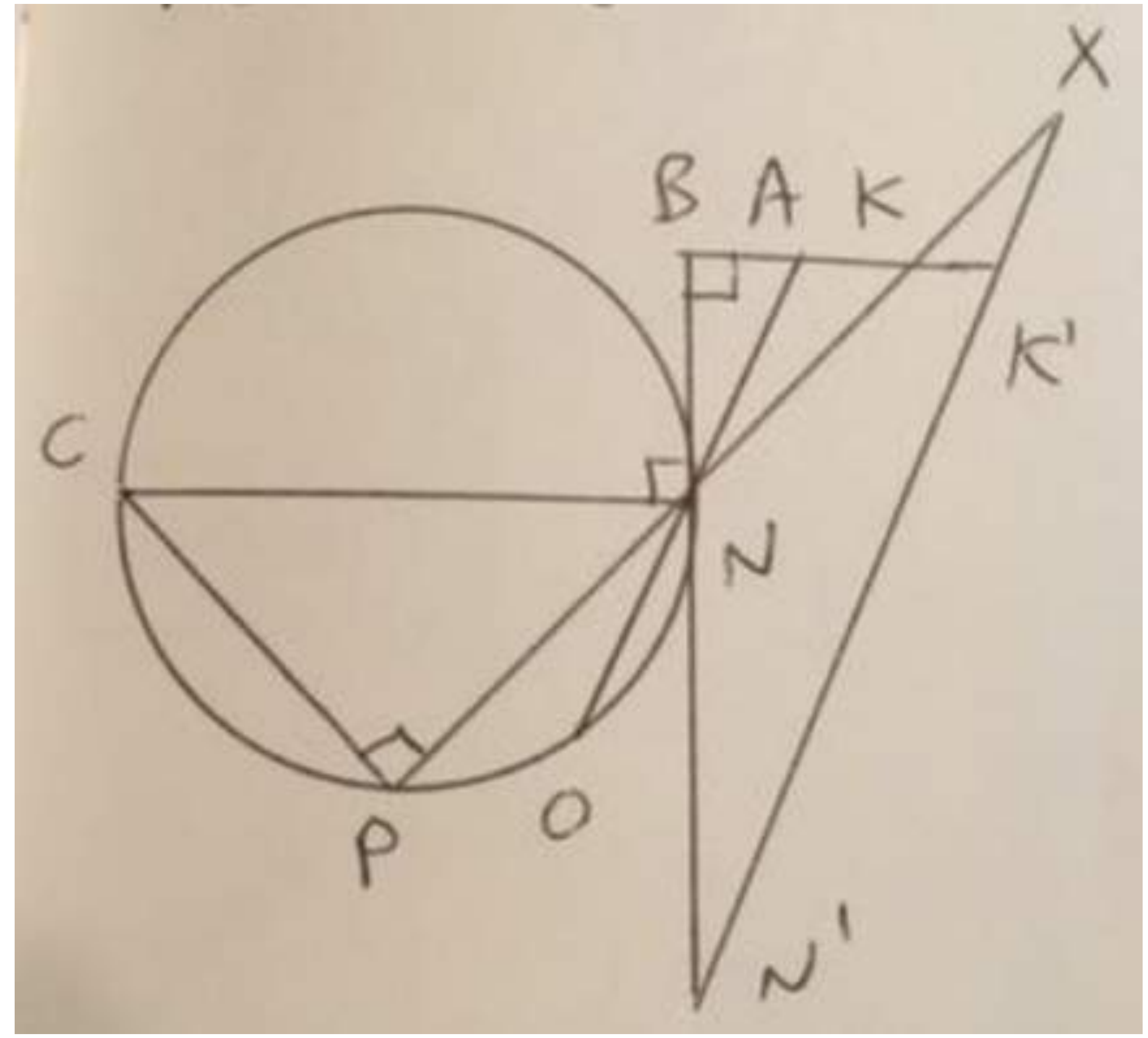


and Z is the clear image of object A refracted at N along BN



given ΔBAO:
use ΔBKW or ΔQBY to find ΔBNY
use ΔBNY to find ΔBKW or ΔQBY

3). refraction along a circle



ΔKNA ≅ ΔOCP

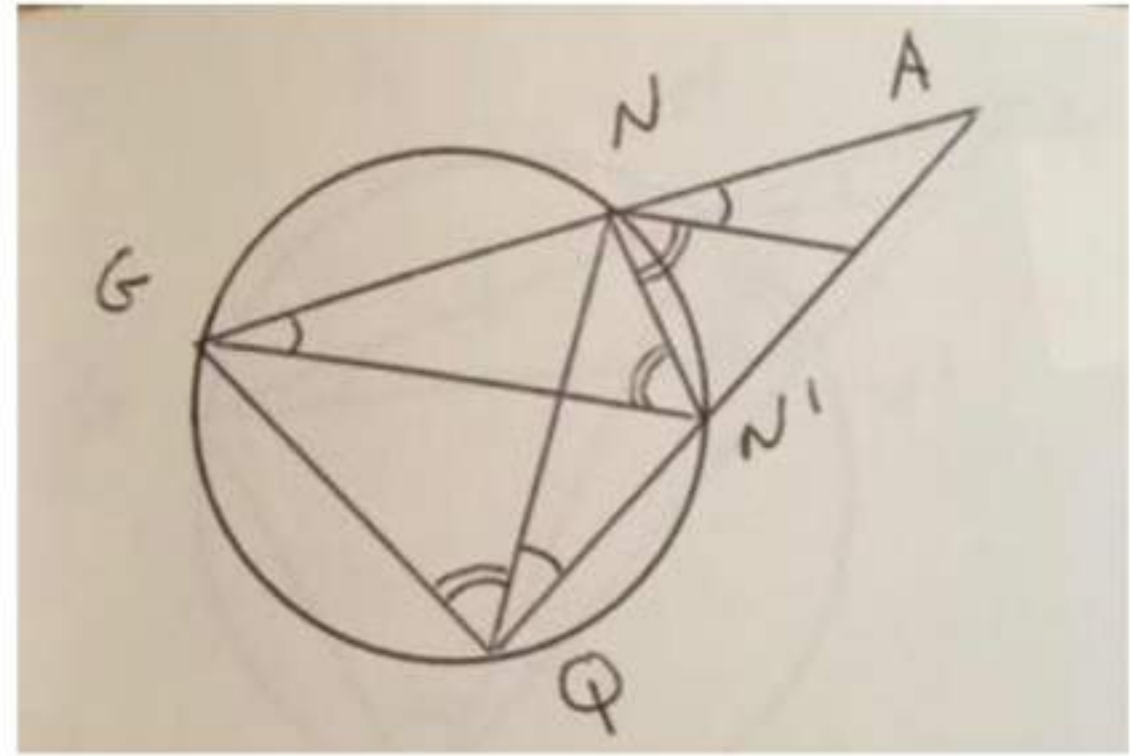
R = NK/NA = N'K'/N'A = CO/CP

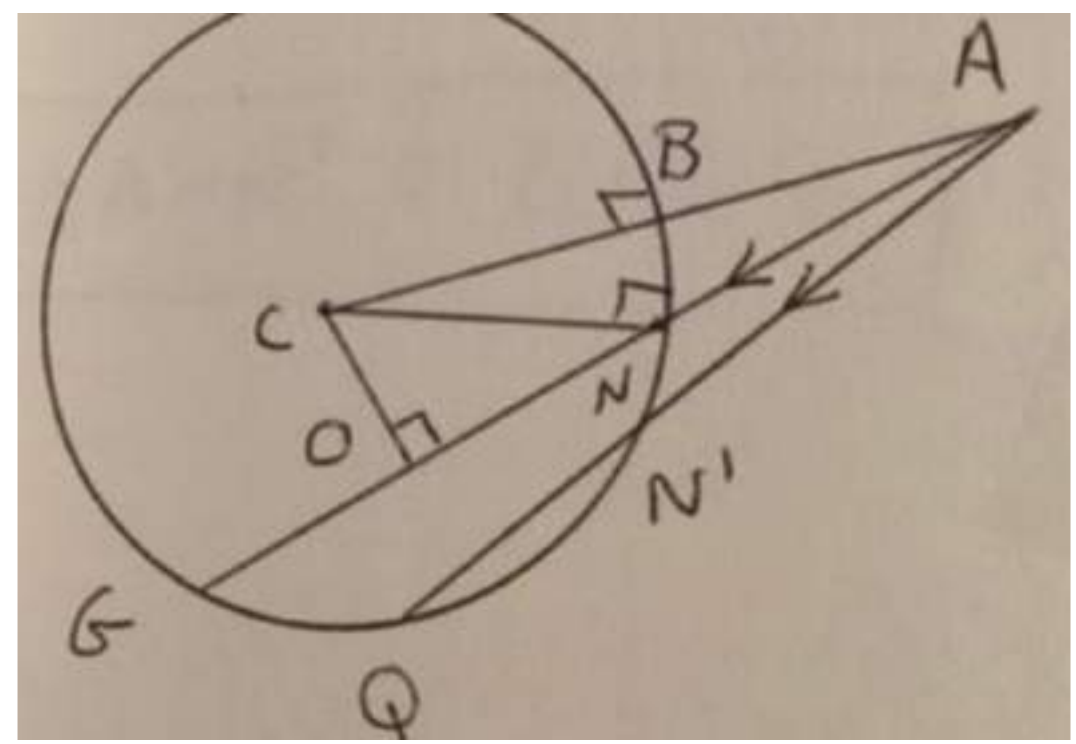
ΔANN’ ≅ ΔAQG

AG/AN’ = QG/NN’

(AG + AN’)/2AN’
= (QG + NN’)/2NN’

Real object A



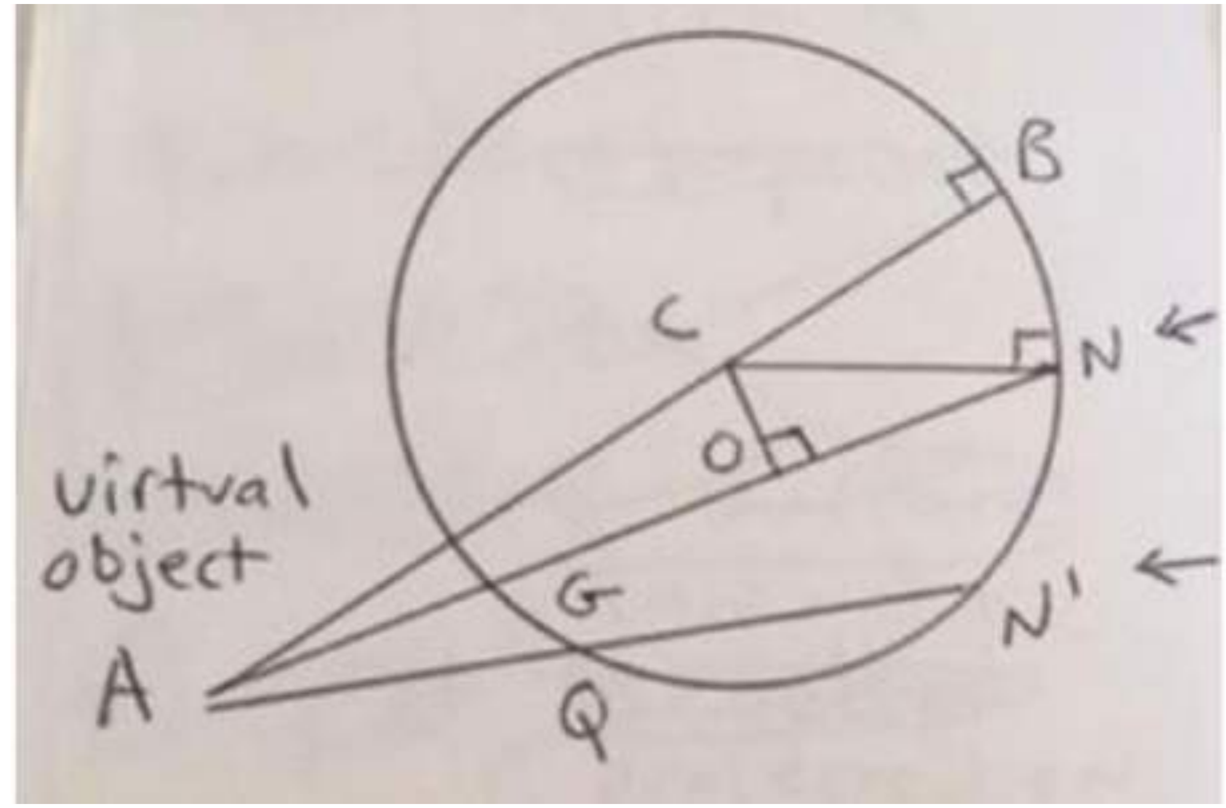


$\Delta ANN' \cong \Delta AQG$

$AG/AN' = QG/NN'$

$(AG + AN')/2AN'$
$= (QG + NN')/2NN'$

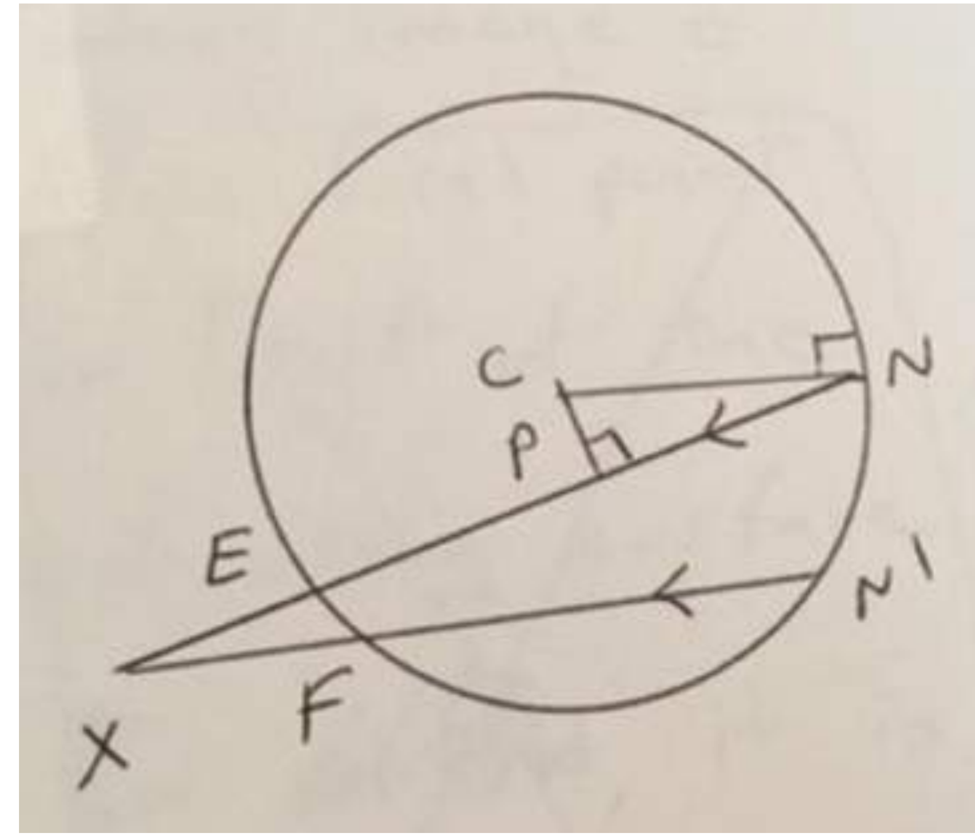
Virtual object A can not be projected on a screen due to refraction at BN.



$\Delta XNN' \cong \Delta XFE$

$XE/XN' = EF/NN'$

$(XE + XN')/2XN'$
$= (EF + NN')/2NN'$

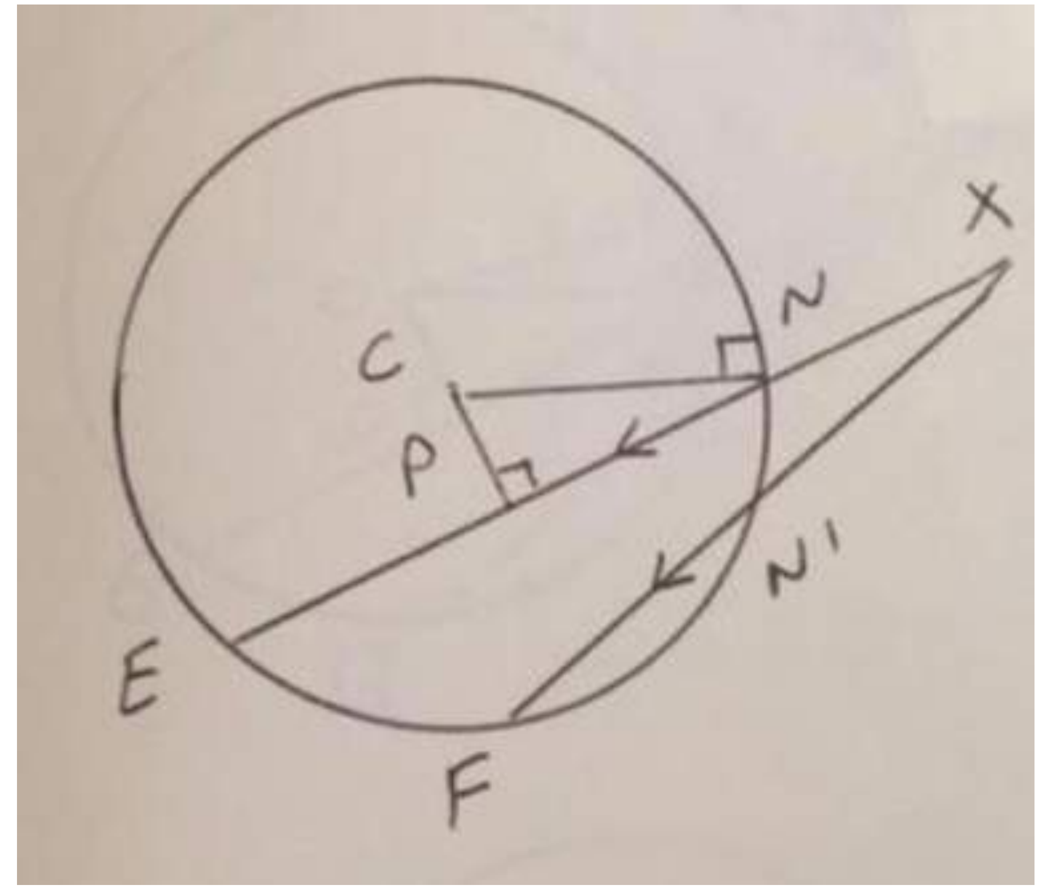


Real image at (X = Z) can be projected on a screen.

ΔXNN’ ≅ ΔXFE

XE/XN’ = EF/NN’

(XE + XN’)/2XN’
= (EF + NN’)/2NN’



Virtual image at (X = Z) can not be projected on a screen.

$$(AG + AN')/2AN' = (QG + NN')/2NN'$$
$$(XE + XN')/2XN' = (EF + NN')/2NN'$$

$$(QG + NN')/(EF + NN')$$
$$= [(AG + AN')/2AN'][2XN'/(XE + XN')]$$

As $N' \Rightarrow N$, $X \Rightarrow Z$, and:

$$(\sim QG + \sim NN')/(\sim EF + \sim NN')$$
$$\Rightarrow (QG + NN')/(EF + NN')$$

$$\Rightarrow (AO/AN)(ZN/ZP)$$

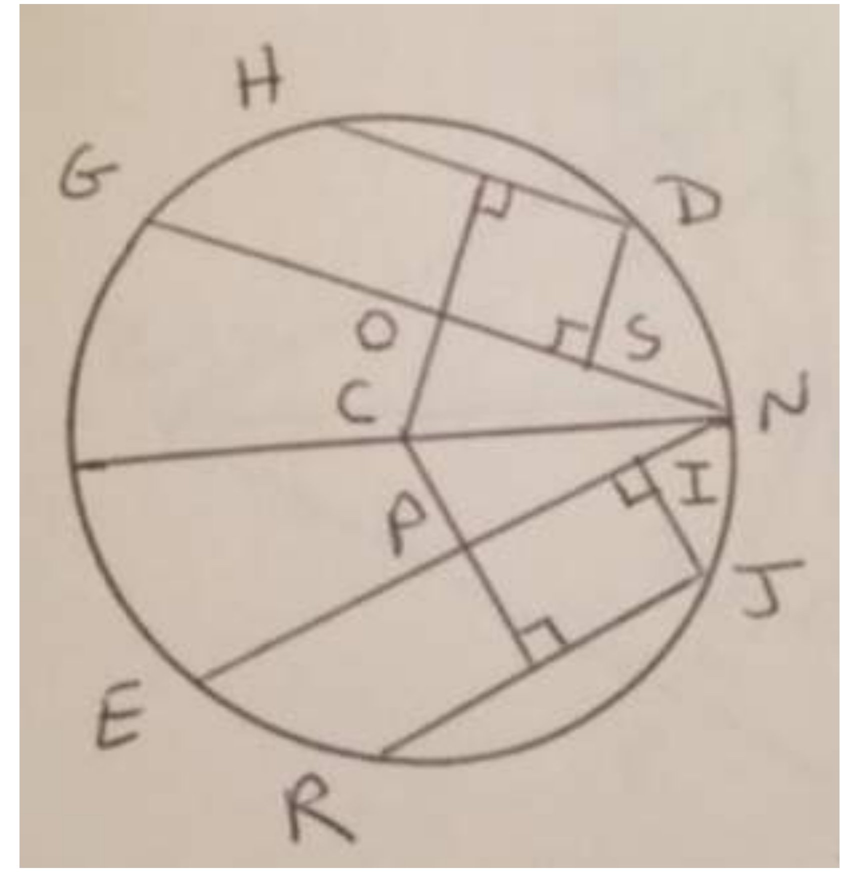
Also, when HD = QN’
and RJ = FN’

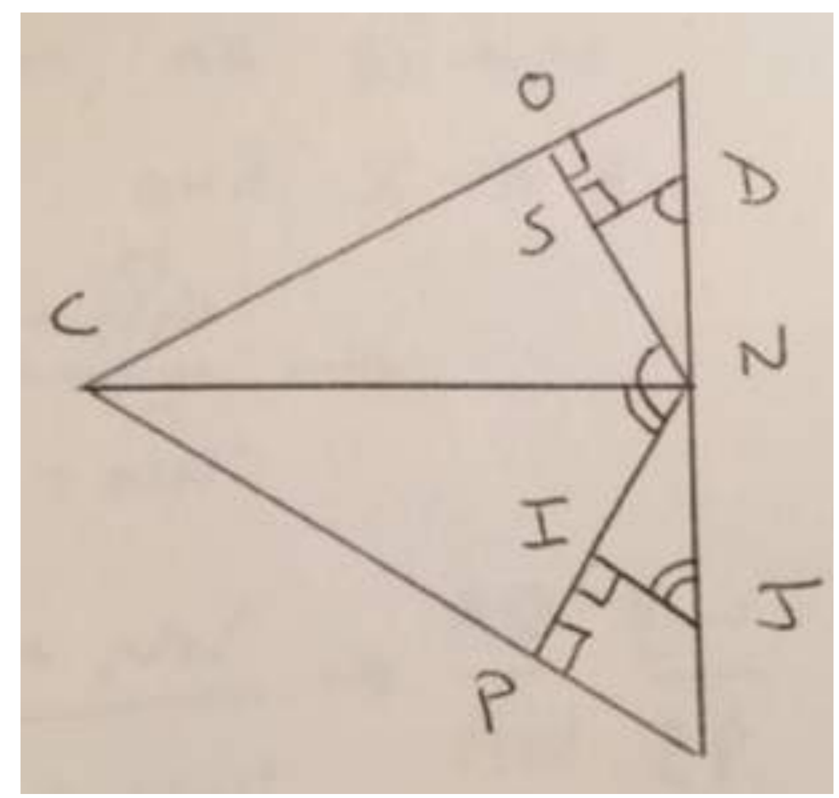
(~QG + ~NN’)/(~EF + ~NN’)
= 2(~ND)/2(~NJ) = ~ND/~NJ

As N’ ⇒ N, X ⇒ Z, and:

~DJ ⇒ line segment DJ, so:

(~QG + ~NN’)/(~EF + ~NN’)
⇒ ND/NJ





$$DS/JI = CO/CP$$
$$JI/JN = NP/NC$$
$$DN/DS = NC/NO$$
$$ND/NJ = (NP/NO)(CO/CP)$$

As $N' \Rightarrow N$, $X \Rightarrow Z$, and:

$$(\sim QG + \sim NN')/(\sim EF + \sim NN')$$
$$\Rightarrow (NP/NO)(CO/CP)$$

and therefore:

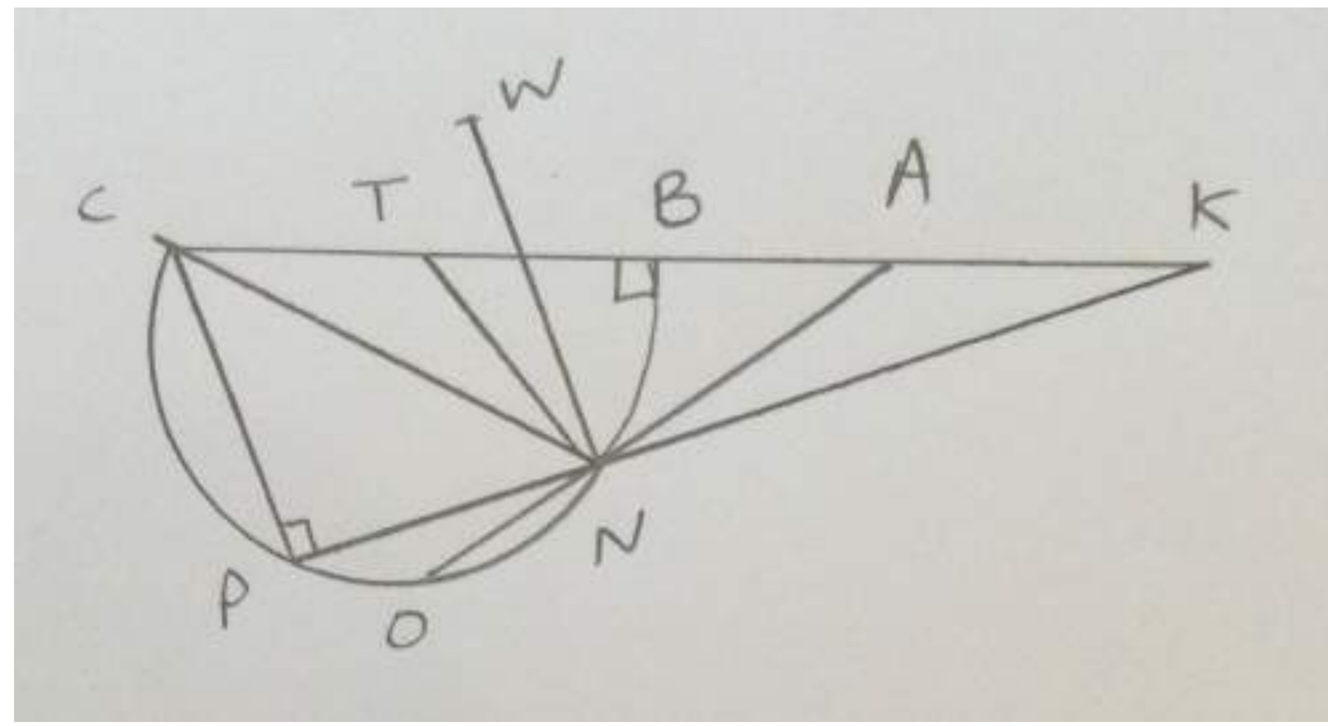
$$(AO/AN)(ZN/ZP) \Rightarrow (NP/NO)(CO/CP)$$

Thus **R** = CO/CP, and Z, (along both NP and CW), is the clear image of A refracted along ~BN, when:

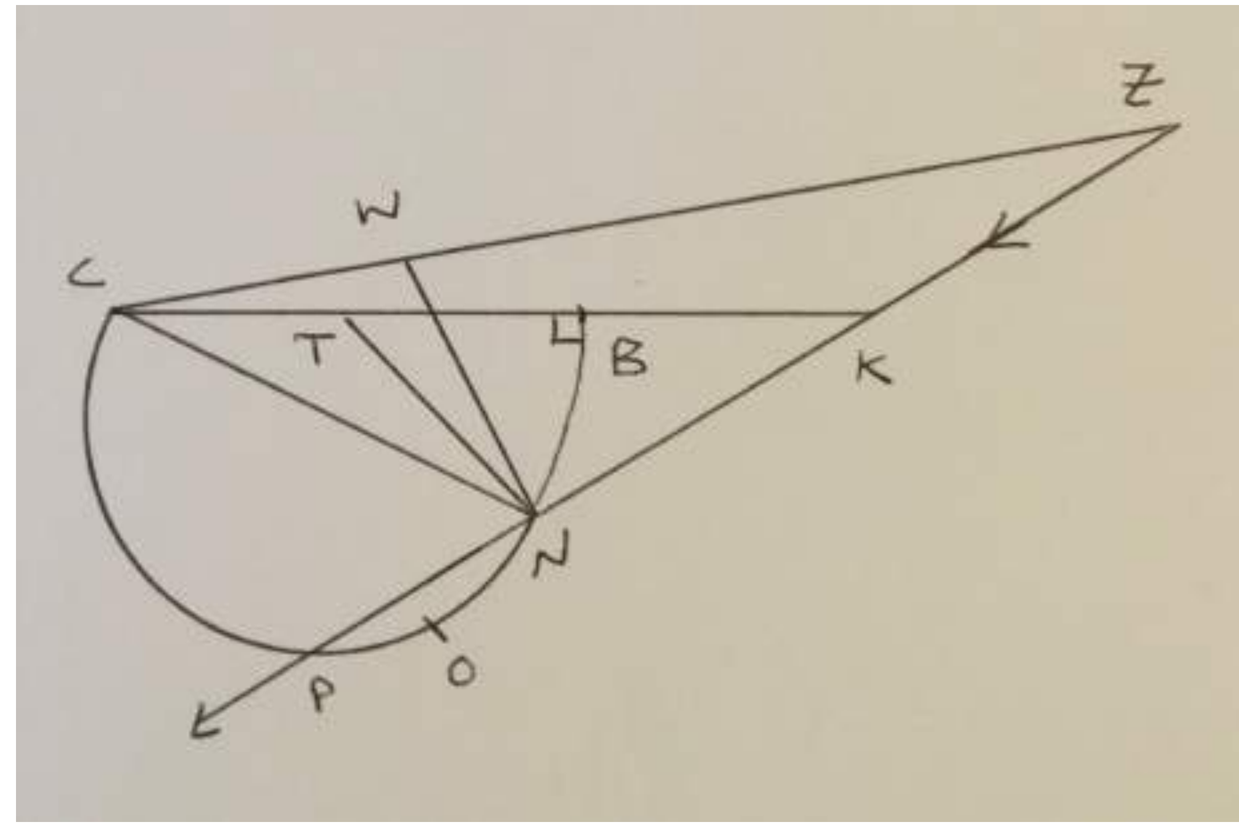
NT||CO, so:
AO/AN = CO/NT and:

NW||CP, so:
ZN/ZP = NW/CP and:

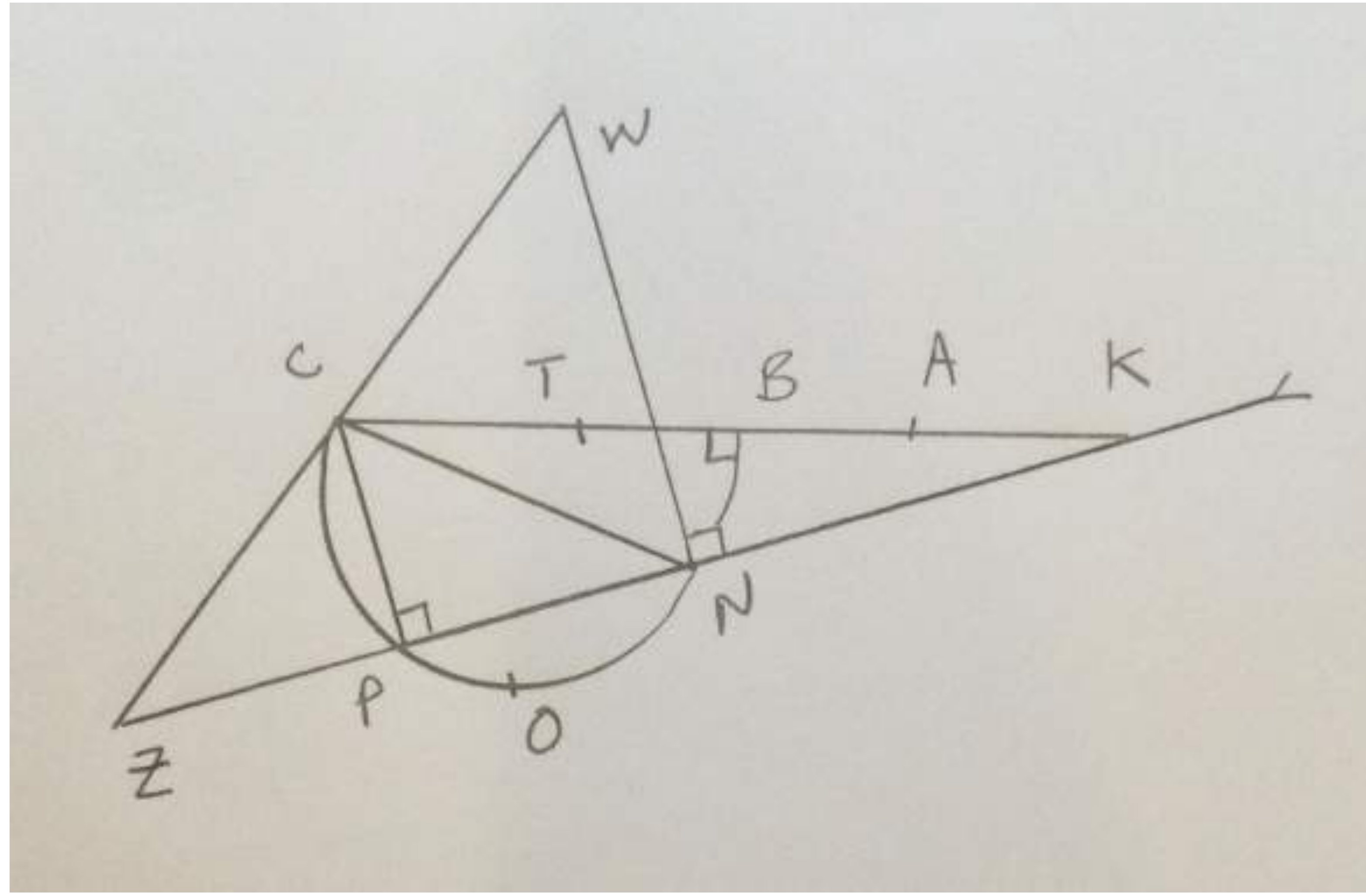
NW/NT = NP/NO
(ΔWNT ≅ ΔPNO)



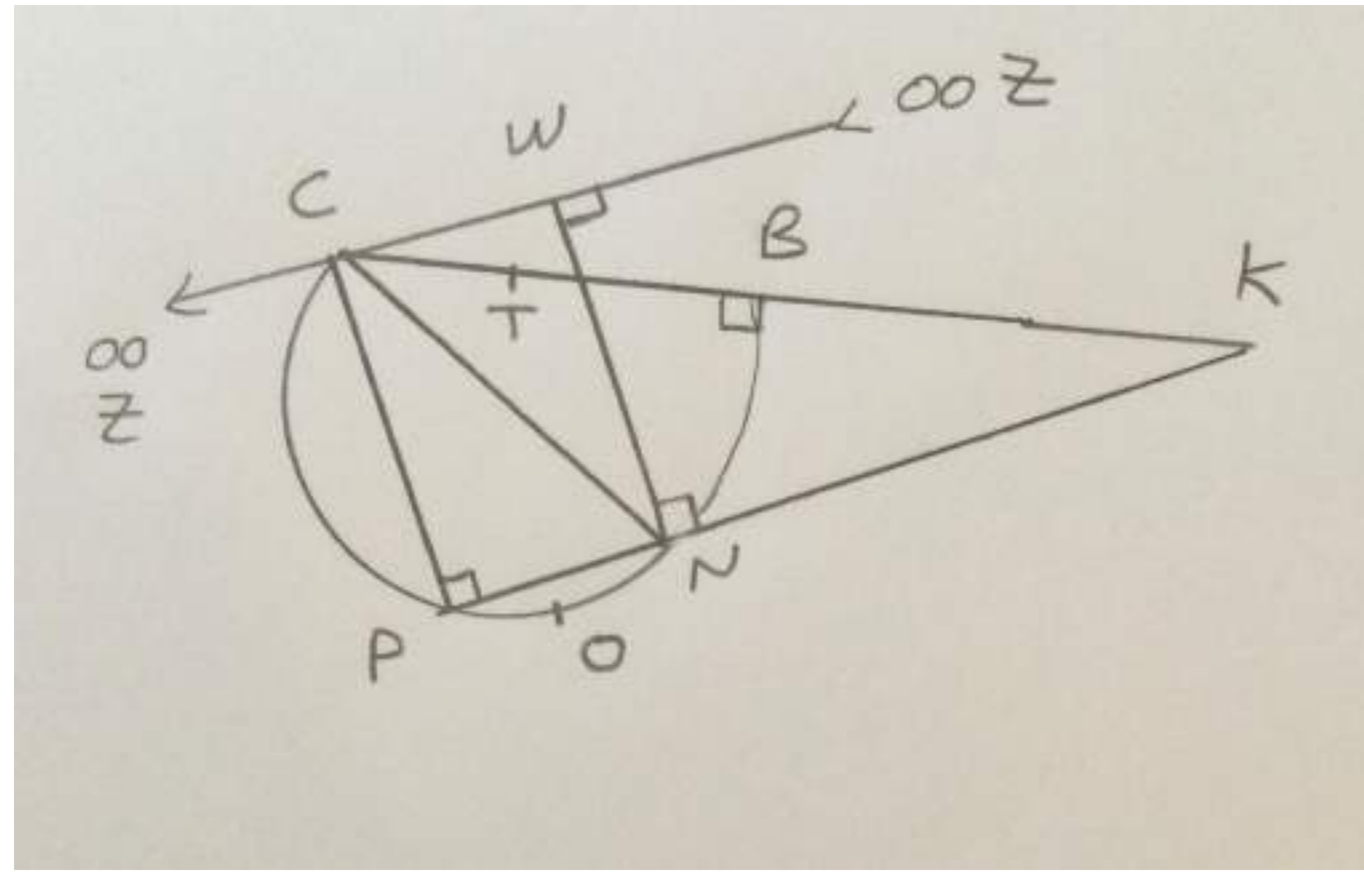
The off-axis rays from any on-axis object A, (real or virtual), can not form a virtual on-axis image at Z because NW must be less than CP for Z to be virtual; but NW must also be greater than NT.



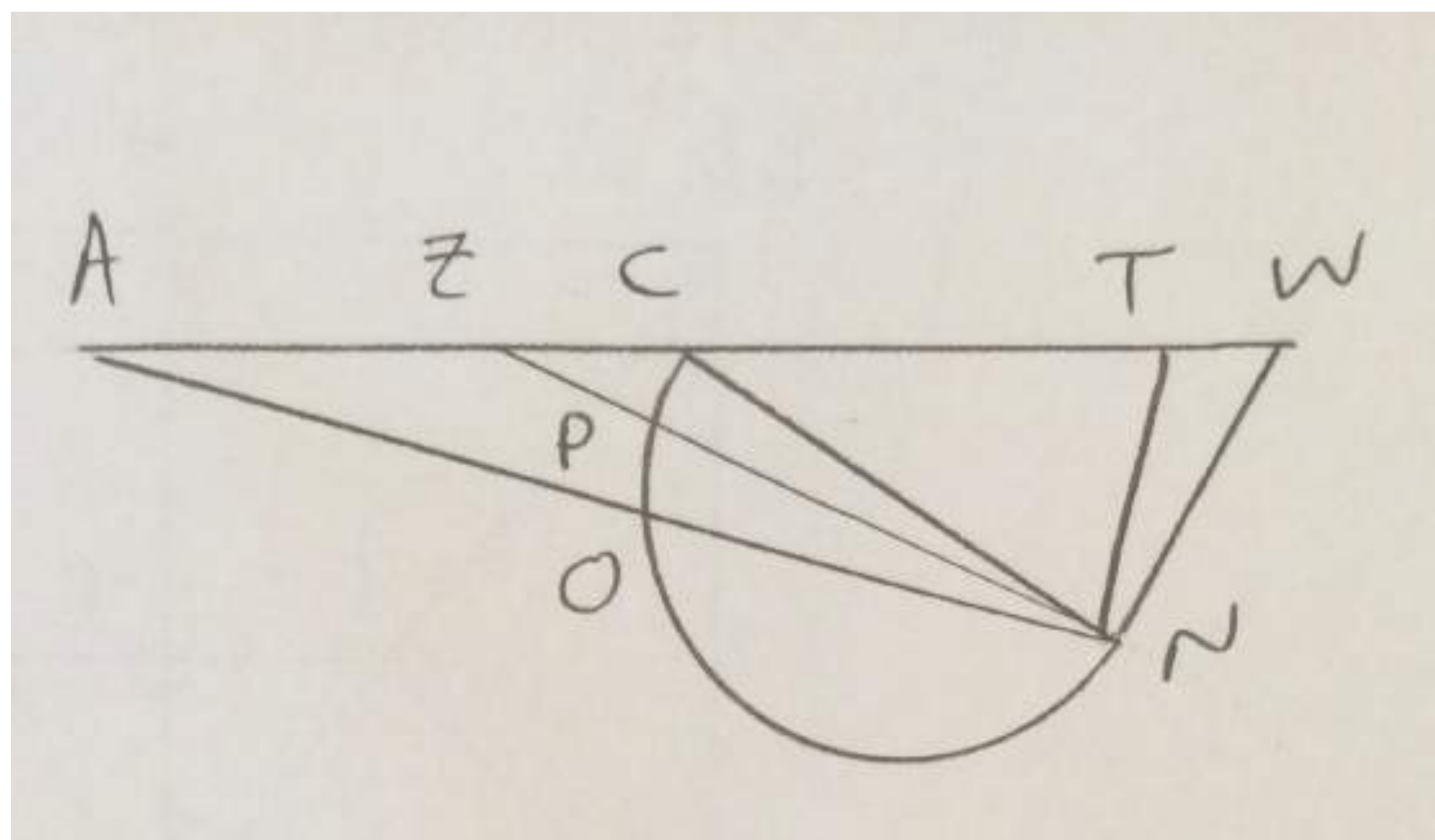
The off-axis rays from any real on-axis object A can not form a real on-axis image at Z because NW must be greater than (or equal to) CP for Z to be real; but NW must also be greater than NT.



The off-axis rays from any real on-axis object A can not form a real on-axis image at Z because NW must be greater than (or equal to, as shown here) CP for Z to be real; but NW must also be greater than NT.



The off-axis rays from a virtual on-axis object A ***can*** form a real on-axis image at Z, if NW is greater than CP, and WT lies along the axis.



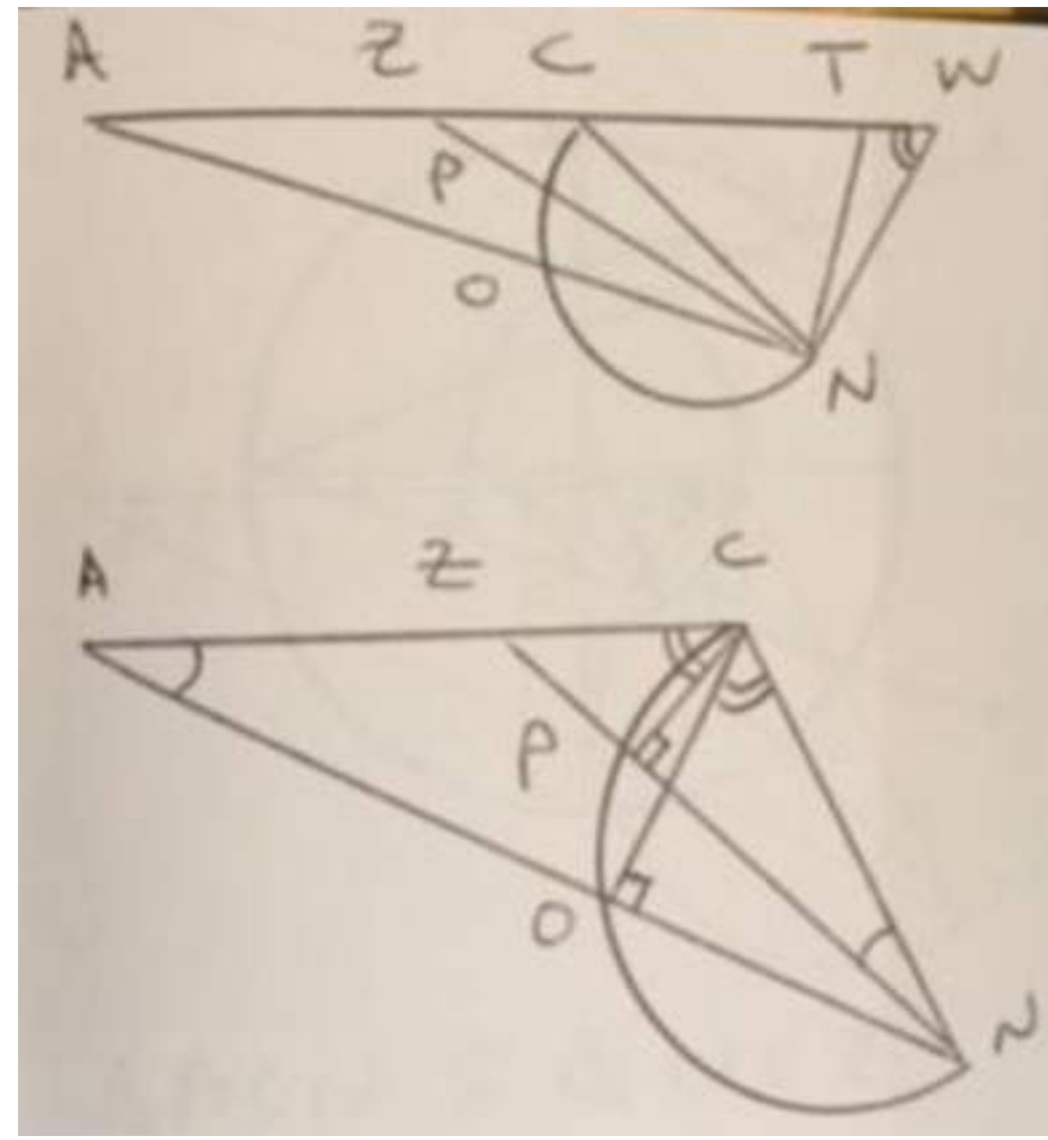
Since:

∠NWT = ∠NPO = ∠NCO

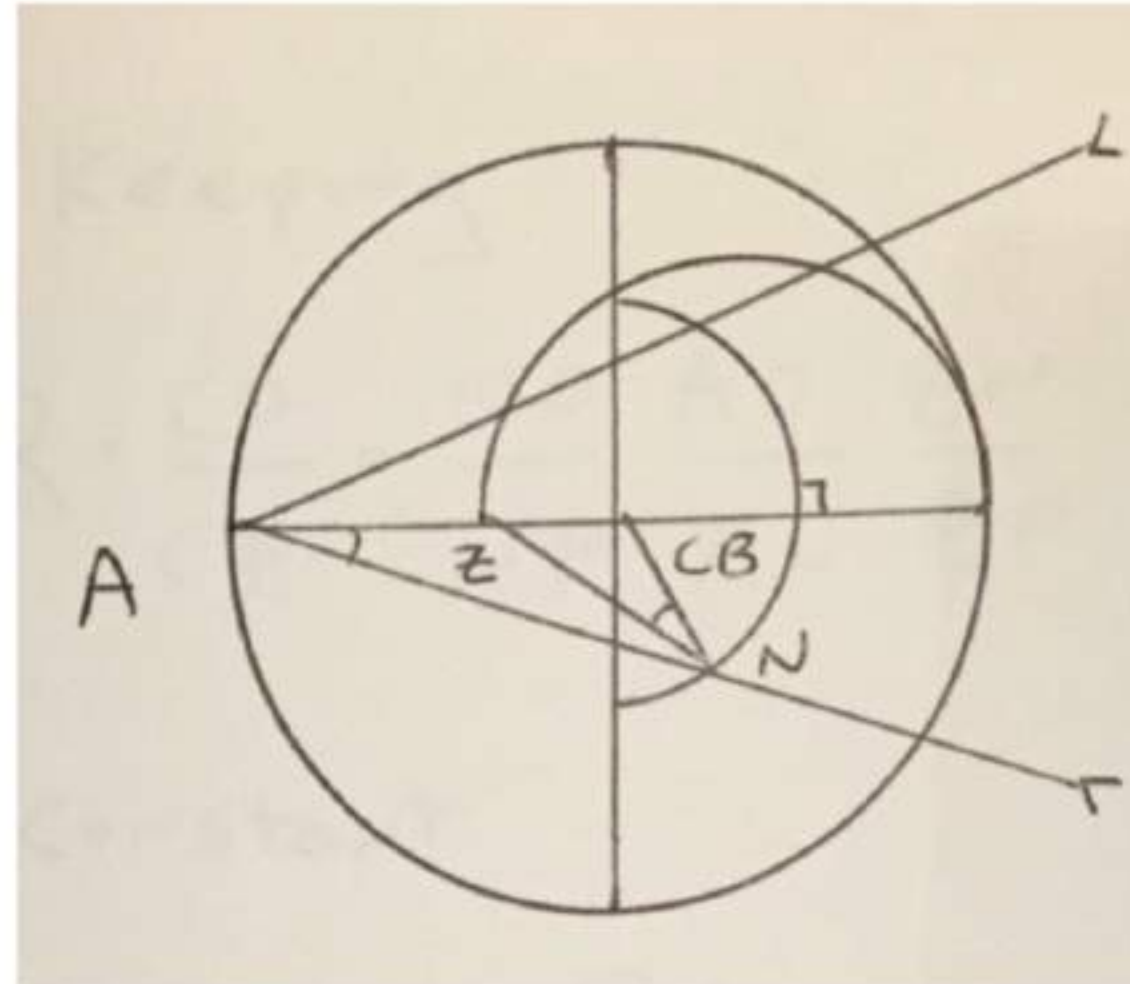
and NW||CP

WT lies along the axis when:

ΔNCO ≅ ΔZCP



When off-axis rays from a virtual on-axis object A form a real on-axis image Z, this occurs at all points N because:



ΔACN ≅ ΔNCZ for all N

4). refraction through a circle's center

Keeping:

$\boldsymbol{R}$ = (CO/CP) =
(NO/NP)(AO/AN)(ZN/ZP)

constant as:
N ⇒ B:

(BC/BC)(AC/AB)(ZB/ZC) ⇒ $\boldsymbol{R}$

Refraction through a circle’s center occurs when N lies at B, so that an object’s ray from A to N lies along ABC, and an image ray lies along BCZ. The locations of the object A and image Z along the optic axis BC are described by the equation:

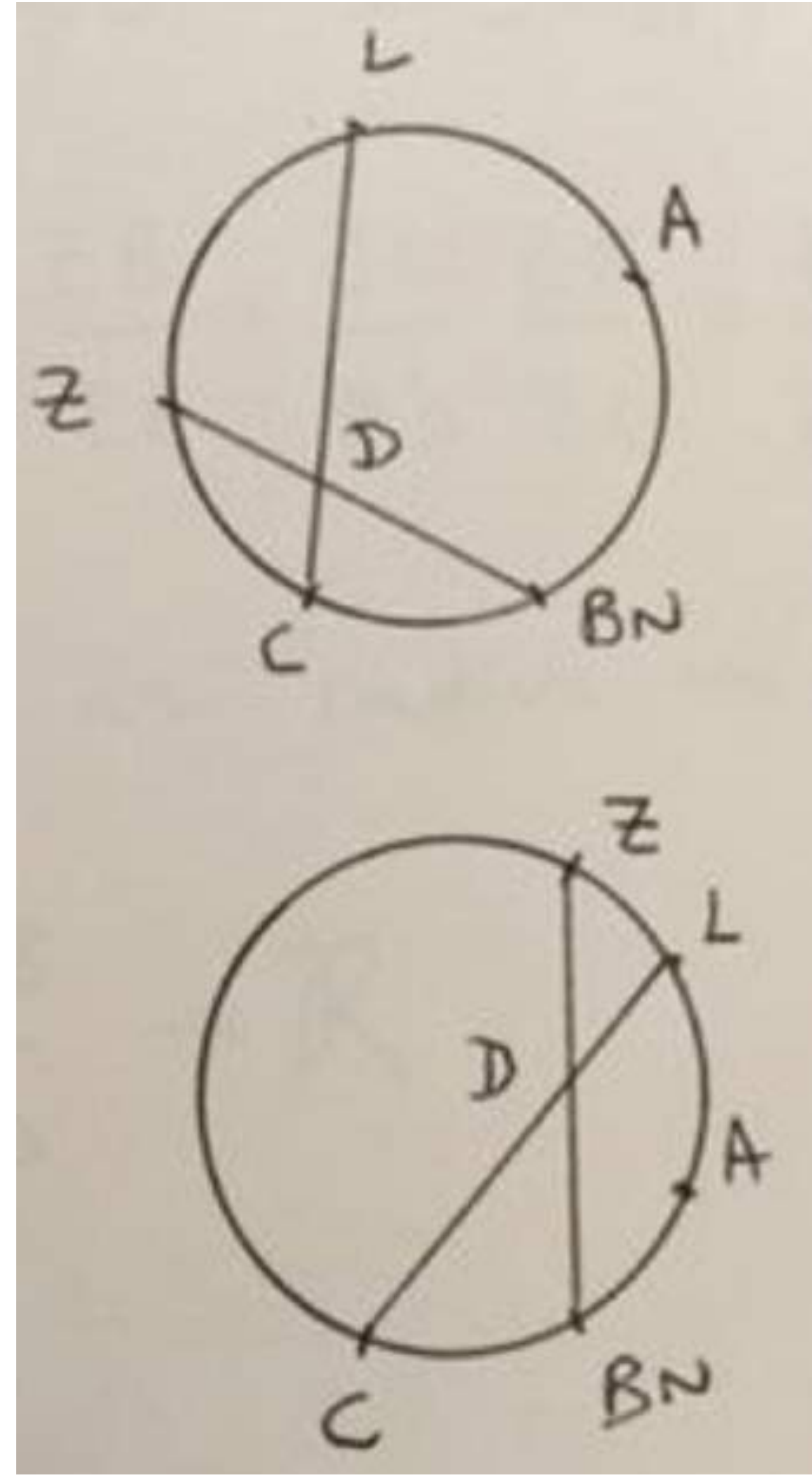
$\boldsymbol{R}$ = CO/CP = (AC/AB)(ZB/ZC)

If we draw A and Z along the optic axis BC ***as if*** it were a circle, and draw CDL so that AL || ZB:

ΔACB ≅ ΔZCD, and:

(AC/AB)(ZB/ZC) =
(ZC/ZD)(ZB/ZC) =
(ZB/ZD)
so as the reference circle's radius ⇒ ∞

(ZB/ZD) ⇒ ***R***

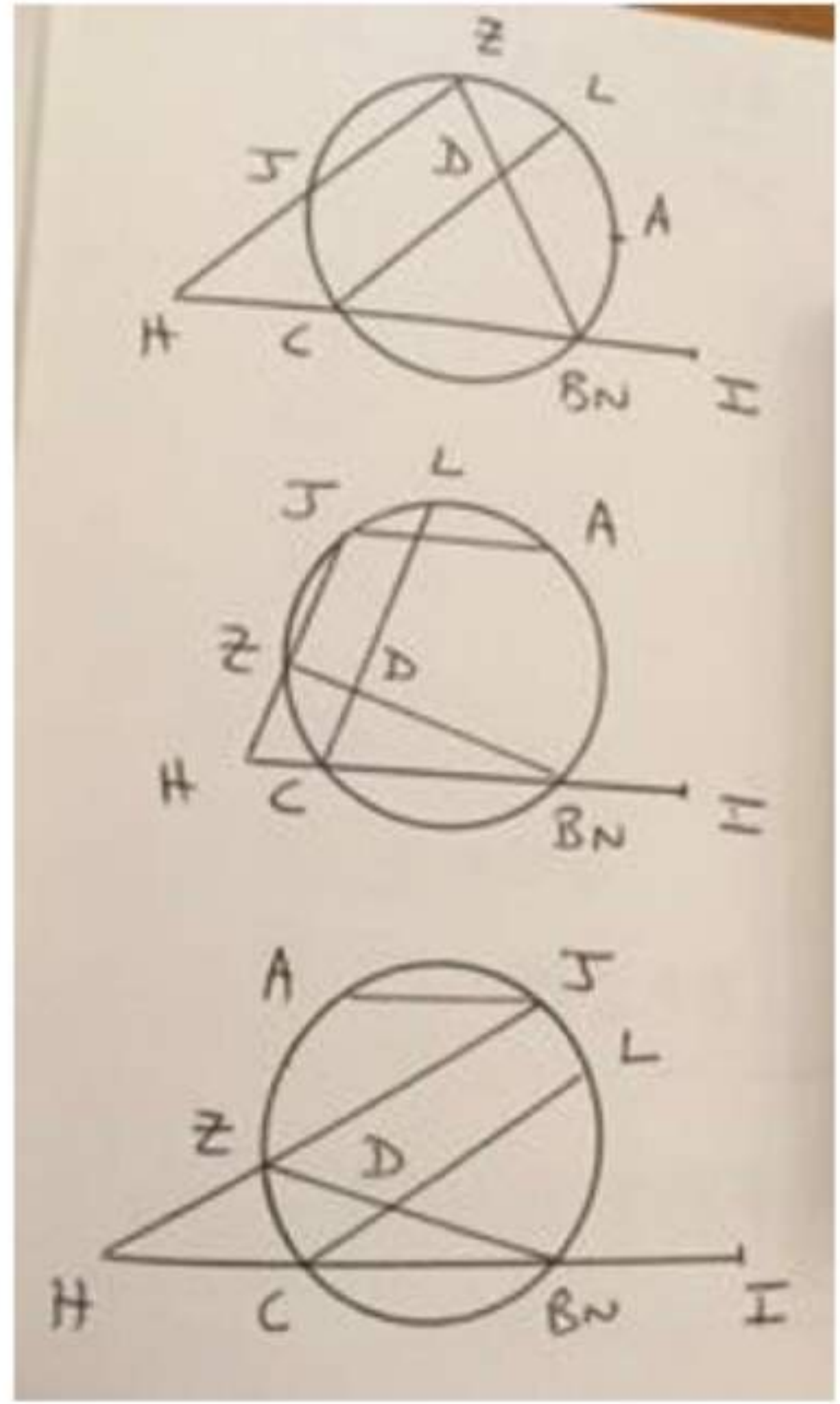


AL || ZB
AZ = BL
~AZ = ~BL

HZ || CL
ZC = LJ
~ZC = ~LJ

~AZ + ~ZC = ~AZC
~BL + ~LJ = ~BLJ

~AZC = ~BLJ
AJ || CB



HZ ll CL

ZB/ZD = HB/HC

ΔHBZ ≅ ΔHJC

when ΔHJC = ΔIAB:

HC = IB, and:

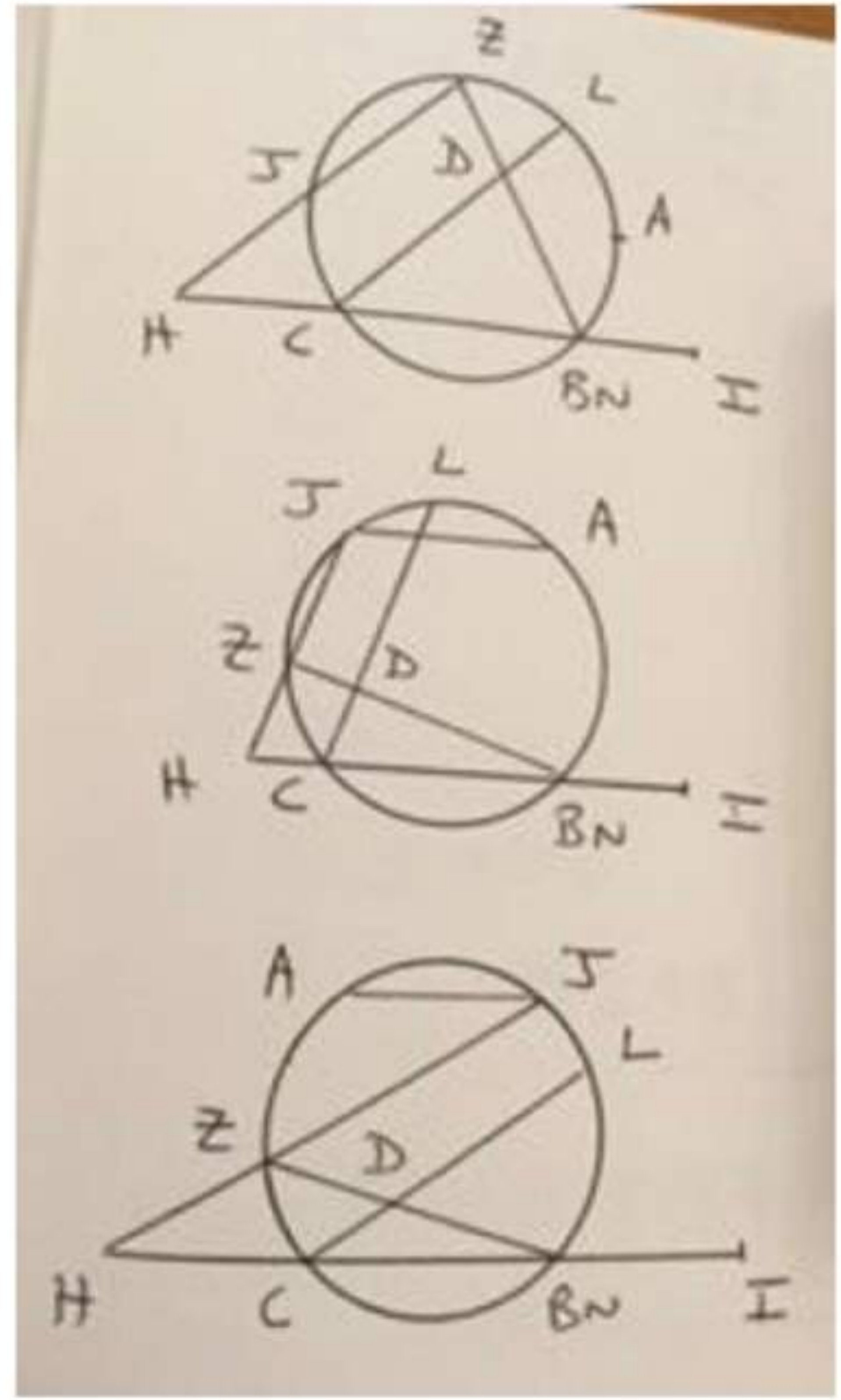
IB/IA = HZ/HB

This results in

Newton's Equation

as the reference circle's radius ⇒ ∞:

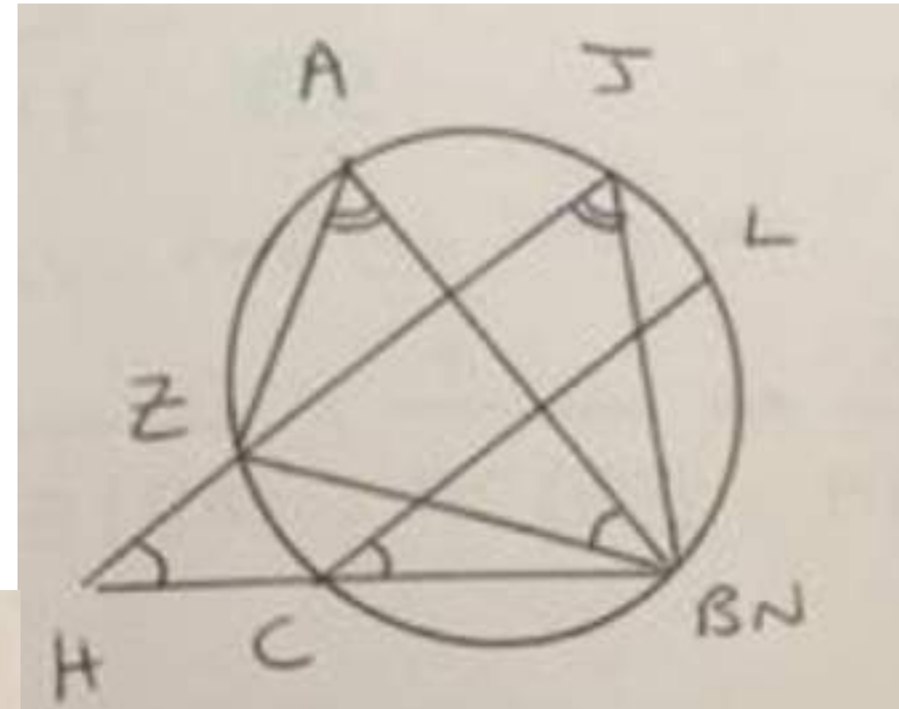
(AI)(ZH) = (BI)(BH)

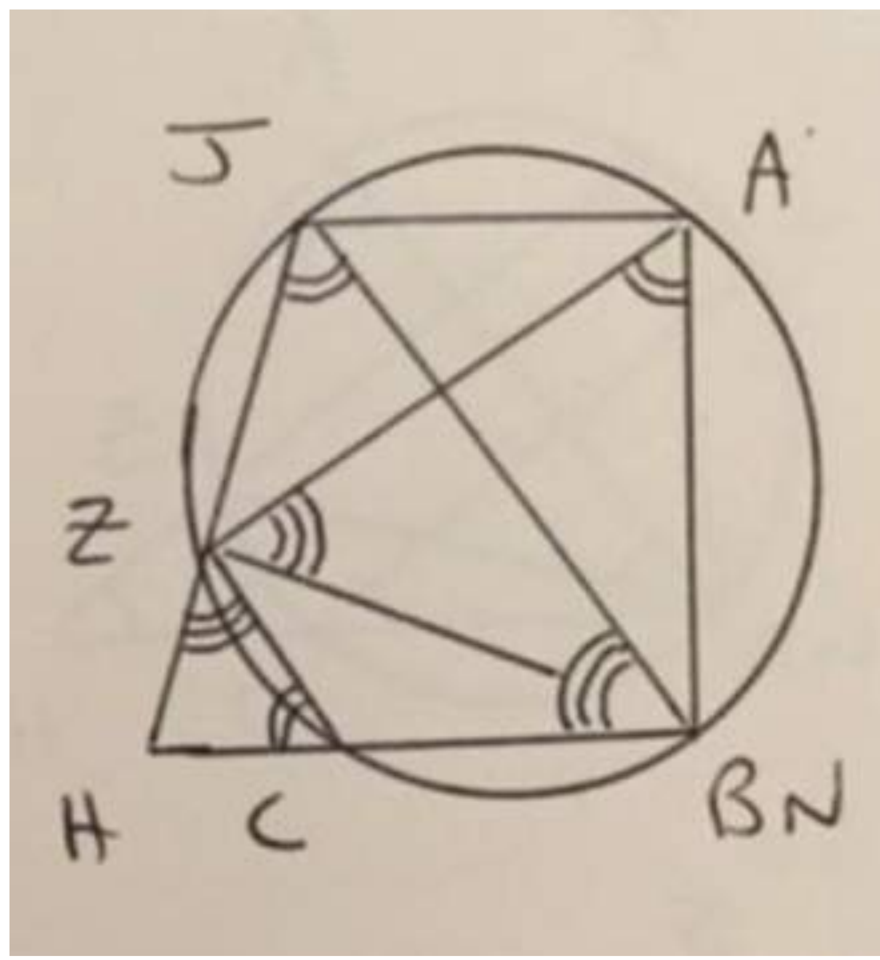


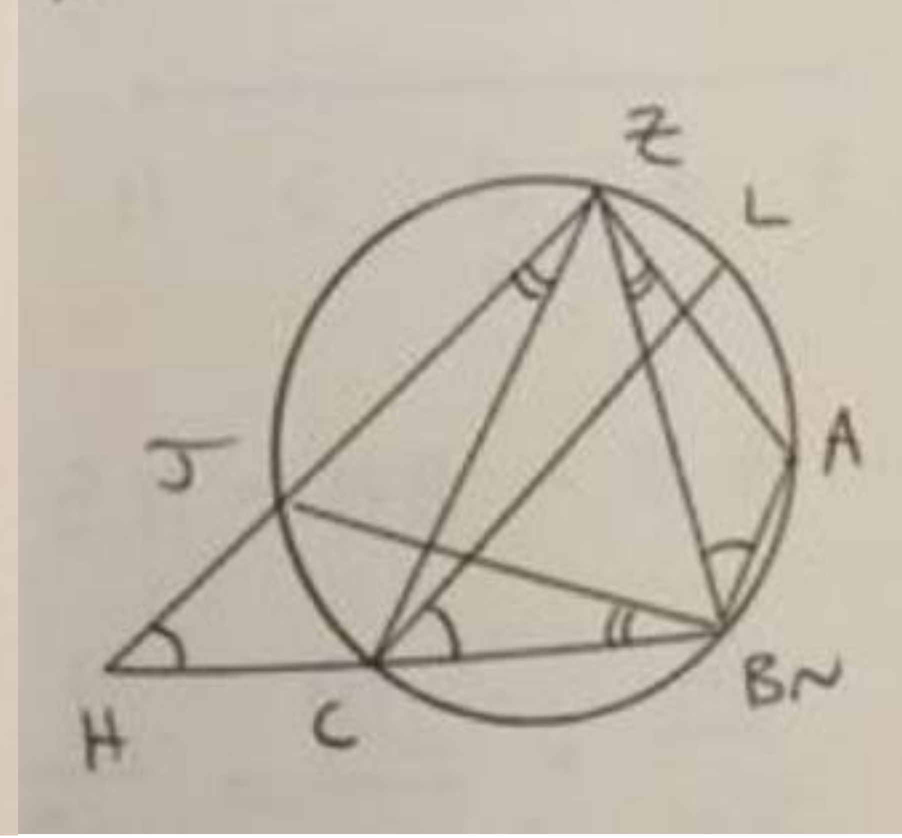
$$\Delta HCZ \cong \Delta HJB \cong \Delta BAZ$$

$$(HC/HZ) = (BA/BZ)$$

$$[1/(HZ)(BA)] = [1/(HC)(BZ)]$$







as the reference circle’s radius $\Rightarrow \infty$:

$[1/(HZ)(BA)] = [1/(HC)(BZ)] \Rightarrow \boldsymbol{R}/(HB)(BZ)$

and the resulting possible sums occur:

$HZ = HB + BZ$
$HB = HZ + BZ$
$BZ = HZ + HB$

which, when multiplied by the above three factors, form the **conjugate foci equations**.

The conjugate foci equations allow for the effect of axial refraction at a circle to be expressed as the term:

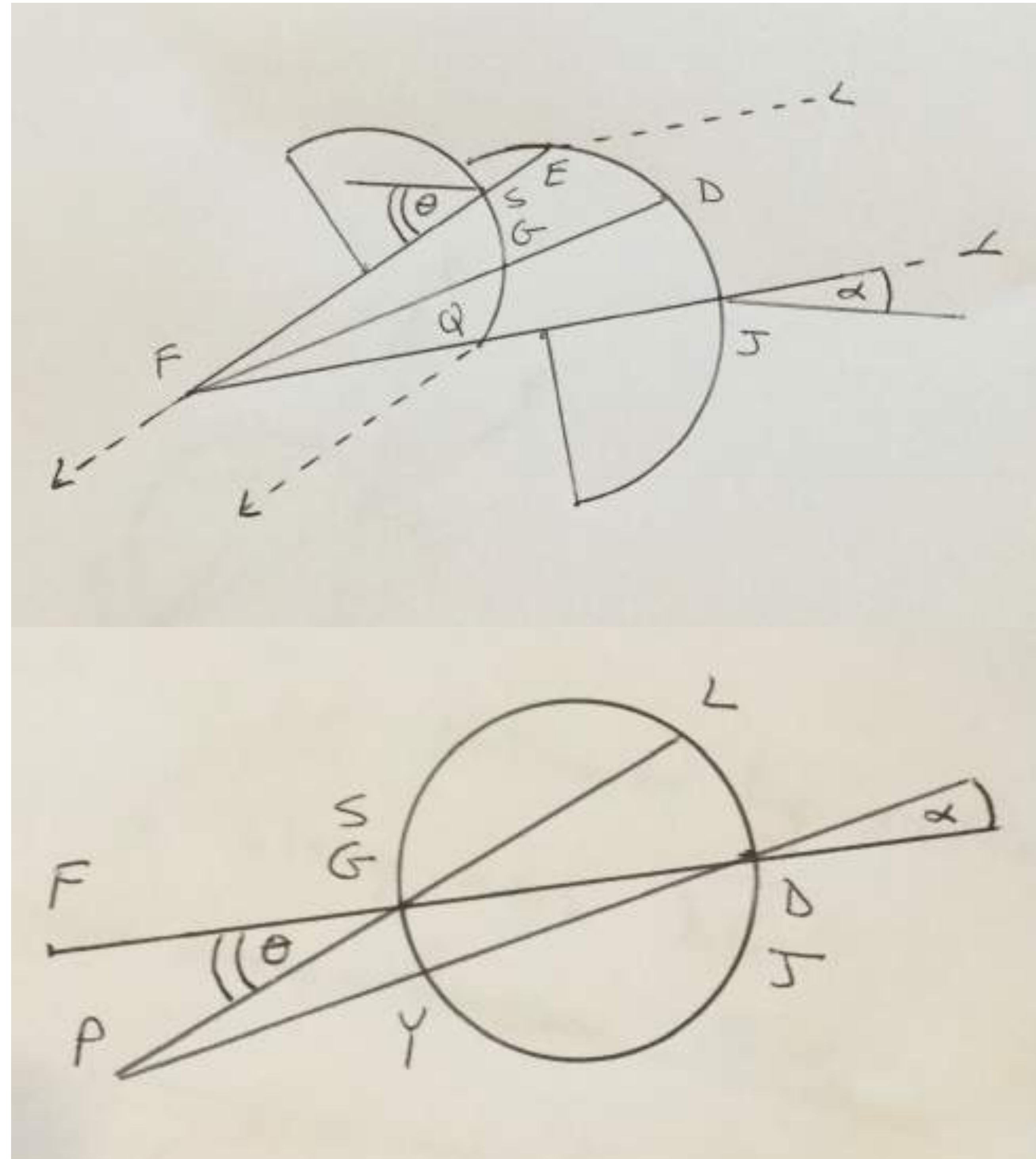
$$(1/HC) = (\boldsymbol{R}/HB)$$

which is then additive with object vergence, defined as (1/BA); or image vergence, defined as (***R***/BZ).

5). afocal angular magnification/minification

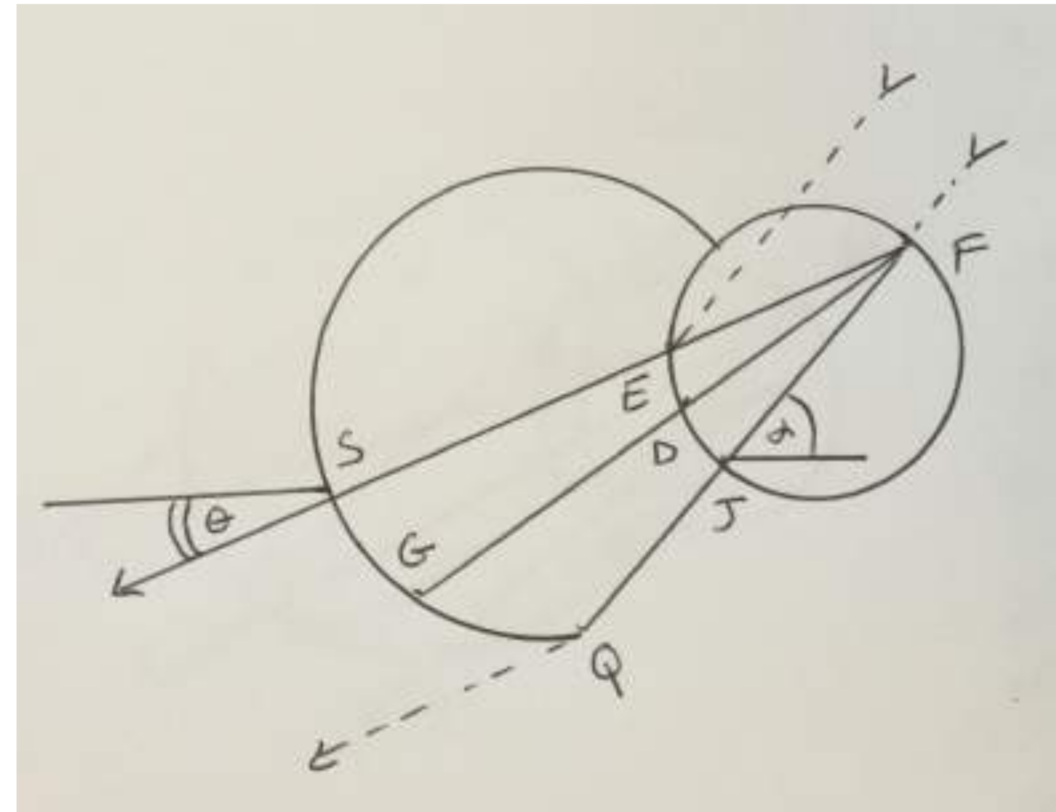
Afocal Angular Magnification

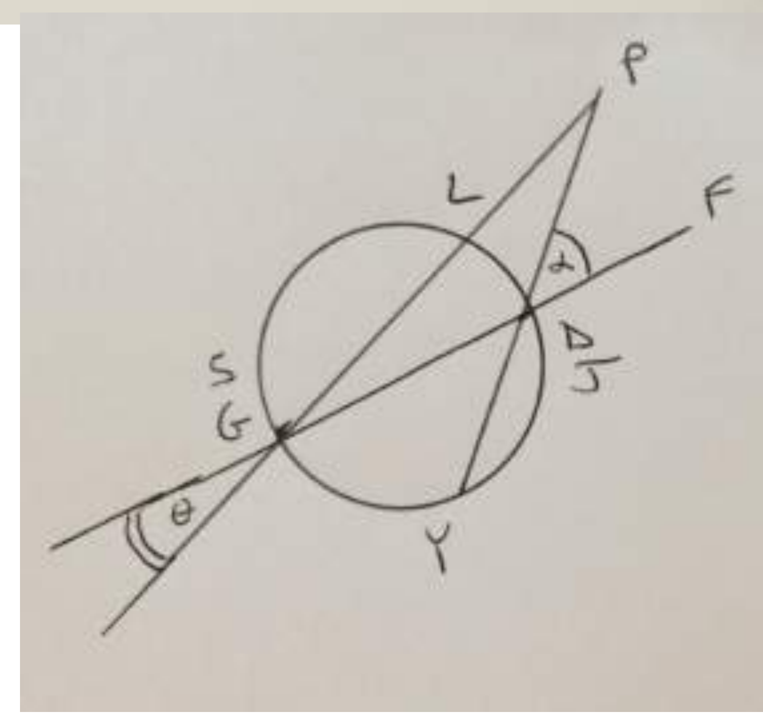
When distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis DGF as shown;
as ∠JFD = ∠SFG,
and both approach zero:



Afocal Angular Minification

Or when distance refraction at ~JDE is followed by refraction into distance at ~QGS along axis FDG, as shown;
as ∠JFD = ∠SFG,
and both approach zero:





$\theta/\alpha \Rightarrow$ (~LD/GD)/(~YG/GD) as P $\Rightarrow$ F

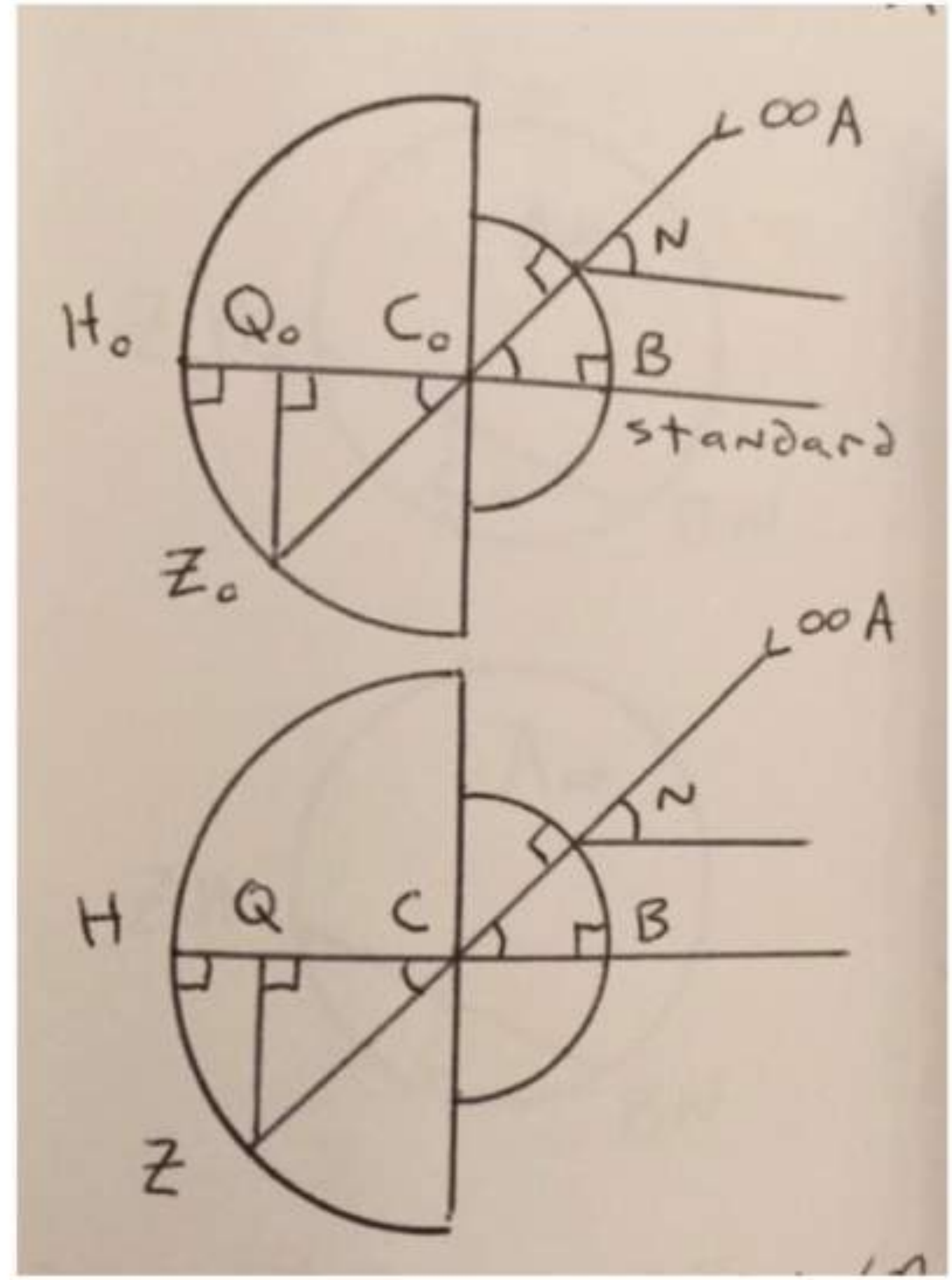
$\theta/\alpha \Rightarrow$ (FD/FG) as P $\Rightarrow$ F

so that **afocal *axial* angular magnification/minification** equals:

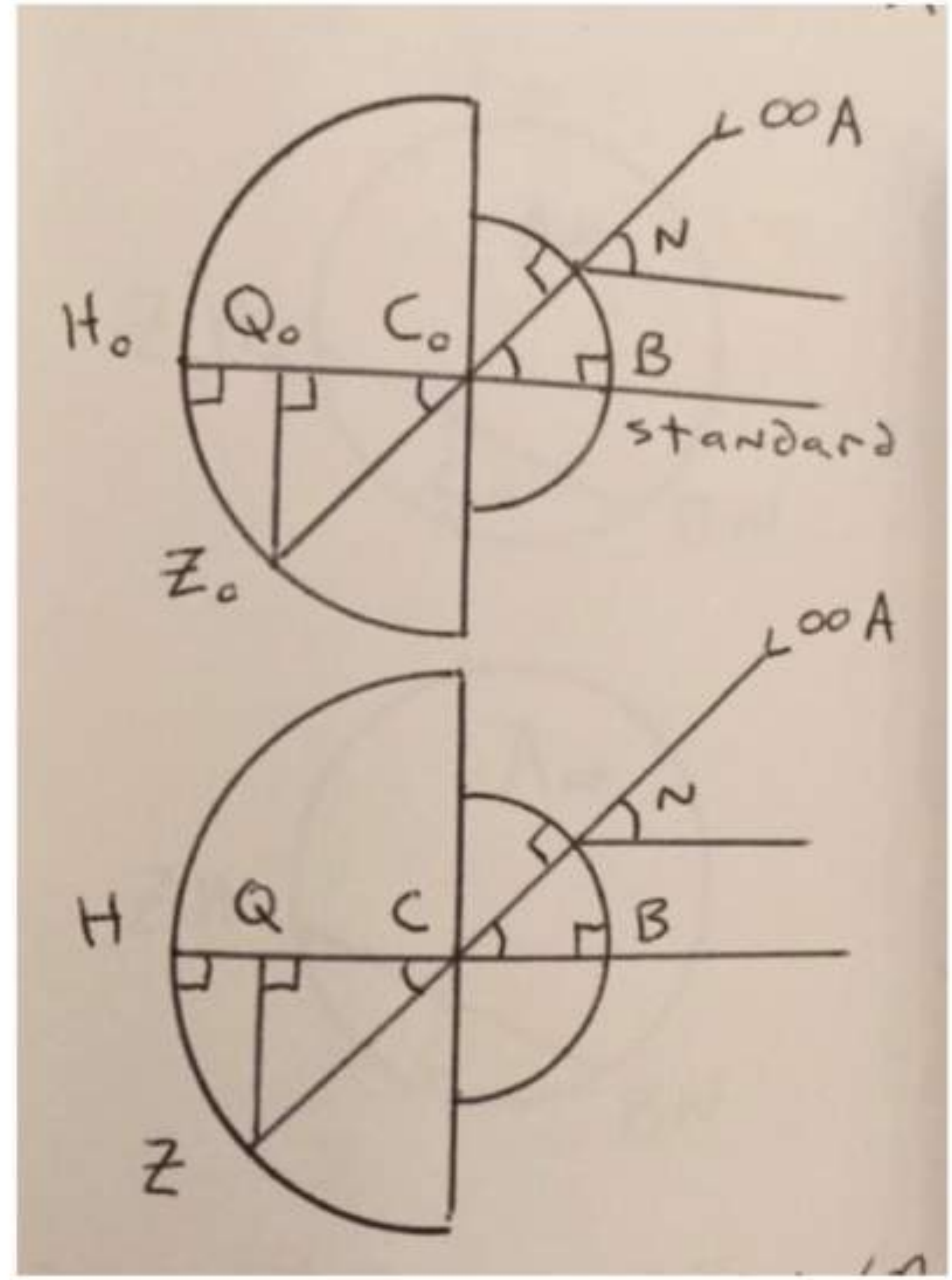
FD/FG

6). retinal image size magnification

The top diagram illustrates a standard single-surfaced eye with a distant object A, and resulting retinal image size H_oZ_o.



The bottom diagram illustrates any single-surfaced eye with a distant object A, and resulting retinal image size HZ.



As $N \Rightarrow B$, the retinal image size

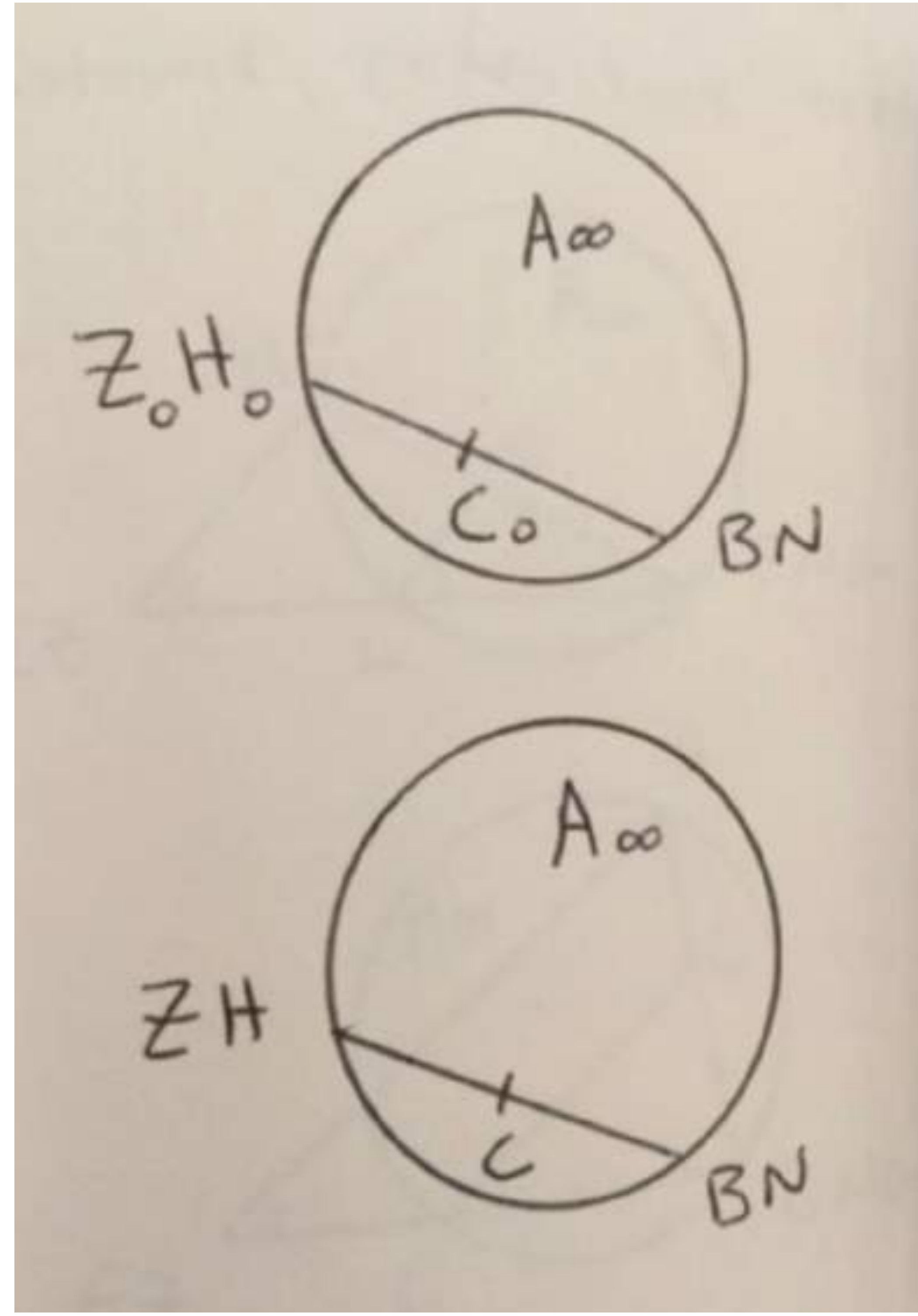
magnification, ZH/Z_oH_o, (relative to an arbitrary standard which factors out with subsequent comparisons), then approaches its *axial* value:

$$ZQ/Z_oQ_o = ZC/Z_oC_o = HC/H_oC_o$$

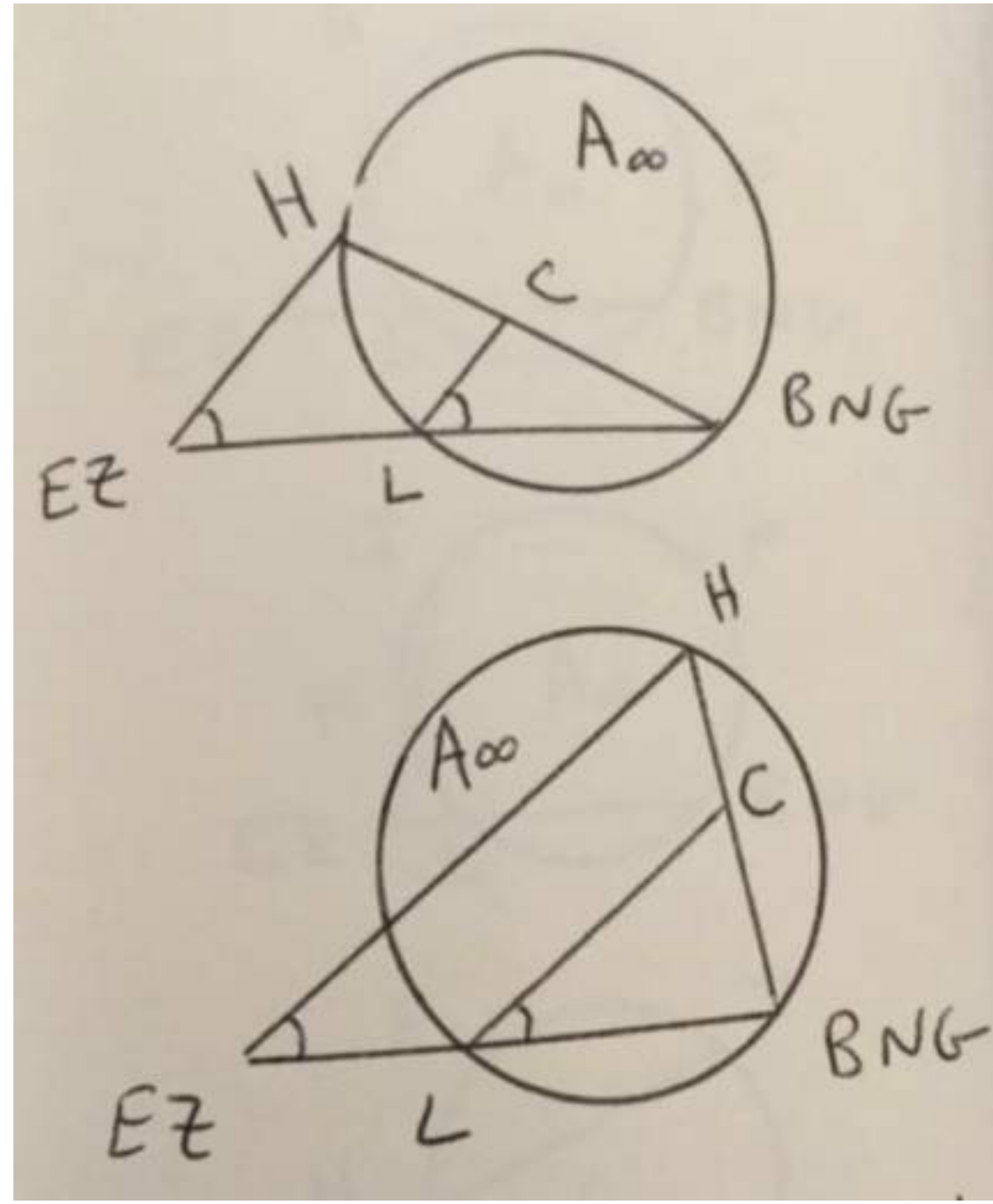
$$= (BH/\boldsymbol{R})/(BH_o/\boldsymbol{R}) = BH/BH_o$$

7). axial magnification of distance correction

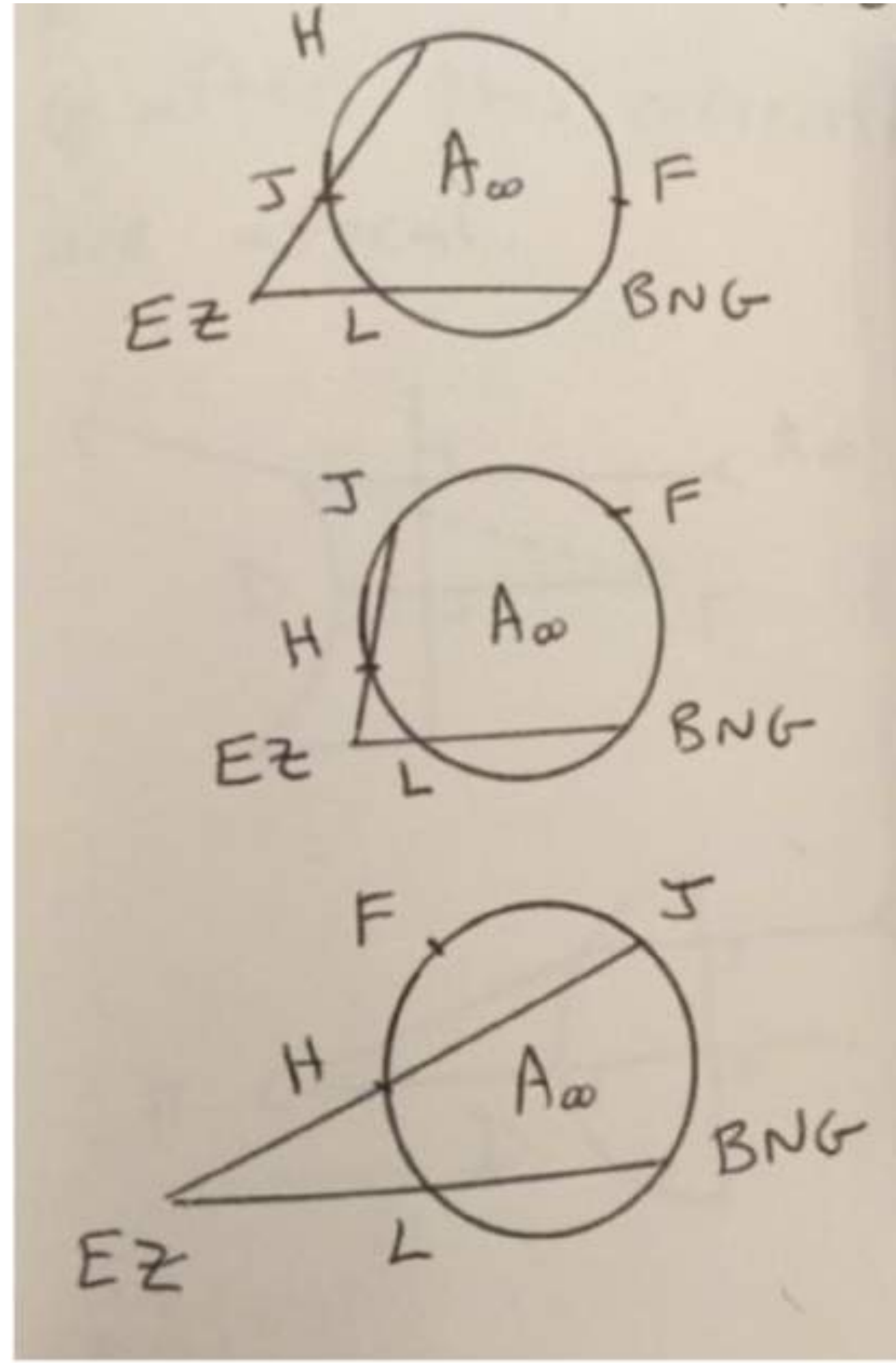
Once again representing the optic axis BCZ as a circle of infinite radius, the distant object A is focused by the curve of radius BC towards the axial object Z, (which lies at the retina H when there is no distance refractive error).



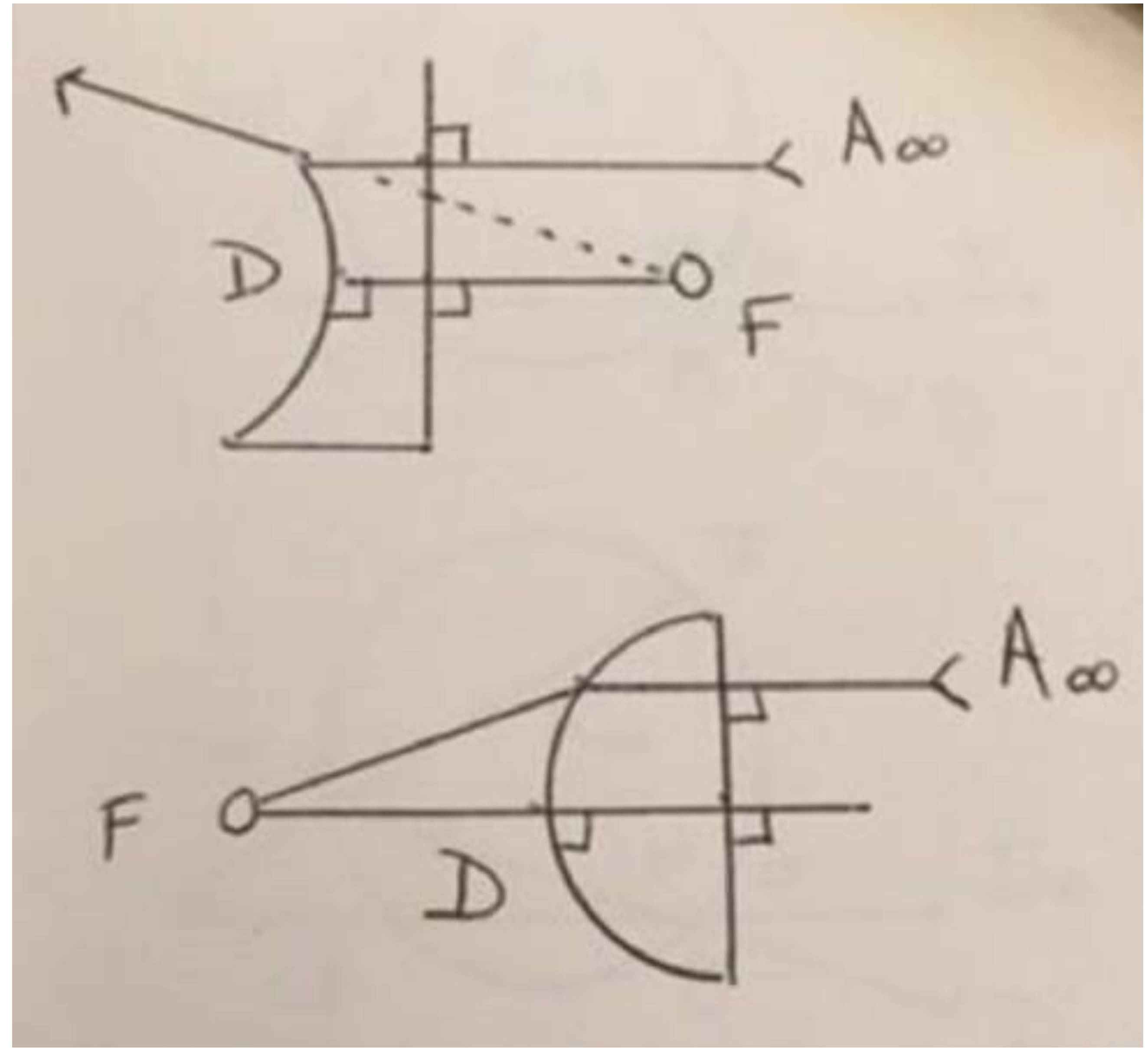
additional refraction at G (at B) will create distance refractive error and a combined single refractive surface of radius BL.



A distance correction must focus the distant object A towards the focal point F of the refractive error G, so that JF || BE, in order to move Z back to H.

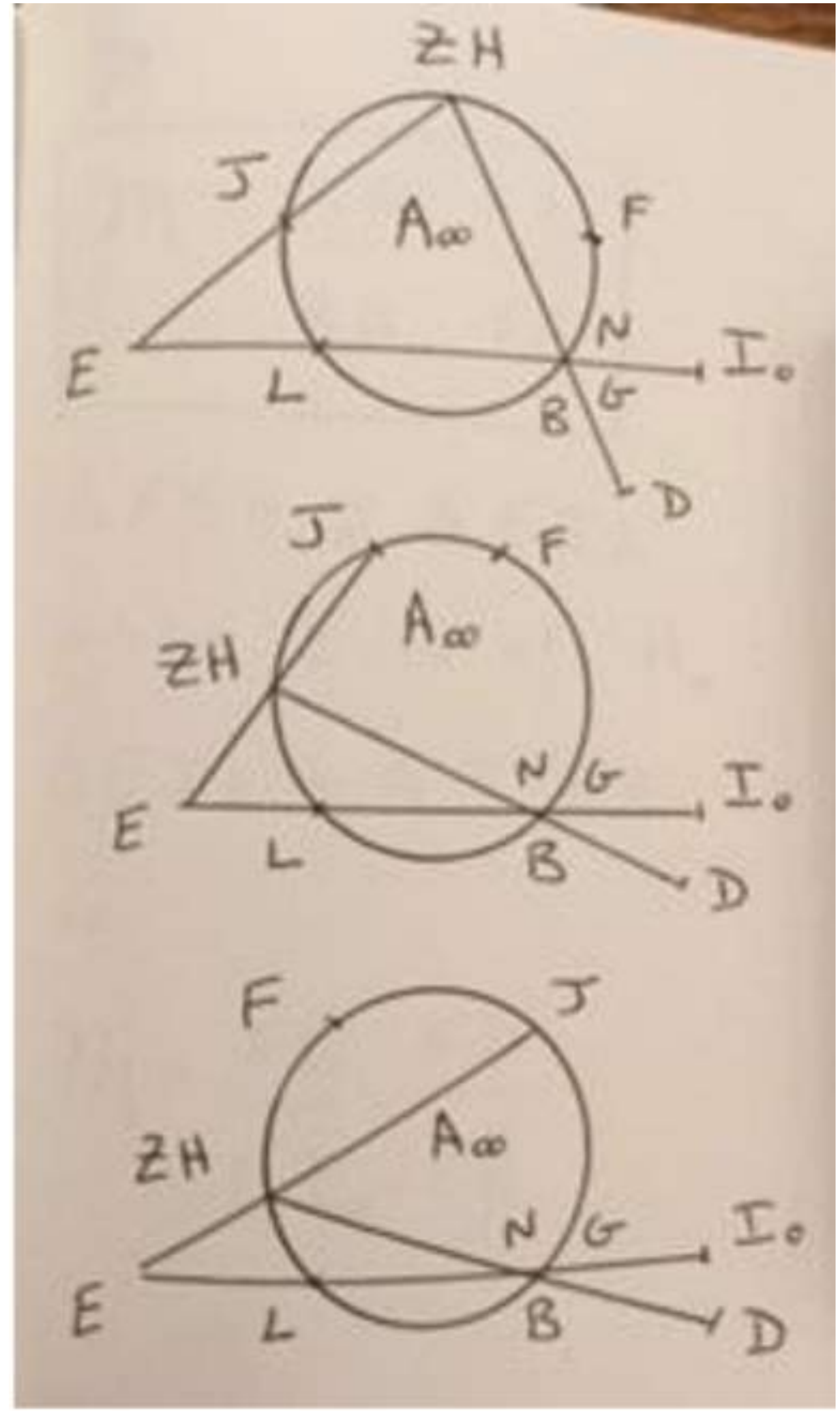


The distance correction at D:



Since the distance correction at D moves Z to H, rays leaving G after this correction must be afocal, resulting in afocal axial angular magnification equaling:

FD/FG (= FD/FB)



The (total) axial magnification of distance correction equals:

$$M = (BH/BH_o)(FD/FB)$$

$\Delta EBH \cong \Delta EJL$

If E is at H_o, the distance refractive error is completely due to an axial length that is not standard.

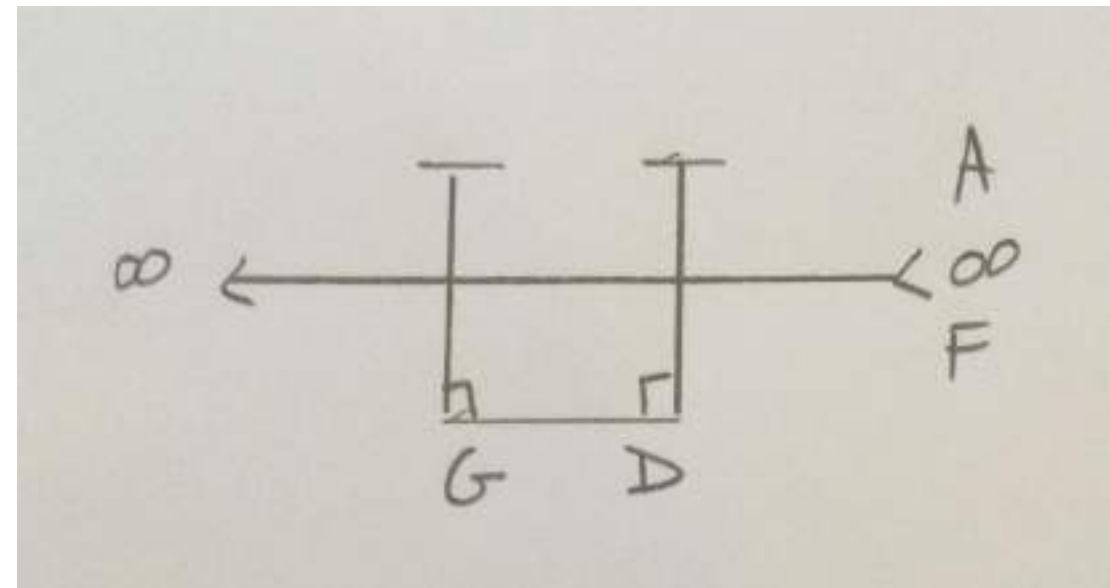
If $\Delta EJL \cong \Delta I_oFB$, then:

$$M = (FB/FI_o)(FD/FB) = FD/FI_o$$

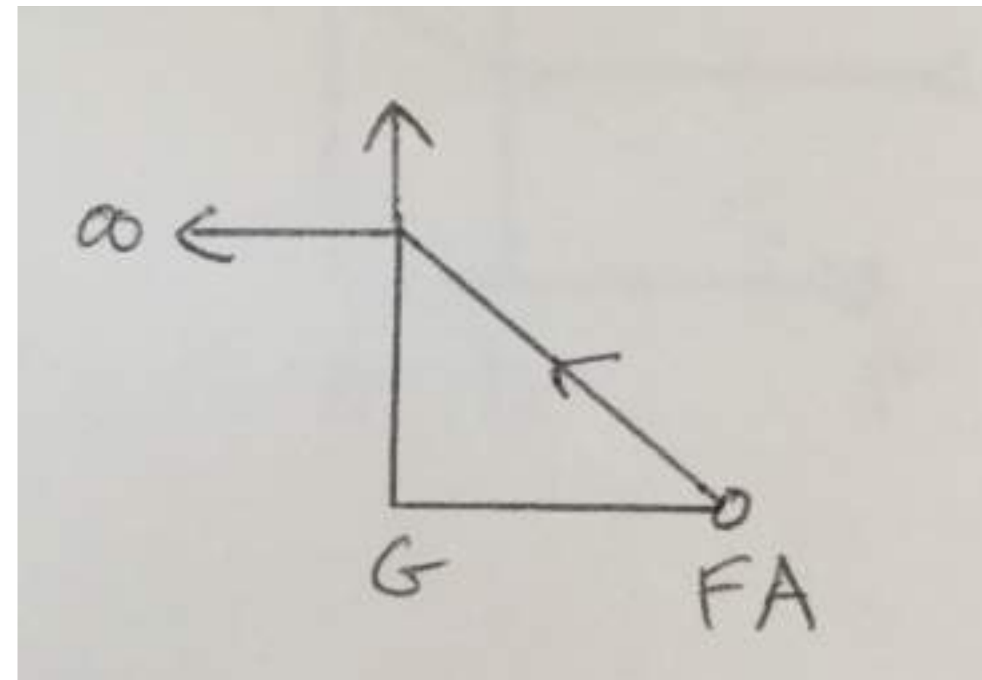
There is then no (total) axial magnification of distance correction if the correction D lies at I_o, the front focal point of the standard eye.

8). axial magnification of near correction

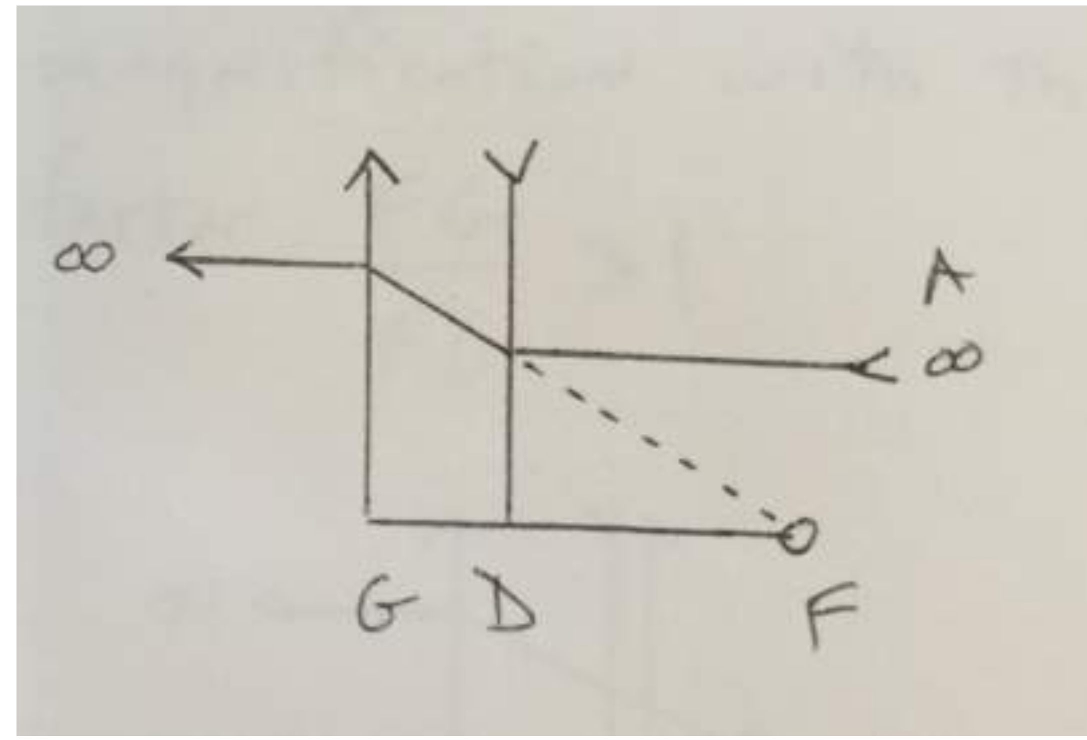
There is no afocal axial angular magnification FD/FB when object A is at distance with an emetropic eye.
(The refractive error at G, (at B), is zero; and the focal point F of that refractive error lies at infinity).



There is also no afocal axial angular magnification when object A is at the front focal point of an uncorrected myopic eye. (The system is not afocal, and involves only one refracting element).



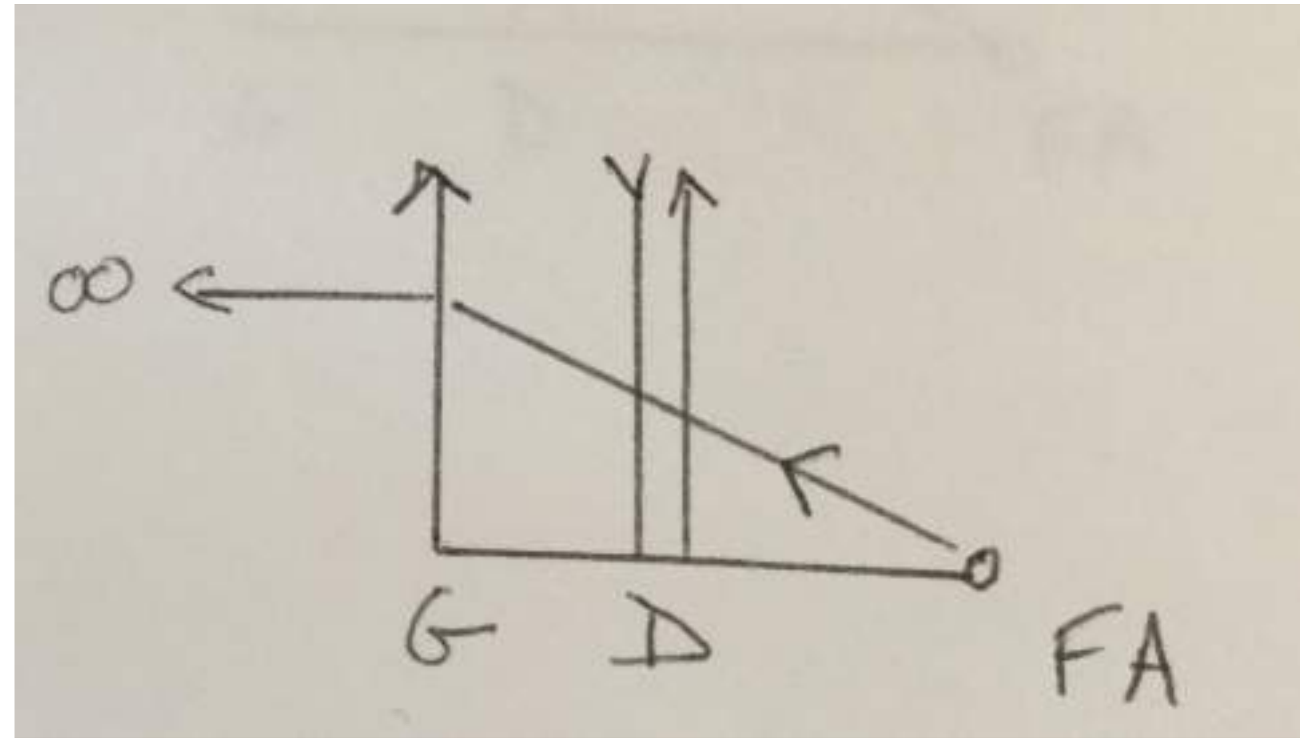
As discussed, a distance myopic correction at D creates afocal axial angular minification:



FD/FG $<$ 1

and this is relative to either the myopic eye with object A at its front focal point F, or the emetropic eye with object A at distance.

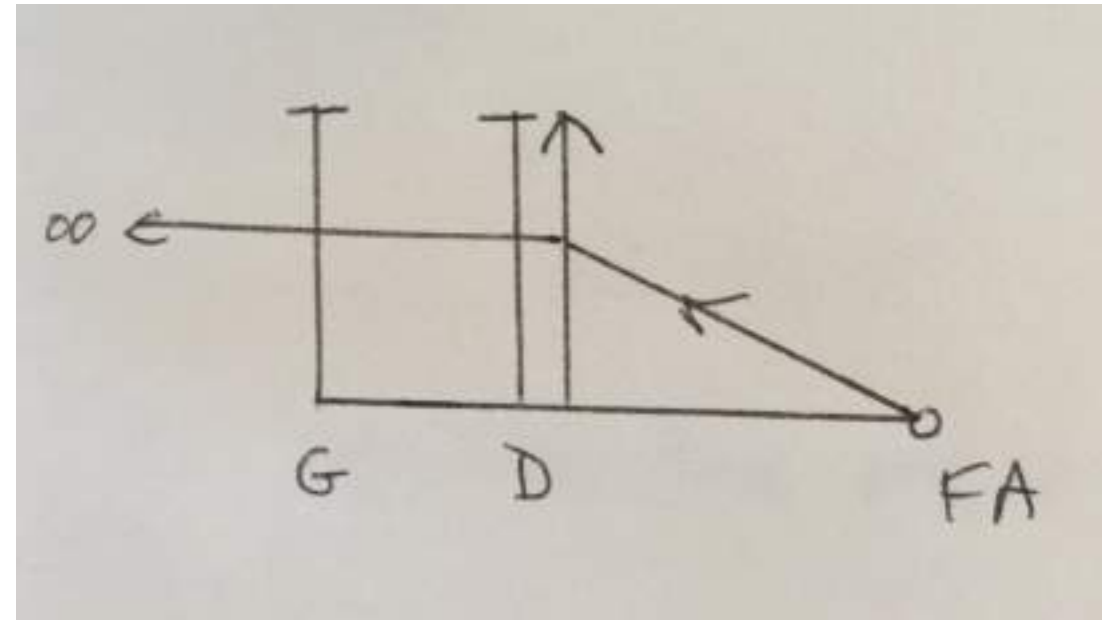
Removing the myopic distance correction at D with a converging lens at D removes this afocal axial angular magnification with the factor:



FG/FD > 1

and this magnification of near correction is relative to the distance corrected myope.

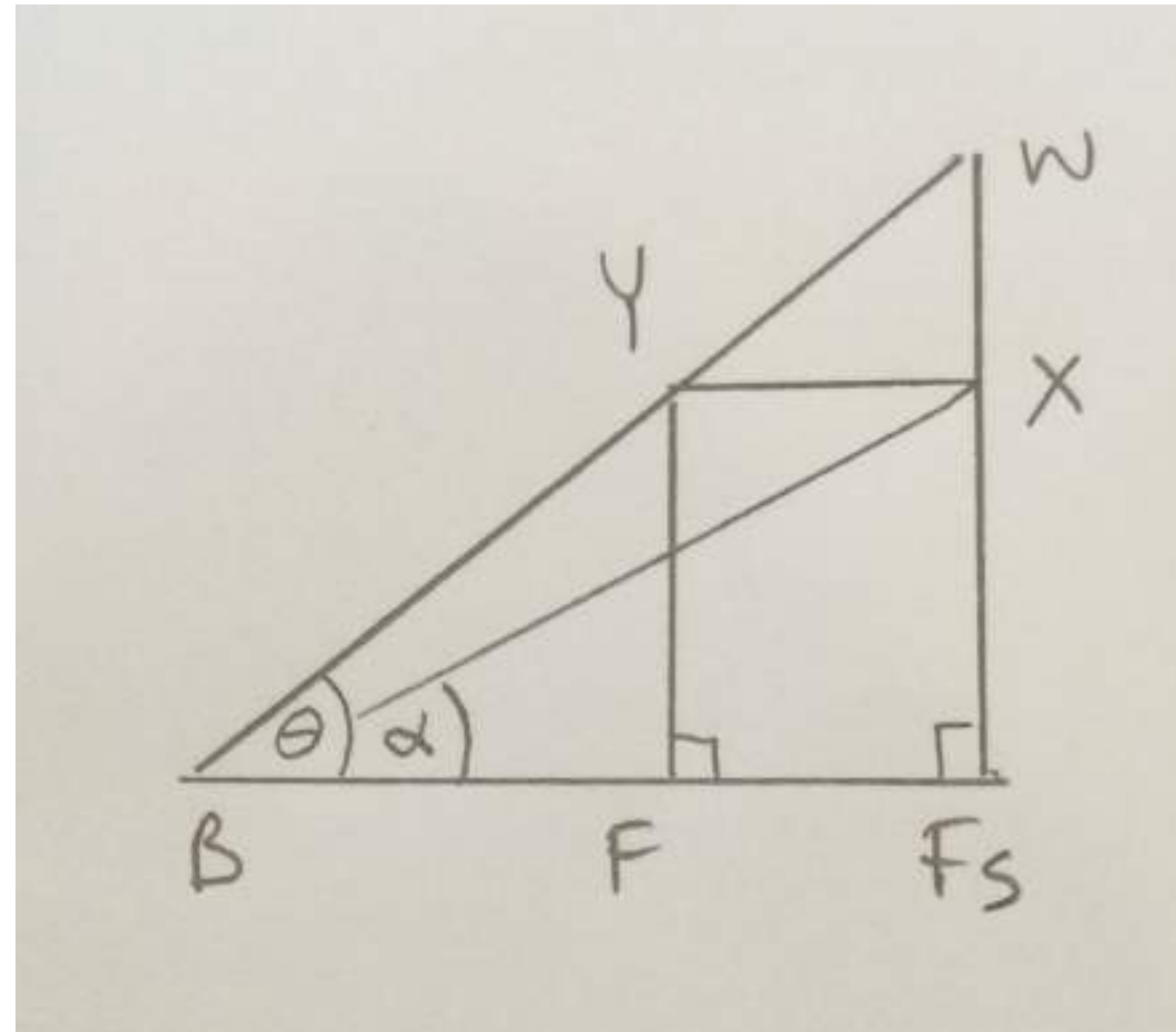
If additional converging power is added to the converging lens so that the near focal point is in focus for an *emetropic* eye, which we then consider to be the reference eye, the magnification of near correction is still that which is removed with the factor:



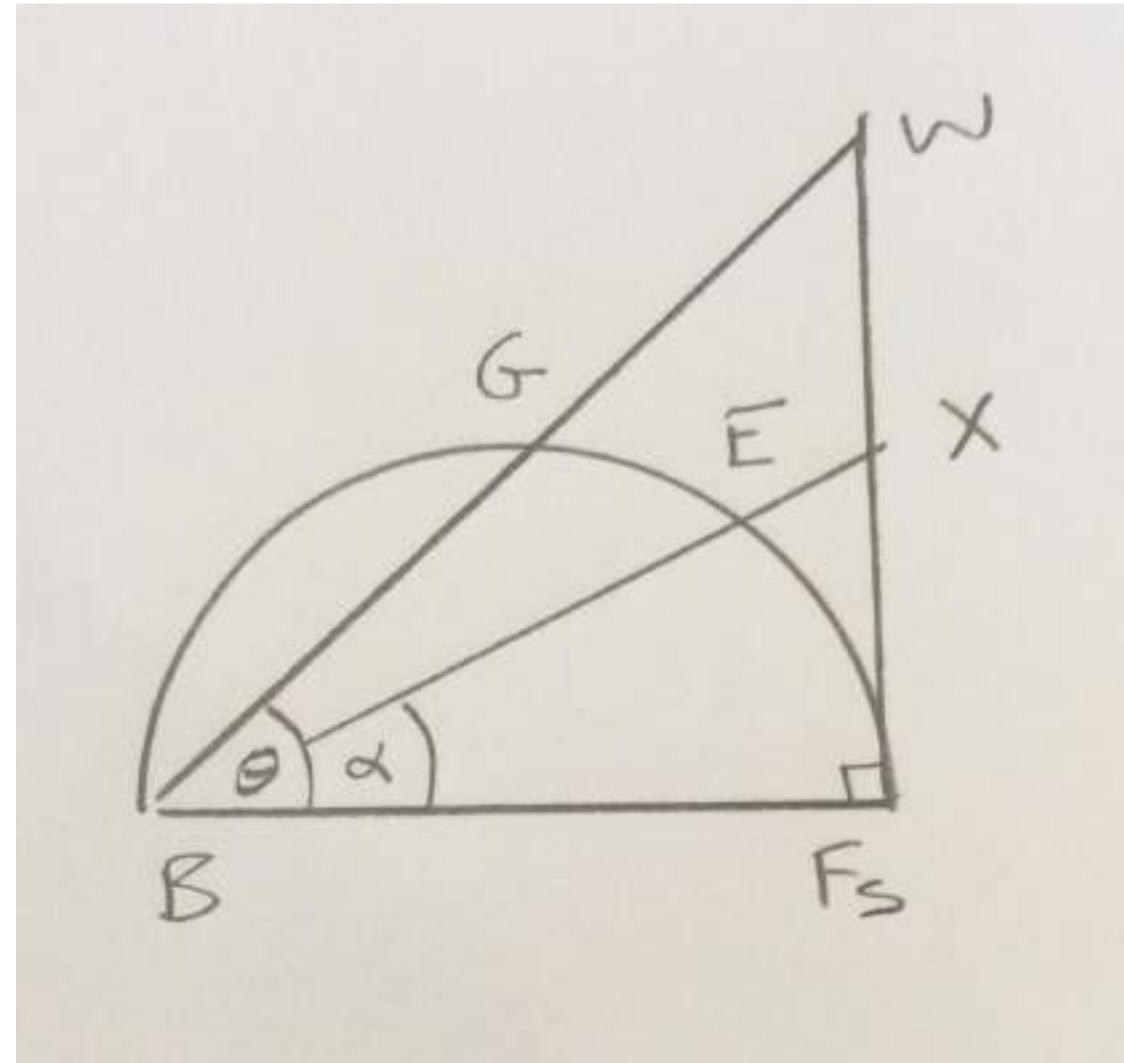
FG/FD > 1

9). object angular subtense magnification

When an object at a standard distance Fs is moved to F:



The object angular subtense magnification equals:



$\theta/\alpha = (\sim GFs/BFs)/(\sim EFs/BFs)$

as XFs $\Rightarrow$ 0

the object angular subtense magnification approaches its axial value:

$\theta/\alpha \Rightarrow WFs/XFs = WFs/YF = BFs/BF$

which equals the *axial* object angular subtense magnification.

The ratio describing axial object angular subtense magnification:

BFs/BF

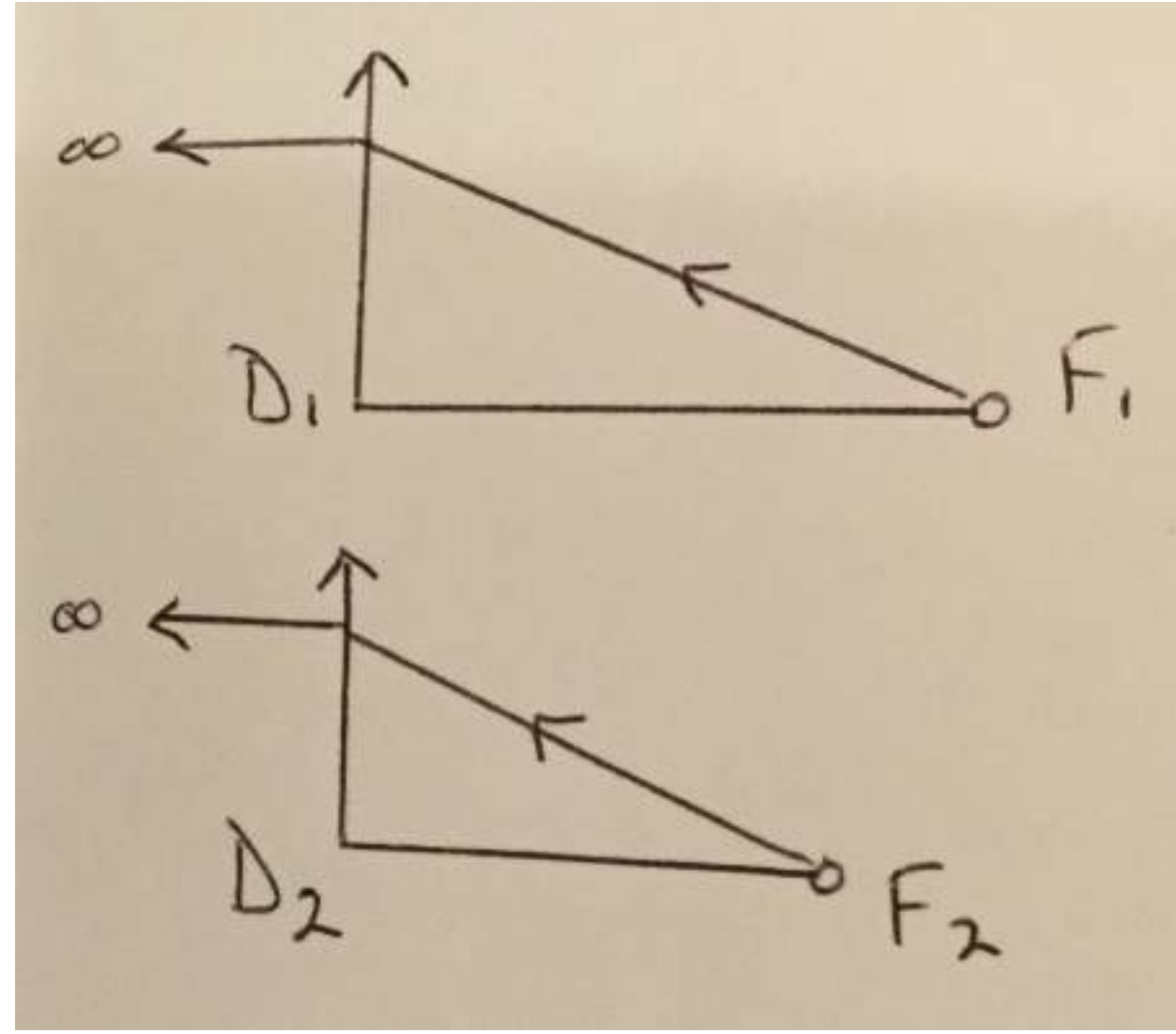
when multiplied by the ratio describing near magnification due to a single converging lens producing parallel light for an emmetropic eye:

FB/FD

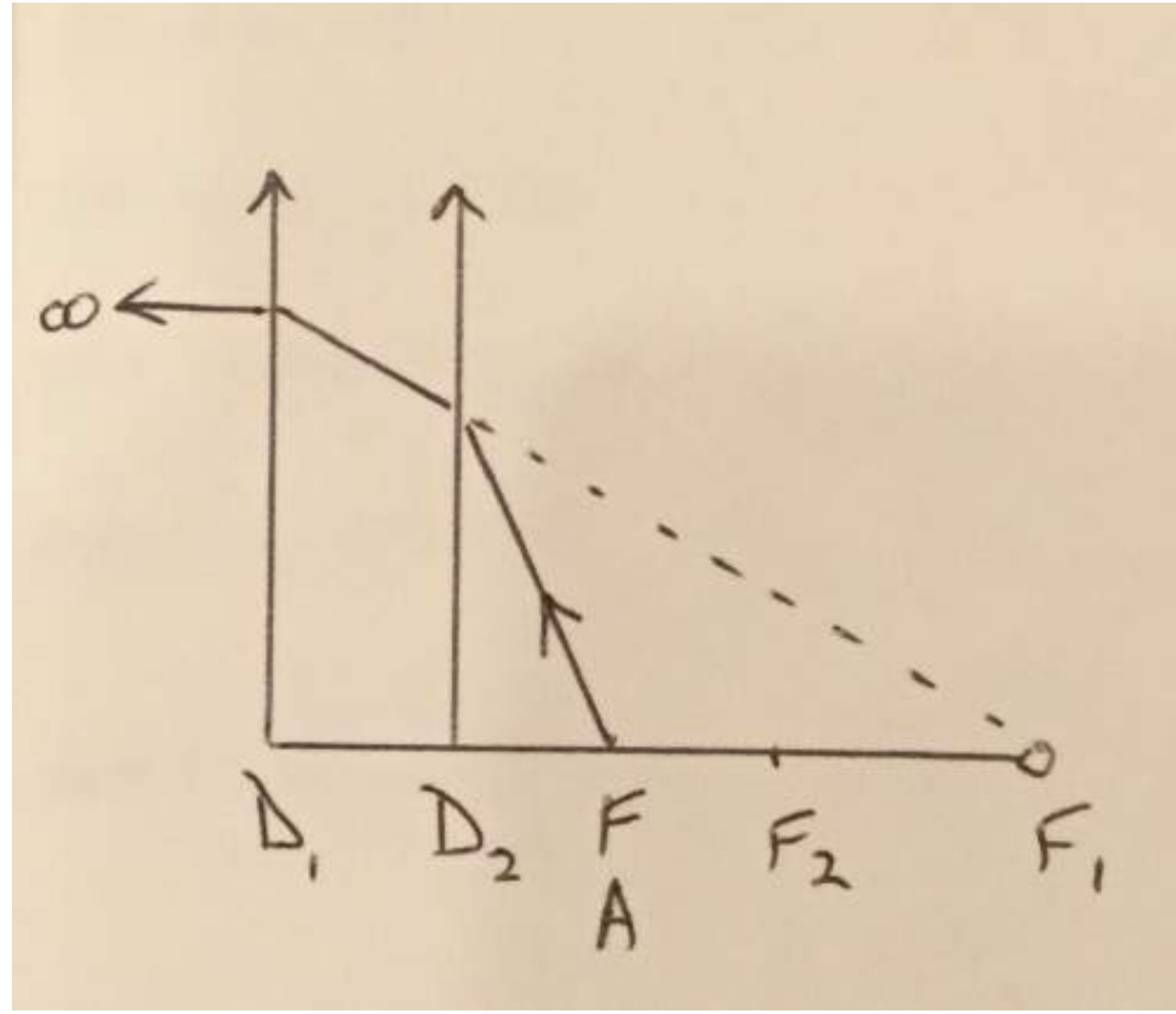
produces a ratio which factors out the object's actual distance to the eye, confirming that when a converging lens is used with its front focal point at the object, so parallel light leaves the converging lens from the object, the image size is the same regardless of the object-to-eye distance.

10). stand magnifier magnification

When the converging lens at D is split into two converging lenses:



with the same combined focus F:



the ratio describing axial near magnification due to a single converging lens producing parallel light for an emmetropic eye:

FB/FD

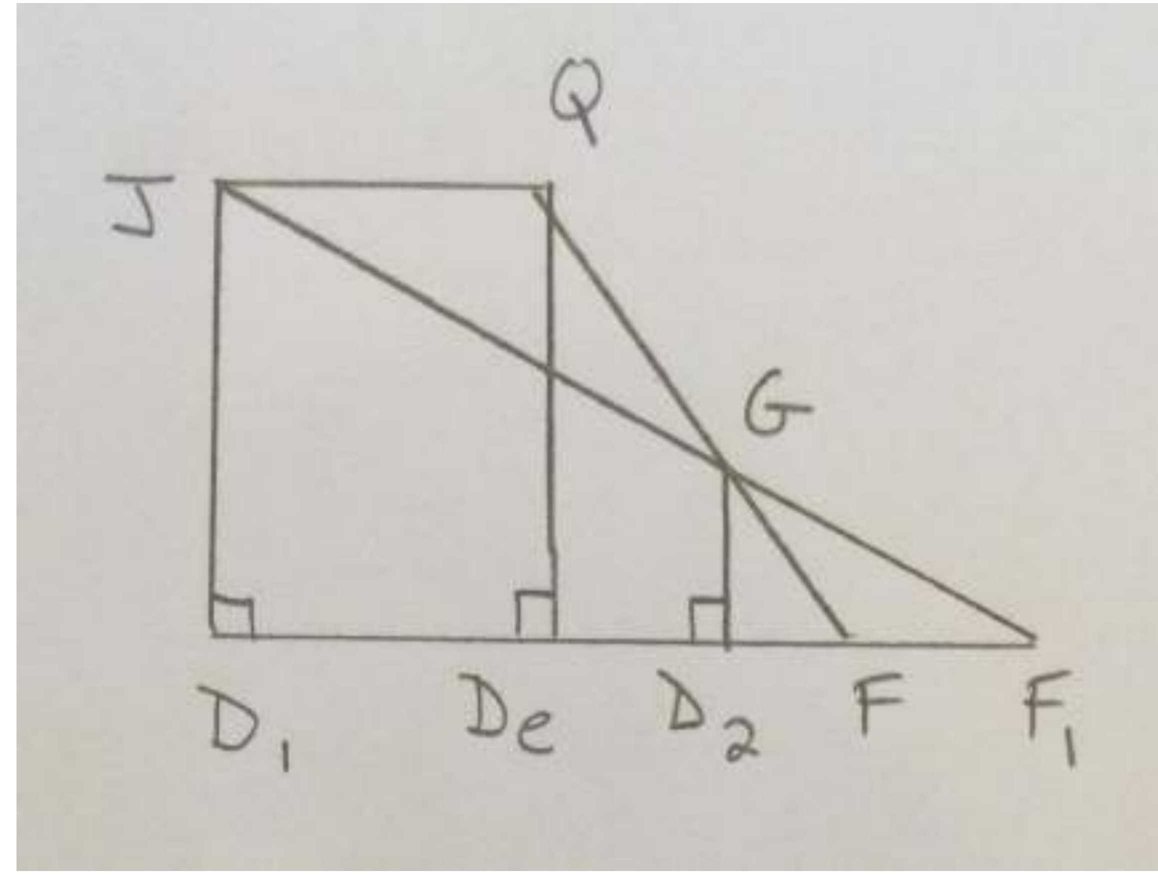
must be expressed ***as if*** all convergence occurred at a single unknown axial point De:

FB/FDe

De can be located using triangles.

$D_2G/D_2F = DeQ/DeF$

$D_2G/D_2F_1 = D_1J/D_1F_1$



$D_2F(DeQ/DeF) = D_2F_1(D_1J/D_1F_1)$

$DeQ/DeF = (D_2F_1/D_2F)(D_1J/D_1F_1)$

$1/DeF = (D_2F_1/D_2F)(1/D_1F_1)$

$FB/FDe = (D_2F_1/D_2F)(FB/D_1F_1)$

Multiplying the axial object subtense magnification by the axial magnification of near correction (relative to the same eye without refractive error) produces:

$$BFs/FDe \;=\; (D_2F_1/D_2F)(BFs/D_1F_1)$$

The converging lens D_2 creates a virtual image F_1 of an object at F. When considering a stand magnifier with lens D_2, constant stand height D_2F, and reading spectacle add or ocular accommodation D_1, the stand magnifier's (constant) enlargement of the object at F equals:

$$E = D_2F_1/D_2F$$

The stand magnifier's axial magnification is its (constant) enlargement factor E, multiplied by what would be produced by D_1 alone, if the object A were at F_1.